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Hybrid iterative algorithms for the split common fixed point problems

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Abstract

In this paper, we introduce two iterative algorithms (one implicit algorithm and one explicit algorithm) based on the hybrid steepest descent method for solving the split common fixed point problems. We establish the strong convergence of the sequences generated by the proposed algorithms to a solution of the split common fixed point problems, which is also a solution of a certain variational inequality. In particular, the minimum norm solution of the split common fixed point problems is obtained. As applications, variational problems and equilibrium problems are considered.

Keywords: Split common fixed point problem, firmly nonexpansive mapping, nonexpansive mapping, variational inequality, minimum-norm, ρ-Lipschitzian and η-strongly monotone operator, bounded linear operator, variational problems, equilibrium problems, iterative algorithms.
2010 MSC: 47J05, 49J40, 49J52, 47J20, 47H10.

1. Introduction

The split feasibility problem (in short, SFP) is formulated as

$$\text{find } \ x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The SFP (1.1) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [8] for modeling inverse problems which arise in phase retrievals and in medical image reconstruction [4]. In [7, 9, 10], it has been shown that the SPF (1.1) can also be used to model the intensity-modulated radiation therapy. Various iterative algorithms have been studied to solve the SFP (1.1), see, e.g., [8, 14, 16, 18, 22, 25, 27, 30–32] and the references therein. In particular, Jung [16] introduced iterative algorithms based on the Yamada's hybrid steepest descent method [28] for solving SFP (1.1). He established strong convergence of sequences generated by the proposed algorithms to a solution of SFP (1.1), which is a solution of a certain variational inequality defined over the set of solutions of SFP (1.1).

Recently, several split type feasibility problems have been considered because of their applications in science, engineering, medical sciences, etc. One of the split type problems is the split common fixed point

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problem (in short, SCFPP) which is to find a fixed point of an operator such that its image under the bounded linear operator is a fixed point of another operator, that is,

$$\text{find } x^* \in \mathrm{Fix}(T) \quad \text{such that} \quad Ax^* \in \mathrm{Fix}(S), \tag{1.2}$$

where $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ denote the set of fixed points of the operators $T : H_1 \to H_1$ with $\mathrm{Fix}(T) \neq \emptyset$ and $S : H_2 \to H_2$ with $\mathrm{Fix}(S) \neq \emptyset$, respectively. We denote by Ω the set of solutions of the SCFPP (1.2) and assume that $\Omega \neq \emptyset$. The SCFPP (1.2) was introduced by Censor and Segal [11]. They considered a parallel algorithm for solving the SCFPP (1.2) for a class of directed operators in finite dimensional spaces. Later, Ansari et al. [2], Cui and Wang [13], Krailkaew and Saejung [17] and Moudafi [20, 21] proposed different kinds of algorithms for solving SCFPP (1.2) in the Hilbert space setting.

In this paper, motivated by the works [2, 16], we present two iterative algorithms based on Yamada's the hybrid steepest descent method [28] for solving the SCFPP (1.2). First, we introduce an implicit algorithm. Next, by discretizing the continuous implicit algorithm, we provide an explicit algorithm. Under some appropriate conditions, we show the strong convergence of proposed algorithms to some solution of the SCFPP (1.2) which solves a certain variational inequality. As special cases, we obtain two algorithms which converges strongly to the minimum norm solution of the SCFPP (1.2). As applications, using our iterative algorithms, we study some variational inequality problem and equilibrium problems. The paper can be considered as a continuation of study for solving the SCFPP (1.2) via fixed point methods.

2. Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let K be a nonempty closed convex subset of H. Recall that the (nearest point or metric) projection from H onto K, denoted by P_K, is defined in such a way that, for each $x \in H$, $P_K x$ is the unique point in K with the property

$$\|x - P_K(x)\| = \min\{\|x - y\| : y \in K\}.$$

We recall ([1, 5, 6, 29]) that

(1) a mapping $f : H \to H$ is k-*contractive* if $\|fx - fy\| \leqslant k\|x - y\|$ for a constant $k \in [0, 1)$ and $\forall x,\ y \in H$;
(2) a mapping $V : H \to H$ is l-*Lipschitzian* if $\|Vx - Vy\| \leqslant l\|x - y\|$ for a constant $l \in [0, \infty)$ and $\forall x,\ y \in H$;
(3) a mapping $T : H \to H$ is *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x,\ y \in H$;
(4) a mapping $T : H \to H$ is *strongly nonexpansive* if T is nonexpansive and

$$\lim_{n\to\infty} \|(x_n - y_n) - (Tx_n - Ty_n)\| = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in H and $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$;
(5) a mapping $T : H \to H$ is *firmly quasi-nonexpansive* if $\|Tx - p\|^2 \leqslant \|x - p\|^2 - \|Tx - p\|^2$ for all $x \in H$ and $p \in \mathrm{Fix}(T)$;
(6) a mapping $T : H \to H$ is *averaged* if $T = (1 - \nu)I + \nu G$, where $\nu \in (0, 1)$ and $G : H \to H$ is nonexpansive. In this case, we also say that T is ν-averaged;
(7) a mapping $A : H \to H$ is monotone if $\langle Ax - Ay, x - y\rangle \geqslant 0$, $\forall x,\ y \in H$;
(8) a mapping $T : H \to H$ is α-*inverse strongly monotone* (α-ism) if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geqslant \alpha\|Tx - Ty\|^2, \forall x,\ y \in H;$$

(9) an operator $F : H \to H$ is κ-*Lipschitzian* and η-*strongly monotone* with constants $\kappa > 0$ and $\eta > 0$ if $\|Fx - Fy\| \leqslant \kappa\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2$, $\forall x,\ y \in H$, respectively.

The following result is well-known.

Proposition 2.1 ([5]). *Let H be a real Hilbert space, and let $T : H \to H$ be an operator.*

(a) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
(b) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $(I - T)$ is $\frac{1}{2\alpha}$-ism.*
(c) *The composite of finitely many averaged mappings is averaged.*
(d) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

As in [2], using Proposition 2.1, we can prove the following. So we omit its proof.

Proposition 2.2. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 - H_2$ be a bounded linear operator, let A^* be the adjoint A, and let $S : H_2 \to H_2$ be a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. Then*

(i) *$A^*(I - S)A$ is $\frac{1}{2\|A\|^2}$-ism;*
(ii) *$U := I - \gamma A^*(I - S)A$ is averaged for $\gamma \in (0, \frac{1}{\|A\|^2})$ and hence U is nonexpansive;*
(iii) *$Ax \in \mathrm{Fix}(S)$ implies $x \in \mathrm{Fix}(U)$, and $x \in \mathrm{Fix}(U)$ implies $Ax \in \mathrm{Fix}(S)$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.3 ([1]). *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.4 (Demiclosedness principle, [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $(I - S)x_n \to y$, then $(I - S)x = y$.*

Lemma 2.5 ([19]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments (i.e., $F(x + ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, x - x^*\rangle \geqslant 0, \quad x \in C,$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, x - x^*\rangle \geqslant 0, \quad x \in C.$$

Lemma 2.6 ([23]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leqslant \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n$, $n \geqslant 0$, and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

Then $\lim_n \|z_n - x_n\| = 0$.

Lemma 2.7 ([26]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leqslant (1 - \lambda_n)s_n + \lambda_n \delta_n, \quad \forall n \geqslant 0,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemma can be easily proven, and therefore, we omit the proof (see also [28]).

Lemma 2.8. *Let* H *be a real Hilbert space* H*. Let* $F : H \to H$ *be a* κ*-Lipschizian and* η*-strongly monotone operator with constants* $\kappa > 0$ *and* $\eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t < \xi \leqslant 1$*. Then* $G := \xi I - t\mu F : H \to H$ *is a contractive mapping with constant* $\xi - t\tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$*.*

From now, we will use the following notations:

- $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;
- $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

3. Iterative algorithms

Throughout the rest of this paper, we always assume the followings:

- H_1 and H_2 are real Hilbert spaces;
- $A : H_1 \to H_2$ is a bounded linear operator and A^* is the adjoint of A;
- $T : H_1 \to H_1$ is a firmly nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$;
- $S : H_2 \to H_2$ is a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$;
- $V : H_1 \to H_1$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $F : H_1 \to H_1$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$;
- constants μ, σ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \sigma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$, and $0 < \gamma < \frac{1}{\|A\|^2}$;
- Ω is the set of solutions of SCFPP (1.2).

