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The existence of infinitely many solutions for nonlinear elliptic equations involving p-Laplace type operators in $\mathbb{R}^N$

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Abstract

We are concerned with the following nonlinear elliptic equations

$$-\operatorname{div}(\varphi(x, \nabla u)) + b(x)|u|^{p-2}u = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N,$$

where the function $\varphi(x, \nu)$ is of type $|\nu|^{p-2}\nu$, $b : \mathbb{R}^N \to (0, \infty)$ is a continuous potential function, λ is a real parameter, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. In this paper, under suitable assumptions, we show the existence of infinitely many weak solutions for the problem above without assuming the Ambrosetti and Rabinowitz condition, by using the fountain theorem. Next, we give a result on the existence of a sequence of solutions for the problem above converging to zero in the L^∞-norm by employing the Moser iteration under appropriate conditions. ©2017 All rights reserved.

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1. Introduction

The celebrated mountain pass theorem of Ambrosetti and Rabinowitz [2] has become one of main tools for solving elliptic equations of variational type. Ambrosetti and Rabinowitz [2] proved the existence of infinitely many distinct pairs of solutions of second order uniformly elliptic equations, subject to the Dirichlet boundary condition as applications of their mountain pass theorem. Due to a generalization of the mountain pass theorem in [2], the existence of infinitely many solutions of the semilinear elliptic Dirichlet boundary value problem was investigated by Bartsch [3].

The multiplicity result of solutions for the following semilinear elliptic problem

$$- \Delta u + b(x)u = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

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was studied by Bartsch and Wang [5], as applying the fountain theorem which was first introduced in [4, Theorem 2.25] (see also [34]). It is well-known that the main difficulty in treating the problem (1.1) in $\mathbb{R}^N$ arises from the lack of the compactness of the Sobolev embedding, which prevents from checking directly that the energy functional associated with (1.1) satisfies the Palais-Smale condition. To overcome the difficulty of the noncompact embedding, they established a new compact embedding result under suitable conditions on the potential $b(x)$. To this end, as showing the compactness of the Palais-Smale sequence for an energy functional which plays an important role in applying the fountain theorem, they proved the existence of infinitely many solutions for the problem (1.1). Motivated by the work of [5], many authors considered elliptic problems involving the p-Laplacian in the whole space. We refer the readers to [22, 26]; see also [1, 24] for the case of variable exponents.

In the present paper, we deal with the following nonlinear elliptic equations of the p-Laplace type

$$- \operatorname{div}(\varphi(x, \nabla u)) + b(x)|u|^{p-2}u = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where the function $\varphi(x, v)$ is of type $|v|^{p-2}v$, $b : \mathbb{R}^N \to (0, \infty)$ is a continuous potential function, λ is a real parameter, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. As a natural extension of the p-Laplacian operator, many researchers have been widely studied nonlinear elliptic equations of the p-Laplace type operator in various ways; see [8, 10, 11, 22, 23] and references therein.

The first aim of this paper is to demonstrate the existence of an unbounded sequence of solutions for the problem (1.2) without assuming the following Ambrosetti and Rabinowitz condition ((AR)-condition for short) which is imposed on the nonlinearity $f(x, t)$:

(AR) There exist positive constants N and ζ such that $\zeta > p$ and

$$0 < \zeta F(x, t) \leqslant f(x, t)t, \quad \text{for} \quad x \in \Omega \quad \text{and} \quad |t| \geqslant N,$$

where $F(x, t) = \int_0^t f(x, s)\, ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

The main role of (AR)-condition in applying critical point theory is to ensure the boundedness of the Palais-Smale sequence of the functional with respect to the problem (1.2). However, this condition is very restrictive, and thus undoubtedly eliminates many nonlinearities. Accordingly, many authors in [18, 25, 27–29, 36] have made an effort to drop (AR)-condition in recent years. In particular, Jeanjean [17] obtained the existence and multiplicity of solutions for the Laplacian equation on the whole space $\mathbb{R}^N$ assuming the following condition (Je) below:

(Je) There exists $\eta \geqslant 1$ such that

$$\eta \mathcal{F}(x, t) \geqslant \mathcal{F}(x, st),$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0, 1]$, where $\mathcal{F}(x, t) = f(x, t)t - pF(x, t)$.

This condition (Je) is weaker than (AR)-condition and in past years there are numerous studies dealing with the p-Laplacian problem by assuming (Je); see [7, 27, 28]. We refer to [1, 31] for the case of variable exponents and [32] for the fractional p-Laplacian. From this point of view, we discuss the existence of an unbounded sequence of solutions for the problem (1.2) without assuming (AR)-condition under a suitable assumption for f which is slightly different from (Je). The method of approach is to apply the fountain theorem given in [34, Theorem 3.6].

The second aim is to prove the existence of small solutions for the problem (1.2), which means that the sequence of solutions converges to 0 in the L^∞-norm, relies only on local behavior under the conditions on $f(x, u)$ only for sufficiently small u. To do this, we use a method by means of global variational formulations introduced by Wang [33]. He obtained the existence of infinitely many solutions for nonlinear boundary value problems which is rather a local phenomenon by modifying and extending the functional $f(x, u)$. Utilizing the argument in [33], Naimen [30] showed that nonlinear Neumann problems have infinitely many solutions whose L^∞-norms converge to zero. In this direction, many authors considered the results for the nonlinear equations on a bounded domain in $\mathbb{R}^N$; see [9, 15, 19, 31]. To the best of

our knowledge, such a result on the whole space $\mathbb{R}^N$ is very rare. The strategy to obtain this multiplicity result involves the following two steps: Firstly, we give a regularity type result on the L^∞-bound of solutions for the problem (1.2) by applying the Moser iteration method based on the work of Drábek, Kufner, and Nicolosi in [12]. It is worth noticing that the problem (1.2) includes the potential term, so more complicated analysis has to be carefully carried out in comparison with nonlinear equations given in [12]. Secondly, we establish the existence of a sequence of solutions which converges to zero in the L^∞-norm with the aid of the regularity result.

This paper is organized as follows: In Section 2, under certain conditions on φ and f, we establish the existence of an unbounded solutions for the problem (1.2) by employing the fountain theorem (see Theorem 2.11). In Section 3, we give a result on the existence of a sequence of solutions for the problem (1.2) converging to zero in the L^∞-norm via the Moser iteration under the appropriate conditions (see Proposition 3.1 and Theorem 3.5).

2. Existence of solutions

In this section, we shall give the proof of the existence of nontrivial weak solutions for the problem (1.2), by applying the mountain pass theorem and the fountain theorem.

Let $1 < p < N$ and $p^* := Np/(N-p)$ denotes the Sobolev conjugate of p. Let the potential $b \in C(\mathbb{R}^N, \mathbb{R})$ be continuous. Suppose that

(B) $b \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x \in \mathbb{R}^N} b(x) \geqslant b_0 > 0$, $\text{meas}\{x \in \mathbb{R}^N : -\infty < b(x) \leqslant M\} < +\infty$, $\quad \forall M \in \mathbb{R}$.

Define the linear subspace

$$X := \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} b(x) |u|^p \, dx < \infty \right\},$$

with b satisfying the condition (B), then X is a reflexive separable Banach space with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} b(x) |u|^p \, dx \right)^{\frac{1}{p}},$$

which is equivalent to the norm $\| \cdot \|_{W^{1,p}(\mathbb{R}^N)}$ given by

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}}.$$

Lemma 2.1. *The following statements hold:*

(i) *There is a continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [p, p^*]$.*

(ii) *If b satisfies the assumption (B), then there is a compact embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [p, p^*)$.*

Throughout this paper, let X be the completion of $C_0^\infty(\mathbb{R}^N)$ and let X^* be the dual space of X. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^* and Euclidean scalar product on $\mathbb{R}^N$, respectively.

Definition 2.2. We say that $u \in X$ is a weak solution of the problem (1.2), if

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} b(x)|u|^{p-2} uv \, dx = \lambda \int_{\mathbb{R}^N} f(x, u)v \, dx, \tag{2.1}$$

for all $v \in X$.