First, we introduce the following iterative algorithm that generates a net $\{x_t\}_{t\in(0,\frac{1}{\tau-\sigma l})}$ in an implicit way:

$$x_t = T[I - \gamma A^*(I - S)A]T[t\sigma V x_t + (I - t\mu F)x_t]. \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a x^* which is a solution of the following variational inequality:

$$x^* \in \Omega \quad \text{such that} \quad \langle \sigma V x^* - \mu F x^*, \tilde{x} - x^* \rangle \leqslant 0, \quad \forall \tilde{x} \in \Omega. \tag{3.2}$$

Now, for $t \in (0, \frac{1}{\tau-\sigma l})$, consider a mapping $W_t : C \to C$ defined by

$$W_t x := T[I - \gamma A^*(I - S)A]T[t\sigma V x + (I - t\mu F)x], \quad x \in C.$$

It is easy to see that W_t is a contractive mapping with constant $1 - t(\tau - \sigma l)$. Indeed, note that T and $I - \gamma A^*(I - S)A$ are nonexpansive (by Proposition 2.2). Thus, by Lemma 2.8, we have for $x,\ y \in C$,

$$\begin{aligned}
\|W_t x - W_t y\| &= \|T[I - \gamma A^*(I - S)A]T[t\sigma V x + (I - t\mu F)x] - T[I - \gamma A^*(I - S)A]T[t\sigma V y + (I - t\mu F)y]\| \\
&\leqslant t\sigma\|Vx - Vy\| + \|(I - \mu t F)x - (I - \mu t F)y\| \\
&\leqslant t\sigma l\|x - y\| + (1 - t\tau)\|x - y\| \\
&= [1 - t(\tau - \sigma l)]\|x - y\|.
\end{aligned}$$

Therefore W_t is a contractive mapping when $t \in (0, \frac{1}{\tau-\sigma l})$. By the Banach contraction principle, W_t has a unique fixed point in C, denoted by x_t, that is,

$$x_t = T[I - \gamma A^*(I - S)A]T[t\sigma V x_t + (I - t\mu F)x_t],$$

which is exactly (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Let $\Omega \neq \emptyset$, and let $\{x_t\}$ be defined via (3.1). Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0, \frac{1}{\tau-\sigma l})$;

(ii) $\lim_{t\to 0} \|x_t - T[I-\gamma A^*(I-S)A]Tx_t\| = 0$;

(iii) x_t *defines a continuous path from* $(0, \frac{1}{\tau-\sigma l})$ *into* H_1.

Proof.
(i) Let $\widetilde{x}$ be any point in Ω. Then $\widetilde{x} \in \mathrm{Fix}(T)$ and $A\widetilde{x} \in \mathrm{Fix}(S)$. Set

$$U = I - \gamma A^*(I-S)A.$$

Then, from Proposition 2.2 (iii), we have $\widetilde{x} \in \mathrm{Fix}(U)$, and we can rewrite (3.1) as

$$x_t = TUT[t\sigma Vx_t + (I-t\mu F)x_t], \quad t \in \left(0, \frac{1}{\tau-\sigma l}\right).$$

It follows that

$$\begin{aligned}
\|x_t - \widetilde{x}\| &= \|TUT[t\sigma Vx_t + (I-t\mu F)x_t] - \widetilde{x}\| \\
&\leqslant \|t\sigma(Vx_t - V\widetilde{x})\| + \|(I-t\mu F)x_t - (I-t\mu F)\widetilde{x}\| + \|t\sigma V\widetilde{x} - t\mu F\widetilde{x}\| \\
&\leqslant t\sigma\|x_t - \widetilde{x}\| + (1-t\tau)\|x_t - \widetilde{x}\| + t\|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&= [1-(\tau-\sigma l)t]\|x_t - \widetilde{x}\| + t\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|.
\end{aligned}$$