We assume that $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $\Phi_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The equality

$$\Phi_0(x, \mathbf{0}) = 0,$$

holds for almost all $x \in \mathbb{R}^N$.

(J2) There are a function $a \in L^{p'}(\mathbb{R}^N)$ and a nonnegative constant a_0 such that

$$|\varphi(x, v)| \leqslant a(x) + a_0 |v|^{p-1},$$

holds for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$, where $1/p + 1/p' = 1$.

(J3) $\Phi_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

(J4) The relations

$$d|v|^p \leqslant \varphi(x, v) \cdot v \quad \text{and} \quad d|v|^p \leqslant p\Phi_0(x, v),$$

hold for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where d is a positive constant.

(J5) There exists a constant $\theta \geqslant p$ such that

$$H(x, sv) \leqslant H(x, v),$$

for $v \in \mathbb{R}^N$ and $s \in [0, 1]$, where $H(x, v) = \theta\Phi_0(x, v) - \varphi(x, v) \cdot v$, for all $x \in \mathbb{R}^N$.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0(x, \nabla u) \, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u|^p \, dx.$$

Under the assumptions (J1), (J2), and (J4), it follows from [24, Lemma 3.2] that the functional Φ is well-defined on X, $\Phi \in C^1(X, \mathbb{R})$, and its Fréchet derivative is given by

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} b(x)|u|^{p-2}uv \, dx.$$

Let us set $F(x, t) = \int_0^t f(x, s) \, ds$. For $1 < p < q < p^*$, we assume that

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot, t)$ is measurable for all $t \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^N$.

(F2) f satisfies the following growth condition: for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$,

$$|f(x, t)| \leqslant \sigma(x) |t|^{q-1},$$

where $\sigma \in L^{\gamma_1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\gamma_1 = \frac{p^*}{p^* - q}$.

(F3) $\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^p} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

(F4) There exists a constant $\mu \geqslant 1$ such that

$$\mu\mathcal{F}(x, t) \geqslant \mathcal{F}(x, st),$$

for $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0, 1]$, where $\mathcal{F}(x, t) = f(x, t)t - \theta F(x, t)$ and θ is given in (J5).

(F5) $f(x, -t) = -f(x, t)$ holds for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

Define the functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u) \, dx.$$

Then it is easy to check that $\Psi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} f(x, u)v\,dx,$$

for any $u, v \in X$ (see [34]). Next we define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

Then it follows that the functional $I_\lambda \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle I_\lambda'(u), v \rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v\,dx + \int_{\mathbb{R}^N} b(x)|u|^{p-2}uv\,dx - \lambda \int_{\mathbb{R}^N} f(x, u)v\,dx,$$

for any $u, v \in X$.

Since the assumption (F4) is weaker than the following assumption that

$$\frac{f(x, t)}{|t|^{\theta - 2} t} \quad \text{is increasing in } t > 0 \text{ and decreasing in } t < 0, \tag{2.2}$$

for any $x \in \mathbb{R}^N$, we check that the following examples satisfy the assumption (F4) by applying the assumption (2.2).

Example 2.3. Let us consider

$$\varphi(x, v) = |v|^{p-2} v, \quad \text{and} \quad f(x, t) = |t|^{q-2} t \log(1 + |t|),$$

for all $v \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$. It is clear that the function φ satisfies the assumptions (J1)-(J5) when $\theta = p$. Also, it is obvious that the function f satisfies the conditions (F1)-(F3) hold. Since the relation

$$\frac{f(x, t)}{|t|^{p-2} t} = \frac{|t|^{q-2} t \log(1 + |t|)}{|t|^{p-2} t} = |t|^{q-p} \log(1 + |t|),$$

is increasing in $t > 0$ and decreasing in $t < 0$, if $q > p = \theta$ for all $x \in \mathbb{R}^N$, it follows that the assumption (F4) holds.

Example 2.4. Let us consider

$$\varphi(x, v) = \left(1 + \frac{|v|^p}{\sqrt{1 + |v|^{2p}}} \right) |v|^{p-2} v, \quad \text{and} \quad f(x, t) = |t|^{q-2} t \log(1 + |t|),$$

for all $v \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$. In this case, put

$$\Phi_0(x, v) = \frac{1}{p} \left(|v|^p + \sqrt{1 + |v|^{2p}} - 1 \right),$$

for all $v \in \mathbb{R}^N$. It is immediately apparent that the conditions (J1)-(J4) hold. A similar argument in [6] implies that the assumption (J5) holds when $\theta = 2p$. For the sake of convenience, we give the detailed proof. In fact, we have, for all $s \in [0, 1]$

$$H(x, v) - H(x, sv) = 2p\Phi_0(x, v) - \varphi(x, v) \cdot v - 2p\Phi_0(x, sv) + \varphi(x, sv) \cdot sv$$

$$= 2p \left(\frac{1}{p} |v|^p + \frac{\sqrt{1 + |v|^{2p}}}{p} - \frac{1}{p} |sv|^p - \frac{\sqrt{1 + |sv|^{2p}}}{p} \right)$$

$$-\left(1 + \frac{|v|^p}{\sqrt{1 + |v|^{2p}}}\right)|v|^p + \left(1 + \frac{|sv|^p}{\sqrt{1 + |sv|^{2p}}}\right)|sv|^p$$

$$= |v|^p - |sv|^p$$

$$+ 2\sqrt{1 + |v|^{2p}} - \frac{|v|^{2p}}{\sqrt{1 + |v|^{2p}}} - 2\sqrt{1 + |sv|^{2p}} + \frac{|sv|^{2p}}{\sqrt{1 + |sv|^{2p}}}$$

$$\geqslant 2\sqrt{1 + |v|^{2p}} - \frac{|v|^{2p}}{\sqrt{1 + |v|^{2p}}} - 2\sqrt{1 + |sv|^{2p}} + \frac{|sv|^{2p}}{\sqrt{1 + |sv|^{2p}}}.$$

Define the functional $h : \mathbb{R}^N \times (\mathbb{R}^+ \cup \{0\}) \to \mathbb{R}$ by

$$h(x, \ell) = 2\sqrt{1 + \ell^{2p}} - \frac{\ell^{2p}}{\sqrt{1 + \ell^{2p}}}.$$

Then we obtain

$$\frac{\partial}{\partial \ell} h(x, \ell) = \left(2p\ell^{2p-1} - 2p\ell^{2p-1}\right) \frac{1}{\sqrt{1 + \ell^{2p}}} + \frac{p\ell^{4p-1}}{(1 + \ell^{2p})\sqrt{1 + \ell^{2p}}} \geqslant 0,$$

for all $(x, \ell) \in \mathbb{R}^N \times (\mathbb{R}^+ \cup \{0\})$. This implies that

$$H(x, v) - H(x, sv) \geqslant 0.$$

Also the same argument as in the previous example shows that f satisfies the assumption (F4) if we set $q > 2p = \theta$ for all $x \in \mathbb{R}^N$.

Next, taking inspiration from the argument given in [21, Theorem 4.1], we will show that the operator Φ' is of type (S_+) when X is the Sobolev space on the whole space $\mathbb{R}^N$; see also [8, 24]. For bounded domains in $\mathbb{R}^N$, we refer to [10, 21].

Lemma 2.5. *Assume that* (B) *and* (J1)-(J4) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and*

$$\limsup_{n \to \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leqslant 0,$$

then $u_n \to u$ *in* X *as* $n \to \infty$.

Proof. To show that Φ' is of type (S_+), let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and

$$\limsup_{n \to \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leqslant 0. \tag{2.3}$$

Since Φ is strictly convex by the assumption (J3), it is obvious that the operator Φ' is monotone, that is,

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \geqslant 0. \tag{2.4}$$

By (2.3) and (2.4), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (\varphi(x, \nabla u_n) - \varphi(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx$$

$$+ \lim_{n \to \infty} \int_{\mathbb{R}^N} b(x)(|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) \, dx$$

$$= \lim_{n \to \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = 0,$$

which yields that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (\varphi(x, \nabla u_n) - \varphi(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx = 0. \tag{2.5}$$

Now, we prove that the sequence $\{|\nabla u_n|^p\}$ is uniformly integrable in $\mathbb{R}^N$, i.e., for each $\varepsilon > 0$, there exists $\delta > 0$ such that if H is a measurable subset of $\mathbb{R}^N$ with meas(H) $\leqslant \delta$ (meas(H) is the Lebesgue measure of H), then

$$\int_H |\nabla u_n|^p \, dx \leqslant \varepsilon,$$

for every $n \in \mathbb{N}$. Given $\varepsilon > 0$, the equation (2.5) implies that there exists $n_1 \in \mathbb{N}$ such that for $n \geqslant n_1$, we get

$$\int_H (\varphi(x, \nabla u_n) - \varphi(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \leqslant \frac{\varepsilon d}{8},$$

for any measurable subset H of $\mathbb{R}^N$, which says

$$\int_H \varphi(x, \nabla u_n) \cdot \nabla u_n \, dx \leqslant \frac{\varepsilon d}{8} + \int_H \varphi(x, \nabla u) \cdot \nabla u_n + \varphi(x, \nabla u_n) \cdot \nabla u - \varphi(x, \nabla u) \cdot \nabla u \, dx. \tag{2.6}$$

According to the assumption (J4) we obtain that

$$\int_H \varphi(x, \nabla u_n) \cdot \nabla u_n \, dx \geqslant \int_H d \, |\nabla u_n|^p \, dx. \tag{2.7}$$

Given $\varepsilon_0 > 0$, there exists a constant $C(\varepsilon_0) > 0$ depending on ε_0 such that for all $a, b \in \mathbb{R}$,

$$ab \leqslant \varepsilon_0 |a|^p + C(\varepsilon_0) |b|^{p'}. \tag{2.8}$$

By using this inequality, (J2), and (J4), we deduce that

$$\varphi(x, \nabla u) \cdot \nabla u_n + \varphi(x, \nabla u_n) \cdot \nabla u - \varphi(x, \nabla u) \cdot \nabla u$$
$$\leqslant |a(x)| \nabla u_n + a_0 |\nabla u|^{p-1} \nabla u_n + |a(x)| \nabla u + a_0 |\nabla u_n|^{p-1} \nabla u - d \, |\nabla u|^p$$
$$\leqslant C\left(\frac{d}{4}\right) |a(x)|^{p'} + \frac{d}{4} |\nabla u_n|^p + a_0 C\left(\frac{d}{4a_0}\right) |\nabla u|^p + \frac{d}{4} |\nabla u_n|^p$$
$$+ \frac{d}{4a_0} |a(x)|^{p'} + C\left(\frac{d}{4a_0}\right) |\nabla u|^p + \frac{d}{4} |\nabla u_n|^p + a_0 C\left(\frac{d}{4a_0}\right) |\nabla u|^p$$
$$= \left(C\left(\frac{d}{4}\right) + \frac{d}{4a_0}\right) |a(x)|^{p'} + \frac{3d}{4} |\nabla u_n|^p + (2a_0 + 1) C\left(\frac{d}{4a_0}\right) |\nabla u|^p,$$

where C (d/4) and C (d/4a$_0$) are the constants given in (2.8). Combining this relation and (2.7), it follows from (2.6) that

$$\frac{d}{4} \int_H |\nabla u_n|^p \, dx \leqslant \int_H \left(C\left(\frac{d}{4}\right) + \frac{d}{4a_0}\right) |a(x)|^{p'} \, dx + \int_H (2a_0 + 1) C\left(\frac{d}{4a_0}\right) |\nabla u|^p \, dx + \frac{\varepsilon d}{8}. \tag{2.9}$$

Note that $a \in L^{p'}(\mathbb{R}^N)$ and $u \in X$. Then there exists $\delta > 0$ such that if meas(H) $\leqslant \delta$, then

$$\int_H \left(C\left(\frac{d}{4}\right) + \frac{d}{4a_0}\right) |a(x)|^{p'} \, dx + \int_H (2a_0 + 1) C\left(\frac{d}{4a_0}\right) |\nabla u|^p \, dx \leqslant \frac{\varepsilon d}{8}.$$

This together with (2.9) implies that

$$\frac{d}{4} \int_H |\nabla u_n|^p \, dx \leqslant \frac{\varepsilon d}{8} + \frac{\varepsilon d}{8},$$

as claimed.

On the other hand, there exists a constant $\mathcal{C} > 0$ such that

$$|\nabla u_n - \nabla u|^p \leqslant \mathcal{C}(|\nabla u_n|^p + |\nabla u|^p),$$

for all $n \in \mathbb{N}$ and for almost all $x \in \mathbb{R}^N$. Since $|\nabla u|^p \in L^1(\mathbb{R}^N)$ and $\{|\nabla u_n|^p\}$ is uniformly integrable in $\mathbb{R}^N$, $\{|\nabla u_n - \nabla u|^p\}$ is also uniformly integrable in $\mathbb{R}^N$.

Next, we claim that

$$|\nabla u_n(x) - \nabla u(x)| \to 0 \quad \text{as} \quad n \to \infty, \tag{2.10}$$

for almost all $x \in \mathbb{R}^N$. From (2.5), it follows that the sequences $\{(\varphi(x, \nabla u_n) - \varphi(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\}$ and $\{b(x)(|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u)\}$ converge to 0 in $L^1(\mathbb{R}^N; \mathbb{R}^N)$ and $L^1(\mathbb{R}^N)$ as $n \to \infty$, respectively. Hence, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a subset U of $\mathbb{R}^N$ of measure 0 such that

$$\lim_{k \to \infty} (\varphi(x, \nabla u_{n_k}) - \varphi(x, \nabla u)) \cdot (\nabla u_{n_k} - \nabla u) = 0, \tag{2.11}$$

for every $x \in \mathbb{R}^N \setminus U$. Let $x \in \mathbb{R}^N \setminus U$. By a similar argument above, we can show that $\{|\nabla u_{n_k}(x)|\}$ is bounded and so $\{\nabla u_{n_k}(x)\}$ is bounded in $\mathbb{R}^N$. By passing to a subsequence, we can assume that $\nabla u_{n_k}(x) \to \xi$ as $k \to \infty$ for some $\xi \in \mathbb{R}^N$. Then we obtain $\varphi(x, \nabla u_{n_k}) \to \varphi(x, \xi)$ in $\mathbb{R}^N$ as $k \to \infty$ and the relation (2.11) implies that

$$0 = \lim_{k \to \infty} (\varphi(x, \nabla u_{n_k}) - \varphi(x, \nabla u)) \cdot (\nabla u_{n_k} - \nabla u) = (\varphi(x, \xi) - \varphi(x, \nabla u)) \cdot (\xi - \nabla u).$$

It follows from the strict monotonicity of φ on X due to (J3) that this relation occurs only if $\xi = \nabla u$, that is, $\nabla u_{n_k}(x) \to \nabla u(x)$ in $\mathbb{R}^N$ as $k \to \infty$. Since these arguments hold for any subsequence of $\{\nabla u_n\}$, we conclude that $\nabla u_n(x) \to \nabla u(x)$ in $\mathbb{R}^N$ as $n \to \infty$ for every $x \in \mathbb{R}^N \setminus U$, i.e., (2.10) holds. Hence, we obtain that $\{|\nabla u_n - \nabla u|^p\}$ converges to zero in $\mathbb{R}^N$. Combining this with its uniform integrability, we derive

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^p \, dx = 0, \tag{2.12}$$

by Vitali's convergence theorem, that is, $\|\nabla u_n - \nabla u\|_{L^p(\mathbb{R}^N)} \to 0$ as $n \to \infty$. From the compact embedding theorem, we obtain $u_n \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. It follows from (2.12) that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Therefore, we conclude that $u_n \to u$ in X as $n \to \infty$, because $\|\cdot\|_X$ is equivalent to the norm $\|\cdot\|_{W^{1,p}(\mathbb{R}^N)}$. This completes the proof. $\qquad\square$

Proceeding arguments analogous to [13, Lemma 3.2], we can see that the functionals Ψ and Ψ' are weakly strongly continuous on X. Thus we will omit the proofs here.