Hence

$$\|x_t - \widetilde{x}\| \leqslant \frac{1}{\tau-\sigma l}\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|.$$

Then $\{x_t\}$ is bounded and so are $\{Vx_t\}$, $\{Ux_t\}$, and $\{Fx_t\}$.
(ii) From (3.1), we have

$$\|x_t - T[I-\gamma A^*(I-S)A]Tx_t\| = \|TUT[t\sigma Vx_t + (I-t\mu F)x_t] - TUTx_t\| \leqslant t\|\sigma Vx_t - \mu Fx_t\|.$$

By boundedness of $\{Vx_t\}$ and $\{Fx_t\}$, we obtain

$$\lim_{t\to 0} \|x_t - T[I-\gamma A^*(I-S)A]Tx_t\| = 0.$$

(iii) Let $t,\ t_0 \in (0, \frac{1}{\tau-\sigma l})$. We calculate

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \|T[I-\gamma A^*(I-S)A]T[t\sigma Vx_t + (I-t\mu F)x_t] - T[I-\gamma A^*(I-S)A]T[t_0\sigma Vx_{t_0} + (I-t\mu F)x_{t_0}]\| \\
&\leqslant \|t\sigma Vx_t + (I-t\mu F)x_t - (t_0\sigma Vx_{t_0} + (I-t\mu F)x_{t_0})\| \\
&\leqslant \|t\sigma Vx_t - t_0\sigma Vx_t\| + \|(I-t\mu F)x_t - (I-t\mu F)x_{t_0}\| \\
&\quad + \|t_0\sigma Vx_t - t_0\sigma Vx_{t_0}\| + \|(I-t\mu F)x_{t_0} - (I-t_0\mu F)x_{t_0}\| \\
&\leqslant \sigma\|Vx_t\||t-t_0| + (1-t\tau)\|x_t - x_{t_0}\| + t_0\sigma l\|x_t - x_{t_0}\| + \mu\|Fx_{t_0}\||t-t_0|.
\end{aligned}$$

This implies that

$$\|x_t - x_{t_0}\| \leqslant \frac{\sigma\|Vx_t\| + \mu\|Fx_{t_0}\|}{t\tau - t_0\sigma l}|t-t_0|.$$

This completes the proof. □

Theorem 3.2. *Let $\Omega \neq \emptyset$, and let the net $\{x_t\}$ be defined via (3.1). Then x_t converges strongly to a point x^* as $t \to 0$, which solves the variational inequality (3.2).*

Proof. First, we show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \leqslant \sigma l < \tau$ and $\mu\eta \geqslant \tau \iff \kappa \geqslant \eta$, it follows that

$$\langle(\mu F - \sigma V)x - (\mu F - \sigma V)y, x - y\rangle \geqslant (\mu\eta - \sigma l)\|x - y\|^2.$$

That is, $\mu F - \sigma V$ is strongly monotone for $0 \leqslant \sigma l < \tau \leqslant \mu\eta$. So the variational inequality (3.2) has only one solution.

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. To this end, set $U = I - \gamma A^*(I - S)A$ and let $\widetilde{x}$ be any point in Ω. Then $\widetilde{x} \in \mathrm{Fix}(T)$, $A\widetilde{x} \in \mathrm{Fix}(S)$, and $\widetilde{x} \in \mathrm{Fix}(U)$ (by Proposition 2.2 (iii)). Let $\{t_n\} \subset (0, \frac{1}{\tau - \sigma l})$ be such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From Proposition 3.1 (ii), we have

$$\lim_{n\to\infty} \|x_n - TUTx_n\| = 0. \tag{3.3}$$

Put $z_t = t\sigma Vx_t + (I - t\mu F)x_t$, $y_t = T[t\sigma Vx_t + (I - t\mu F)x_t] = Tz_t$, $z_n := z_{t_n}$, and $y_n := y_{t_n} = Tz_n$. Then we have, for any $\widetilde{x} \in \Omega$,

$$y_t - \widetilde{x} = y_t - z_t + z_t - \widetilde{x} = y_t - z_t + t\sigma(Vx_t - V\widetilde{x}) + (I - t\mu F)x_t - (I - t\mu F)\widetilde{x} + t(\sigma V\widetilde{x} - \mu F\widetilde{x}). \tag{3.4}$$