Lemma 2.6. *Assume that* (B) *and* (F1)-(F2) *hold. Then Ψ and Ψ' are weakly strongly continuous on X.*

With the aid of Lemmas 2.5 and 2.6, we prove that the energy functional I_λ satisfies the Cerami condition ($(C)_c$-condition for short), i.e., for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset X$ such that $I_\lambda(u_n) \to c$ and $\|I'_\lambda(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0$ as $n \to \infty$ has a convergent subsequence. This plays a key role in finding a nontrivial weak solution for the given problem.

Remark 2.7. One of the key assumptions for proving that the functional I_λ satisfies the $(C)_c$-condition (or Palais-Smale condition) for $c \in \mathbb{R}$ is

$$f(x, t) = o(|t|^{p-1}) \text{ as } |t| \to 0, \text{ uniformly for } x \in \mathbb{R}^N, \tag{2.13}$$

see, for instance, [18, 28, 29, 35]. However, we prove the following result without the assumption (2.13).

Lemma 2.8. *Assume that* (B), (J1)-(J5) *and* (F1)-(F4) *hold. Then I_λ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\} \subset X$ be a $(C)_c$-sequence of the functional I_λ, that is,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \|I_\lambda'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0 \quad \text{as} \quad n \to \infty,$$

which shows that

$$c = I_\lambda(u_n) + o(1) \quad \text{and} \quad \langle I_\lambda'(u_n), u_n \rangle = o(1), \tag{2.14}$$

where $o(1) \to 0$ as $n \to \infty$. Since I_λ' is of type (S_+) by Lemmas 2.5, 2.6 and X is reflexive, it suffices to verify that the sequence $\{u_n\}$ is bounded in X.

Assume the contrary that the sequence $\{u_n\}$ is unbounded in X. Then we may suppose that $\|u_n\|_X > 1$ and $\|u_n\|_X \to \infty$ as $n \to \infty$. Define a sequence $\{\ell_n\}$ by $\ell_n = u_n/\|u_n\|_X$. Then it is obvious that $\{\ell_n\} \subset X$ and $\|\ell_n\|_X = 1$. Hence, up to a subsequence, still denoted by $\{\ell_n\}$, we obtain $\ell_n \rightharpoonup \ell$ in X as $n \to \infty$ and according to Lemma 2.1

$$\ell_n(x) \to \ell(x) \text{ a.e. in } \mathbb{R}^N, \quad \ell_n \to \ell \text{ in } L^q(\mathbb{R}^N), \quad \text{and} \quad \ell_n \to \ell \text{ in } L^p(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{2.15}$$

Set $\Omega_1 = \{x \in \mathbb{R}^N : \ell(x) \neq 0\}$. Using the assumption (J4) and (2.14), we assert that

$$
\begin{aligned}
c &= I_\lambda(u_n) + o(1) \\
&= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u|^p\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx + o(1) \\
&\geqslant \frac{d}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p\, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u|^p\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx + o(1) \\
&\geqslant \frac{\min\{d, 1\}}{p} \|u_n\|_X^p - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx + o(1).
\end{aligned}
$$

Since the sequence $\{u_n\}$ is the $(C)_c$-sequence, we get

$$\int_{\mathbb{R}^N} F(x, u_n)\, dx \geqslant \frac{\min\{d, 1\}}{\lambda p} \|u_n\|_X^p - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \quad \text{as} \quad n \to \infty. \tag{2.16}$$

Moreover, the assumptions (B), (J2) and the Hölder inequality imply that

$$
\begin{aligned}
I_\lambda(u_n) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u|^p\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \\
&\leqslant \int_{\mathbb{R}^N} a(x)|\nabla u_n|\, dx + \frac{a_0}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p\, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u|^p\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \\
&\leqslant \|a\|_{L^{p'}(\mathbb{R}^N)} \|\nabla u_n\|_{L^p(\mathbb{R}^N)} + (a_0 + 1)\|u_n\|_X^p - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \\
&\leqslant (C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1)\|u_n\|_X^p - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx, \tag{2.17}
\end{aligned}
$$

for some constant C_1. Then we have

$$C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1 \geqslant \frac{1}{\|u_n\|_X^p} \left(I_\lambda(u_n) + \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \right), \tag{2.18}$$

for sufficiently large n. The condition (F3) implies that there exists $t_0 > 1$ such that $F(x, t) > |t|^p$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. Since $F(x, t)$ is continuous on $\mathbb{R}^N \times [-t_0, t_0]$, there exists a positive constant M such that $|F(x, t)| \leqslant M$, for all $(x, t) \in \mathbb{R}^N \times [-t_0, t_0]$. Therefore we can choose a real number M_0 such that $F(x, t) \geqslant M_0$, for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{F(x, u_n(x)) - M_0}{\|u_n\|_X^p} \geqslant 0, \tag{2.19}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. By the relation (2.15), we know that $|u_n(x)| = |\ell_n(x)| \|u_n\|_X \to \infty$ as $n \to \infty$ for all $x \in \Omega_1$. Furthermore, it follows from the condition (F3) that

$$\lim_{n \to \infty} \frac{F(x, u_n(x))}{\|u_n\|_X^p} = \lim_{n \to \infty} \frac{F(x, u_n(x))}{|u_n(x)|^p} |\ell_n(x)|^p = \infty, \tag{2.20}$$

for all $x \in \Omega_1$. Hence we get that $|\Omega_1| = 0$. Indeed, if $|\Omega_1| \neq 0$, then according to (2.16), (2.18), (2.19), (2.20), and the Fatou lemma, we deduce

$$
\begin{aligned}
\frac{1}{\lambda} \left(C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1 \right) &= \lim_{n \to \infty} \frac{(C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1) \int_{\mathbb{R}^N} F(x, u_n(x)) \, dx}{I_\lambda(u_n) + \lambda \int_{\mathbb{R}^N} F(x, u_n(x)) \, dx} \\
&\geqslant \liminf_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))}{\|u_n\|_X^p} \, dx \\
&\geqslant \liminf_{n \to \infty} \int_{\Omega_1} \frac{F(x, u_n(x))}{\|u_n\|_X^p} \, dx - \liminf_{n \to \infty} \int_{\Omega_1} \frac{M_0}{\|u_n\|_X^p} \, dx \\
&\geqslant \liminf_{n \to \infty} \int_{\Omega_1} \frac{F(x, u_n(x)) - M_0}{\|u_n\|_X^p} \, dx \\
&\geqslant \int_{\Omega_1} \liminf_{n \to \infty} \frac{F(x, u_n(x)) - M_0}{\|u_n\|_X^p} \, dx \\
&= \int_{\Omega_1} \liminf_{n \to \infty} \frac{F(x, u_n(x))}{|u_n(x)|^p} |\ell_n(x)|^p \, dx - \int_{\Omega_1} \lim_{n \to \infty} \frac{M_0}{\|u_n\|_X^p} \, dx \\
&= \infty,
\end{aligned}
$$

which is a contradiction. Thus $\ell(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Since $I_\lambda(tu_n)$ is continuous in $t \in [0, 1]$, for each $n \in \mathbb{N}$, there is $t_n \in [0, 1]$ such that

$$I_\lambda(t_n u_n) := \max_{t \in [0,1]} I_\lambda(tu_n).$$

Let $\{d_k\}$ be a positive sequence of real numbers such that $\lim_{k \to \infty} d_k = \infty$ and $d_k > 1$ for any k. Then it is clear that $\|d_k \ell_n\|_X = d_k > 1$ for any k and n. Fix k, since $\ell_n \to 0$ strongly in the space $L^q(\mathbb{R}^N)$ as $n \to \infty$, it follows from the continuity of Nemytskii operator that $F(x, d_k \ell_n) \to 0$ in $L^1(\mathbb{R}^N)$, which asserts

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} F(x, d_k \ell_n) \, dx = 0. \tag{2.21}$$

Since $\|u_n\|_X \to \infty$ as $n \to \infty$, we have $\|u_n\|_X > d_k$ for sufficiently large n. Hence we know by (J4) and (2.21) that

$$
\begin{aligned}
I_\lambda(t_n u_n) &\geqslant I_\lambda \left(\frac{d_k}{\|u_n\|_X} u_n \right) = I_\lambda(d_k \ell_n) \\
&= \int_{\mathbb{R}^N} \Phi_0(x, \nabla d_k \ell_n) \, dx - \lambda \int_{\mathbb{R}^N} F(x, d_k \ell_n) \, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x) |d_k \ell_n|^p \, dx \\
&\geqslant \frac{d}{p} \int_{\mathbb{R}^N} a(x) |\nabla d_k \ell_n|^p \, dx - \lambda \int_{\mathbb{R}^N} F(x, d_k \ell_n) \, dx + \frac{b_0}{p} \int_{\mathbb{R}^N} |d_k \ell_n|^p \, dx \\
&\geqslant C_2 \|d_k \ell_n\|_X^p - \lambda \int_{\mathbb{R}^N} F(x, d_k \ell_n) \, dx \\
&\geqslant \frac{C_2}{2} d_k^p,
\end{aligned}
$$

for sufficiently large n, where $C_2 := \min\{d, b_0\}/p$. Letting n and k tend to infinity, we conclude that

$$\lim_{n \to \infty} I_\lambda(t_n u_n) = \infty. \tag{2.22}$$

For any n large enough, this implies that $I_\lambda(t_n u_n) > 0 = I_\lambda(0) = I_\lambda(0 u_n)$, and thus $t_n > 0$. If $t_n < 1$, then $\frac{d}{dt} I_\lambda(t u_n)|_{t=t_n} = 0$, and so $\langle I'_\lambda(t_n u_n), t_n u_n \rangle = 0$. If $t_n = 1$, then we have $\langle I'_\lambda(u_n), u_n \rangle = 0$. Hence we always obtain

$$\langle I'_\lambda(t_n u_n), t_n u_n \rangle = o(1),$$

for sufficiently large n.

On the other hand, due to the assumptions (J5) and (F4), for all n large enough we obtain that

$$
\begin{aligned}
\frac{1}{\mu} I_\lambda(t_n u_n) &= \frac{1}{\mu} I_\lambda(t_n u_n) - \frac{1}{\theta\mu} \langle I'_\lambda(t_n u_n), t_n u_n \rangle + o(1) \\
&= \frac{1}{\mu} \int_{\mathbb{R}^N} \Phi_0(x, t_n \nabla u_n)\, dx - \frac{\lambda}{\mu} \int_{\mathbb{R}^N} F(x, t_n u_n)\, dx + \frac{1}{p\mu} \int_{\mathbb{R}^N} b(x)|t_n u_n|^p\, dx \\
&\quad - \frac{1}{\theta\mu} \int_{\mathbb{R}^N} \varphi(x, t_n \nabla u_n) \cdot (t_n \nabla u_n)\, dx + \frac{\lambda}{\theta\mu} \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n\, dx \\
&\quad - \frac{1}{\theta\mu} \int_{\mathbb{R}^N} b(x)|t_n u_n|^p\, dx + o(1) \\
&= \frac{1}{\theta\mu} \int_{\mathbb{R}^N} H(x, t_n \nabla u_n)\, dx + \frac{\lambda}{\theta\mu} \int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)\, dx + \left(\frac{1}{p\mu} - \frac{1}{\theta\mu} \right) \int_{\mathbb{R}^N} b(x)|t_n u_n|^p\, dx + o(1) \\
&\leqslant \frac{1}{\theta} \int_{\mathbb{R}^N} H(x, \nabla u_n)\, dx + \frac{\lambda}{\theta} \int_{\mathbb{R}^N} \mathcal{F}(x, u_n)\, dx + \left(\frac{1}{p} - \frac{1}{\theta} \right) \int_{\mathbb{R}^N} b(x)|u_n|^p\, dx + o(1) \\
&= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x)|u_n|^p\, dx \\
&\quad - \frac{1}{\theta} \left(\int_{\mathbb{R}^N} \varphi(x, \nabla u_n) \cdot \nabla u_n\, dx - \lambda \int_{\mathbb{R}^N} f(x, u_n) u_n\, dx + \int_{\mathbb{R}^N} b(x)|u_n|^p\, dx \right) + o(1) \\
&= I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle + o(1) \to c \quad \text{as} \quad n \to \infty,
\end{aligned}
$$

which contradicts (2.22). This completes the proof. $\qquad\square$

Now, adding the oddity on f and applying the fountain theorem in [34, Theorem 3.6], we shall demonstrate the existence of a sequence of weak solutions for the problem (1.2). To employ the fountain theorem, we consider the following lemma.

Lemma 2.9 ([14]). *Let W be a reflexive and separable Banach space. Then there are $\{e_n\} \subseteq W$ and $\{f_n^*\} \subseteq W^*$ such that*

$$W = \overline{span\{e_n : n = 1, 2, \cdots\}}, \quad W^* = \overline{span\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $W_n = span\{e_n\}$, $Y_k = \bigoplus_{n=1}^k W_n$, and $Z_k = \overline{\bigoplus_{n=k}^\infty W_n}$. As a key tool, we recall the following fountain theorem.

Lemma 2.10 ([34]). *Let X be a real reflexive Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$ and I is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$ such that the following conditions hold:*

(1) $b_k := \inf\{I(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ *as* $k \to \infty$;

(2) $a_k := \max\{I(u) : u \in Y_k, \|u\|_X = \rho_k\} \leqslant 0$.

Then the functional I has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset X$ such that $I'(u_n) = 0$ and $I(u_n) \to \infty$ as $n \to \infty$.

Due to the fountain theorem, we give our main result in this section.

Theorem 2.11. *Assume that* (B), (J1)-(J5), *and* (F1)-(F5) *hold. If* $\Phi_0(x, -v) = \Phi_0(x, v)$ *holds for all* $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$, *then for any* $\lambda > 0$, *the problem* (1.2) *possesses an unbounded sequence of nontrivial weak solutions* $\{u_n\}$ *in* X *such that* $I_\lambda(u_n) \to \infty$ *as* $n \to \infty$.

Proof. Obviously, I_λ is an even functional and satisfies the $(C)_c$-condition. It suffices to show that for every $k \in \mathbb{N}$ there exist $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ as $k \to \infty$;
(2) $a_k := \max\{I_\lambda(u) : u \in Y_k, \|u\|_X = \rho_k\} \leqslant 0$.

Set

$$\alpha_k := \sup_{u \in Z_k, \|u\|_X = 1} \|u\|_{L^q(\mathbb{R}^N)}.$$

Then we have $\alpha_k \to 0$ as $k \to \infty$. In fact, assume the contrary, then there exist $\varepsilon_0 > 0$, $k_0 \in \mathbb{N}$, and a sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_X = 1 \quad \text{and} \quad \|u_k\|_{L^{p^*}(\mathbb{R}^N)} \geqslant \varepsilon_0,$$

for all $k \geqslant k_0$. Since the sequence $\{u_k\}$ is bounded in X, we may suppose that $u_k \rightharpoonup u$ in X as $k \to \infty$ for some $u \in X$ and thus

$$\langle f_j^*, u \rangle = \lim_{k \to \infty} \langle f_j^*, u_k \rangle = 0,$$

for $j = 1, 2, \cdots$. Hence we get $u = 0$. However, we observe that

$$\varepsilon_0 \leqslant \lim_{k \to \infty} \|u_k\|_{L^q(\mathbb{R}^N)} = \|u\|_{L^q(\mathbb{R}^N)} = 0,$$

which provides a contradiction.

For any $u \in Z_k$, it follows from the conditions (J4), (F2), the Sobolev inequality, and the Hölder inequality that

$$\begin{aligned}
I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u) \, dx + \int_{\mathbb{R}^N} \frac{b(x)}{p} |u|^p \, dx - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx \\
&\geqslant \frac{d}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} b(x) |u|^p \, dx - \lambda \int_{\mathbb{R}^N} \frac{|\sigma(x)|}{q} |u|^q \, dx \\
&\geqslant \frac{\min\{d, 1\}}{p} \|u\|_X^p - \frac{\lambda}{q} \|\sigma\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^q(\mathbb{R}^N)}^q \\
&\geqslant \frac{\min\{d, 1\}}{p} \|u\|_X^p - \frac{\lambda C_3}{q} \alpha_k^q \|u\|_X^q,
\end{aligned}$$

where C_3 is a positive constant. Choose $\delta_k = (\lambda C_3 \alpha_k^q / \min\{d, 1\})^{1/(p-q)}$. Since $p < q$ and $\alpha_k \to 0$ as $k \to \infty$, we assert $\delta_k \to \infty$ as $k \to \infty$. Hence, if $u \in Z_k$ and $\|u\|_X = \delta_k$, then we deduce that

$$I_\lambda(u) \geqslant \min\{d, 1\} \left(\frac{1}{p} - \frac{1}{q} \right) \delta_k^p - \lambda C_4 \delta_k \to \infty \quad \text{as} \quad k \to \infty,$$

for some positive constant C_4, which implies (1).

Next, the proof of condition (2) follows the lines of that of [1, Theorem 1.3]. For the sake of convenience, we give the proof. Assume that the condition (2) does not hold for some k. Then there exists a sequence $\{u_n\}$ in Y_k such that

$$\|u_n\|_X \to \infty \quad \text{as} \quad n \to \infty \quad \text{and} \quad I_\lambda(u_n) \geqslant 0. \tag{2.23}$$

Let $\ell_n = u_n/\|u_n\|_X$. Then it is obvious that $\|\ell_n\|_X = 1$. Since $\dim Y_k < \infty$, there exists $\ell \in Y_k \setminus \{0\}$ such that up to a subsequence,

$$\|\ell_n - \ell\|_X \to 0 \quad \text{and} \quad \ell_n(x) \to \ell(x),$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. If $\ell(x) \neq 0$, then $|u_n(x)| \to \infty$ for all $x \in \mathbb{R}^N$ as $n \to \infty$. Hence it follows from the condition (F3) that

$$\lim_{n \to \infty} \frac{F(x, u_n(x))}{\|u_n\|_X^p} = \lim_{n \to \infty} \frac{F(x, u_n(x))}{|u_n(x)|^p} |\ell_n(x)|^p = \infty,$$

for all $x \in \Omega_2 := \{x \in \mathbb{R}^N : \ell(x) \neq 0\}$. The estimate in Lemma 2.8 implies that

$$\int_{\Omega_2} \frac{F(x, u_n(x))}{\|u_n\|_X^p} \, dx \to \infty \quad \text{as } n \to \infty.$$

Therefore, from the relation (2.17), we have

$$I_\lambda(u_n) \leqslant \left(C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1 \right) \|u_n\|_X^p - \lambda \int_{\Omega_2} F(x, u_n) \, dx$$

$$= \|u_n\|_X^p \left(C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1 - \lambda \int_{\Omega_2} \frac{F(x, u_n(x))}{\|u_n\|_X^p} \, dx \right) \to -\infty \quad \text{as } n \to \infty,$$

which contradicts (2.23). This completes the proof. □

3. Existence of a sequence of weak solutions converging to zero

Now, we deal with the existence of a sequence of weak solutions converging to zero for the problem (1.2). For this, we employ a regularity result (Proposition 3.1). First of all, we need the following additional assumptions for φ, Φ_0, and f:

(J6) $\varphi(x, v) \cdot v - p\Phi_0(x, v) \geqslant 0$, for all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(F6) There exists a constant $s_0 > 0$ such that $pF(x, t) - f(x, t)t > 0$, for all $x \in \mathbb{R}^N$ and for $0 < |t| < s_0$.

(F7) $\lim_{|t| \to 0} \frac{f(x, t)}{|t|^{p-2}t} = +\infty$ uniformly for all $x \in \mathbb{R}^N$.

From the assumptions above, we show the existence of a sequence of solutions for the problem (1.2) converging to zero in the L^∞-norm based on the iteration method in [12, Theorem 4.1], which considered only nonnegative weak solutions. Since the problem (1.2) includes the potential term, more complicated analysis has to be carefully carried out in comparison with nonlinear equations given in [12]. Furthermore, we extend the result to the case of any weak solutions. For a bounded domain, see [20, Theorem 4.1].

Proposition 3.1. *Assume that* (B), (J1)-(J2), (J4), *and* (F1)-(F2) *hold. If* u *is a weak solution of the problem* (1.2), *then* $u \in L^\infty(\mathbb{R}^N)$.

Proof. Suppose that u is nonnegative. For a positive constant M, define

$$v_M(x) = \min\{u(x), M\},$$

and choose $v = v_M^{kp+1}$ ($k \geqslant 0$) as a test function in (2.1). Then obviously $v \in X \cap L^\infty(\mathbb{R}^N)$ and it follows from (2.1) that

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v_M^{kp+1} \, dx + \int_{\mathbb{R}^N} b(x) |u|^{p-2} u v_M^{kp+1} \, dx = \lambda \int_{\mathbb{R}^N} f(x, u) v_M^{kp+1} \, dx. \tag{3.1}$$

Due to (J4) and Lemma 2.1, the left-hand side of (3.1) can be estimated as follows:

$$
\int_{\mathbb{R}^N} \varphi(x, \nabla v_M) \cdot \nabla v_M^{kp+1} \, dx + \int_{\mathbb{R}^N} b(x) \, |u|^{p-2} \, u v_M^{kp+1} \, dx \tag{3.2}
$$

$$
\geqslant d(kp+1) \int_{\mathbb{R}^N} v_M^{kp} \, |\nabla v_M|^p \, dx + \int_{\mathbb{R}^N} b(x) v_M^{(k+1)p} \, dx
$$

$$
\geqslant \frac{d(kp+1)}{(k+1)^p} \int_{\mathbb{R}^N} |\nabla v_M^{k+1}|^p \, dx + \frac{1}{(k+1)^p} \int_{\mathbb{R}^N} b(x) v_M^{(k+1)p} \, dx
$$

$$
\geqslant \frac{\min\{d, 1\}}{C_5^p (k+1)^p} \left(\int_{\mathbb{R}^N} |v_M|^{(k+1)p^*} \, dx \right)^{\frac{p}{p^*}},
$$

for some constant $C_5 > 0$. By using the assumption (F2) and the Hölder inequality, the right-hand side of (3.1) can be formally estimated from above and we obtain

$$
\lambda \int_{\mathbb{R}^N} f(x, u) v_M^{kp+1} \, dx \leqslant \lambda \int_{\mathbb{R}^N} |f(x, u)| |u|^{kp+1} \, dx \tag{3.3}
$$

$$
\leqslant \lambda \left(\int_{\mathbb{R}^N} \sigma(x)^{\gamma_1} \, dx \right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{(k+1)p\gamma_1'} |u|^{(q-p)\gamma_1'} \, dx \right)^{\frac{1}{\gamma_1'}}
$$

$$
\leqslant \lambda \left(\int_{\mathbb{R}^N} \sigma(x)^{\gamma_1} \, dx \right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{(k+1)s} \, dx \right)^{\frac{p}{s}} \left(\int_{\mathbb{R}^N} |u|^{(q-p)\gamma_1' \frac{s}{s-p\gamma_1'}} \, dx \right)^{\frac{s-p\gamma_1'}{s\gamma_1'}},
$$

where $s = \frac{pp^*\gamma_1'}{p^* - (q-p)\gamma_1'}$. Obviously $s \leqslant p^*$, $1 < \frac{s}{p\gamma_1'}$, and $\frac{(q-p)\gamma_1' s}{s - p\gamma_1'} = p^*$, and hence (3.3) yields

$$
\lambda \int_{\mathbb{R}^N} f(x, u) v_M^{kp+1} \, dx \leqslant \lambda \left(\int_{\mathbb{R}^N} \sigma(x)^{\gamma_1} \, dx \right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{\frac{s-p\gamma_1'}{s\gamma_1'}} \left(\int_{\mathbb{R}^N} |u|^{(k+1)s} \, dx \right)^{\frac{p}{s}}. \tag{3.4}
$$

Now it follows from (3.1), (3.2), (3.4), and the Sobolev inequality that there exists a constant $C_6 > 0$ (independent of M and $k > 0$) such that

$$
\left(\int_{\mathbb{R}^N} |v_M|^{(k+1)p^*} \, dx \right)^{\frac{p}{p^*}} \leqslant C_6 (k+1)^p \left(\int_{\mathbb{R}^N} |u|^{(k+1)s} \, dx \right)^{\frac{p}{s}},
$$

which implies

$$
\|v_M\|_{L^{(k+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{(k+1)p}} (k+1)^{\frac{1}{k+1}} \|u\|_{L^{(k+1)s}(\mathbb{R}^N)}, \tag{3.5}
$$

for any positive constant M. The expression (3.5) is a starting point for a bootstrap argument which plays an important role in L^∞- estimates. Since $u \in X$ and hence $u \in L^{p^*}(\mathbb{R}^N)$ we can choose $k := k_1$ in (3.5) such that $(k_1 + 1)s = p^*$, i.e., $k_1 = \frac{p^*}{s} - 1$. Then we have

$$
\|v_M\|_{L^{(k_1+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{(k_1+1)p}} (k_1+1)^{\frac{1}{k_1+1}} \|u\|_{L^{(k_1+1)s}(\mathbb{R}^N)}, \tag{3.6}
$$

for any positive constant M. Due to $u(x) = \lim_{M \to \infty} v_M(x)$ for almost every $x \in \mathbb{R}^N$, the Fatou lemma and (3.6) imply

$$
\|u\|_{L^{(k_1+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{(k_1+1)p}} (k_1+1)^{\frac{1}{k_1+1}} \|u\|_{L^{(k_1+1)s}(\mathbb{R}^N)}. \tag{3.7}
$$

Thus, we can choose $k = k_2$ in (3.5) such that $(k_2+1)s = (k_1+1)p^* = \frac{(p^*)^2}{s}$ and repeating the same argument we get

$$
\|u\|_{L^{(k_2+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{(k_2+1)p}} (k_2+1)^{\frac{1}{k_2+1}} \|u\|_{L^{(k_2+1)s}(\mathbb{R}^N)}.
$$

By induction we obtain

$$\|u\|_{L^{(k_n+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{(k_n+1)p}} (k_n+1)^{\frac{1}{k_n+1}} \|u\|_{L^{(k_n+1)s}(\mathbb{R}^N)}, \tag{3.8}$$

for any $n \in \mathbb{N}$, where $k_n + 1 = \left(\frac{p^*}{s}\right)^n$. It follows from (3.7) and (3.8) that

$$\|u\|_{L^{(k_n+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{p}\sum_{j=1}^n \frac{1}{k_j+1}} (k_1+1)^{\frac{1}{k_1+1}} (k_2+1)^{\frac{1}{k_2+1}}$$
$$\cdots (k_n+1)^{\frac{1}{k_n+1}} \|u\|_{L^{(k_1+1)s}(\mathbb{R}^N)}.$$

Since $(k_n+1)^{\frac{1}{k_n+1}} > 1$ and $\lim_{k_n\to\infty} (k_n+1)^{\frac{1}{k_n+1}} = 1$, there exists $C_7 > 1$ (independent of k_n) such that

$$\|u\|_{L^{(k_n+1)p^*}(\mathbb{R}^N)} \leqslant C_6^{\frac{1}{p}\sum_{j=1}^n \frac{1}{k_j+1}} C_7 \|u\|_{L^{(k_1+1)s}(\mathbb{R}^N)}. \tag{3.9}$$

However $\sum_{j=1}^n \frac{1}{k_j+1} = \sum_{j=1}^n \left(\frac{s}{p^*}\right)^{nj}$ and $\frac{s}{p^*} < 1$. Hence it follows from (3.9) that there exists a constant $C_8 > 0$ such that

$$\|u\|_{L^{r_n}(\mathbb{R}^N)} \leqslant C_8 \|u\|_{L^{p^*}(\mathbb{R}^N)}, \tag{3.10}$$

for $r_n = (k_n + 1)p^* \to \infty$ when $n \to \infty$. Let us assume that $\|u\|_{L^\infty(\mathbb{R}^N)} > C_8\|u\|_{L^{p^*}(\mathbb{R}^N)}$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $u(x) \geqslant C_8\|u\|_{L^{p^*}(\mathbb{R}^N)} + \eta$ for $x \in A$. It follows that

$$\liminf_{r_n\to\infty} \left(\int_{\mathbb{R}^N} |u(x)|^{r_n} \, dx\right)^{\frac{1}{r_n}} \geqslant \liminf_{r_n\to\infty} \left(\int_A |u(x)|^{r_n} \, dx\right)^{\frac{1}{r_n}}$$
$$\geqslant \liminf_{r_n\to\infty} \left(C_8\|u\|_{L^{p^*}(\mathbb{R}^N)} + \eta\right) (\text{meas}(A))^{\frac{1}{r_n}}$$
$$= C_8\|u\|_{L^{p^*}(\mathbb{R}^N)} + \eta,$$

which contradicts (3.10). Therefore

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C_8\|u\|_{L^{p^*}(\mathbb{R}^N)} \leqslant C_9,$$

for some constant $C_9 > 0$.

If u changes sign, we set $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$. Then it is clear that $u^+ \in X$ and $u^- \in X$. Define for each $M > 0$, $v_M(x) = \min\{u^+(x), M\}$. Taking again $\varphi = v_M^{kp+1}$ as a test function in X, we obtain

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v_M^{kp+1} \, dx + \int_{\mathbb{R}^N} b(x) |u|^{p-2} u v_M^{kp+1} \, dx = \lambda \int_{\mathbb{R}^N} f(x, u) v_M^{kp+1} \, dx,$$

which implies that

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u^+) \cdot \nabla v_M^{kp+1} \, dx + \int_{\mathbb{R}^N} b(x) |u^+|^{p-2} u^+ v_M^{kp+1} \, dx = \lambda \int_{\mathbb{R}^N} f(x, u^+) v_M^{kp+1} \, dx.$$

Proceeding a similar argument as above, we obtain $u^+ \in L^\infty(\mathbb{R}^N)$. Likewise, we get $u^- \in L^\infty(\mathbb{R}^N)$. Therefore $u = u^+ + u^-$ is in $L^\infty(\mathbb{R}^N)$. This completes the proof. $\square$

The following lemma is quoted from [16].

Lemma 3.2 ([16]). *Let* $I \in C^1(X, \mathbb{R})$ *where* X *is a Banach space. Assume that* I *satisfies the* (PS)*-condition, is even and bounded from below, and* $I(0) = 0$. *If for any* $n \in \mathbb{N}$, *there exists an* n*-dimensional subspace* X_n *and* $\rho_n > 0$ *such that*

$$\sup_{X_n \cap S_{\rho_n}} I < 0,$$

where $S_\rho := \{u \in X : \|u\|_X = \rho\}$, *then* I *has a sequence of critical values* $c_n < 0$ *satisfying* $c_n \to 0$ *as* $n \to \infty$.

Based on the works of [9, 33], we give the following two lemmas.

Lemma 3.3. *Assume that* (J1)-(J2)*,* (J4)*,* (J6)*,* (F1)-(F2)*, and* (F6) *hold. Then*

$$I_\lambda(u) = 0 = \langle I'_\lambda(u), u \rangle, \quad \text{if and only if} \quad u = 0.$$

Lemma 3.4. *Assume that* (F1)-(F2) *and* (F6)-(F7) *hold. Then there exist* $t_0 > 0$ *with* $t_0 < s_0/2$ *and* $\tilde{f} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *such that* $\tilde{f}(x, t)$ *is odd in* t *and satisfy*

$$\tilde{\mathcal{F}}(x, t) := p\tilde{F}(x, t) - \tilde{f}(x, t)t \geqslant 0,$$

$$\tilde{\mathcal{F}}(x, t) = 0, \quad iff \quad t \equiv 0 \quad or \quad |t| \geqslant 2t_0,$$

where $\frac{\partial}{\partial t}\tilde{F}(x, t) = \tilde{f}(x, t)$.

Proof. Let us define a cut-off function $\kappa \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $\kappa(t) = 1$ for $|t| \leqslant t_0$, $\kappa(t) = 0$ for $|t| \geqslant 2t_0$, $|\kappa'(t)| \leqslant 2/t_0$, and $\kappa'(t)t \leqslant 0$. So, we define

$$\tilde{F}(x, t) = \kappa(t)F(x, t) + (1 - \kappa(t))\xi|t|^p \quad \text{and} \quad \tilde{f}(x, t) = \frac{\partial}{\partial t}\tilde{F}(x, t), \tag{3.11}$$

where $\xi > 0$ is a constant. It is straightforward that

$$p\tilde{F}(x, t) - \tilde{f}(x, t)t = \kappa(t)\mathcal{F}(x, t) - \kappa'(t)tF(x, t) + \kappa'(t)t\xi|t|^p,$$

where $\mathcal{F}(x, t) := pF(x, t) - f(x, t)t$. For $0 \leqslant |t| \leqslant t_0$ and $|t| \geqslant 2t_0$ the conclusion follows. Due to (F7), we choose a sufficiently small $t_0 > 0$ such that $F(x, t) \geqslant \xi t^p$ for $t_0 \leqslant |t| \leqslant 2t_0$. By the assumption $\kappa'(t)t \leqslant 0$, we get the conclusion. □

Now, we prove the second main result using Proposition 3.1 and Lemmas 3.2 and 3.4.

Theorem 3.5. *Assume that* (B)*,* (J1)-(J4)*,* (J6)*,* (F1)-(F2)*, and* (F5)-(F7) *hold. Moreover, assume that* $\Phi_0(x, -v) = \Phi_0(x, v)$ *holds for all* $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$*. Then there exists a positive constant* λ^* *such that for every* $\lambda \in [0, \lambda^*)$*, the problem* (1.2) *has a sequence of weak solutions* $\{u_n\}$ *satisfying* $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ *as* $n \to \infty$.

Proof. First of all, we will show that I_λ is coercive on X. Let $u \in X$ and $\|u\|_X > 1$. For the given function $f(x, t)$, we can modify and extend $\tilde{f} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying all properties listed in Lemma 3.4. Define $\tilde{I}_\lambda(u) := \Phi(u) - \lambda \int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx$. Then by Lemma 3.4, it is easy to show that $\tilde{I}_\lambda \in C^1(X, \mathbb{R})$ and is even on X. Moreover, it follows from (F6) and (F7) that for $|u(x)| \leqslant 2t_0$, there exists a positive constant M_1 such that $M_1|u|^p \geqslant |F(x, u)|$. Set $\Omega_3 := \{x \in \mathbb{R}^N : |u(x)| \leqslant t_0\}$, $\Omega_4 := \{x \in \mathbb{R}^N : t_0 \leqslant |u(x)| \leqslant 2t_0\}$, and $\Omega_5 := \{x \in \mathbb{R}^N : 2t_0 \leqslant |u(x)|\}$, where t_0 is given in Lemma 3.4. From (J4), (J6), (3.11), and the conditions of κ, we have

$$\begin{aligned}
\tilde{I}_\lambda(u) &:= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx + \int_{\mathbb{R}^N} \frac{b(x)}{p}|u|^p\, dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx \\
&\geqslant \frac{d}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{1}{p}\int_{\mathbb{R}^N} b(x)|u|^p\, dx \\
&\quad - \lambda \int_{\Omega_3} F(x, u)\, dx - \lambda \int_{\Omega_4} \{\kappa(u)F(x, u) + (1 - \kappa(u))\xi|u|^p\}\, dx - \lambda \int_{\Omega_5} \xi|u|^p\, dx \\
&\geqslant \frac{\min\{d, 1\}}{p}\|u\|_X^p - \lambda \int_{\Omega_3} F(x, u)\, dx - \lambda \int_{\Omega_4} F(x, u)\, dx - \lambda \int_{\Omega_4} \xi|u|^p\, dx - \lambda \int_{\Omega_5} \xi|u|^p\, dx \\
&\geqslant \frac{\min\{d, 1\}}{p}\|u\|_X^p - \lambda \int_{\Omega_3} M_1|u|^p\, dx - \lambda \int_{\Omega_4} M_1|u|^p\, dx - \lambda \int_{\Omega_4} \xi|u|^p\, dx - \lambda \int_{\Omega_5} \xi|u|^p\, dx \\
&\geqslant \frac{\min\{d, 1\}}{p}\|u\|_X^p - \lambda(M_1 + \xi)\int_{\mathbb{R}^N} |u|^p\, dx \\
&\geqslant \frac{\min\{d, 1\}}{p}\|u\|_X^p - \frac{\lambda}{b_0}(M_1 + \xi)\|u\|_X^p.
\end{aligned}$$

If we set

$$\lambda^* := \frac{b_0 \min\{d, 1\}}{p(M_1 + \xi)},$$

then we deduce that for any $\lambda \in [0, \lambda^*)$, $\tilde{I}_\lambda$ is coercive, that is, $\tilde{I}_\lambda(u) \to \infty$ as $\|u\|_X \to \infty$. By a standard argument, $\tilde{I}_\lambda$ satisfies the $(PS)_c$-condition. In order to apply Lemma 3.2, we only need to find for any $n \in \mathbb{N}$, a subspace X_n and $\rho_n > 0$ such that $\sup_{X_n \cap S_{\rho_n}} \tilde{I}_\lambda < 0$. For any $n \in \mathbb{N}$ we find n independent smooth functions ϕ_i for $i = 1, \cdots, n$, and define $X_n := \mathrm{span}\{\phi_1, ..., \phi_n\}$. Due to (J2) and Lemma 3.4, when $\|u\|_X < 1$ we have that

$$\tilde{I}_\lambda(u) = \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx + \int_{\mathbb{R}^N} \frac{b(x)}{p} |u|^p\, dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx$$

$$\leqslant (C_1 \|a\|_{L^{p'}(\mathbb{R}^N)} + a_0 + 1)\|u\|_X^p - \lambda C_{10} \int_{\mathbb{R}^N} F(x, u)\, dx,$$

for a positive constant C_{10}. It follows from the assumption (F7) that for a sufficiently large $M_2 > 0$, there exists $\delta_0 > 0$ such that $|t| < \delta_0$ implies

$$\int_{\mathbb{R}^N} F(x, t)\, dx \geqslant \frac{M_2}{p} \int_{\mathbb{R}^N} |t|^p\, dx.$$

By this relation and the fact that all norms on X_n are equivalent, choosing a suitable constant C_{10} and sufficiently small $\rho_n > 0$, we can obtain

$$\sup_{X_n \cap S_{\rho_n}} \tilde{I}_\lambda < 0.$$

By Lemma 3.2, we get a sequence $c_n < 0$ for $\tilde{I}_\lambda$ satisfying $c_n \to 0$ when n goes to ∞. Then for any $u_n \in X$ satisfying $\tilde{I}_\lambda(u_n) = c_n$ and $\tilde{I}'_\lambda(u_n) = 0$, the sequence $\{u_n\}$ is a $(PS)_0$-sequence of $\tilde{I}_\lambda$ and $\{u_n\}$ has a convergent subsequence. Lemmas 3.3 and 3.4 imply that **0** is the only critical point with 0 energy and the subsequence of $\{u_n\}$ has to converge to **0**. An indirect argument shows the sequence $\{u_n\}$ has to converge to **0**. On the other hand, we have $u_n \in C(\mathbb{R}^N)$ due to Proposition 3.1. Since $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, by Lemma 3.4 again, we deduce $\|u_n\|_{C(\mathbb{R}^N)} \leqslant t_0$. Thus $\{u_n\}$ are weak solutions of the problem (1.2). The proof is complete. $\qquad\square$

Acknowledgment

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