Since T is a firmly nonexpansive mapping with a fixed point $\widetilde{x}$, we have

$$\langle y_t - z_t, y_t - \widetilde{x}\rangle \leqslant 0. \tag{3.5}$$

Combining (3.4) with (3.5) along with Lemma 2.8, we get

$$\begin{aligned}
\|y_t - \widetilde{x}\|^2 &= \langle y_t - \widetilde{x}, y_t - \widetilde{x}\rangle \\
&= \langle y_t - z_t, y_t - \widetilde{x}\rangle + t\sigma\langle Vx_t - V\widetilde{x}, y_t - \widetilde{x}\rangle \\
&\quad + \langle (I - t\mu F)x_t - (I - t\mu F)\widetilde{x}, y_t - \widetilde{x}\rangle + t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\
&\leqslant t\sigma l\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + (1 - t\tau)\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\
&= [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|\|y_t - \widetilde{x}\| + t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle \\
&\leqslant \frac{1 - (\tau - \sigma l)t}{2}\|x_t - \widetilde{x}\|^2 + \frac{1}{2}\|y_t - \widetilde{x}\|^2 + t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.
\end{aligned}$$

It follows that

$$\|y_t - \widetilde{x}\|^2 \leqslant [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|^2 + 2t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.$$

Thus, from Proposition 2.2, we have

$$\begin{aligned}
\|x_t - \widetilde{x}\|^2 = \|TUTz_t - TUT\widetilde{x}\|^2 &\leqslant \|TUy_t - TU\widetilde{x}\|^2 \\
&\leqslant \|y_t - \widetilde{x}\|^2 \leqslant [1 - (\tau - \sigma l)t]\|x_t - \widetilde{x}\|^2 + 2t\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.
\end{aligned}$$

Hence, we obtain

$$\|x_t - \widetilde{x}\|^2 \leqslant \frac{2}{\tau - \sigma l}\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_t - \widetilde{x}\rangle.$$

In particular, we have

$$\|x_n - \widetilde{x}\|^2 \leqslant \frac{2}{\tau - \sigma l}\langle \sigma V\widetilde{x} - \mu F\widetilde{x}, y_n - \widetilde{x}\rangle, \quad \widetilde{x} \in \Omega. \tag{3.6}$$

Note that

$$\|x_t - z_t\| = \|x_t - [t\sigma Vx_t + (I - t\mu F)x_t]\| \leqslant t\|\sigma Vx_t - \mu Fx_t\| \to 0 \quad \text{as } t \to 0.$$

So,

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.7}$$

Observe that

$$\begin{aligned}\|z_n-\widetilde{x}\| &= \|t_n\sigma Vx_n+(I-t_n\mu F)x_n-\widetilde{x}\|\\ &= \|(x_n-\widetilde{x})+t_n(\sigma Vx_n-\mu Fx_n)\|\\ &\leqslant \|x_n-\widetilde{x}\|+t_n\|\sigma Vx_n-\mu Fx_n\|.\end{aligned} \tag{3.8}$$

Then, since every firmly nonexpansive mapping with a fixed point is firmly quasi-nonexpansive, from (3.8) we deduce

$$\begin{aligned}\|x_n-\widetilde{x}\|^2 &= \|TUTz_n-TUT\widetilde{x}\|^2\\ &\leqslant \|Tz_n-\widetilde{x}\|^2\\ &\leqslant \|z_n-\widetilde{x}\|^2-\|Tz_n-z_n\|^2\\ &\leqslant (\|x_n-\widetilde{x}\|+t_n\|\sigma Vx_n-\mu Fx_n\|)^2-\|Tz_n-z_n\|^2\\ &\leqslant \|x_n-\widetilde{x}\|^2+t_nM-\|Tz_n-z_n\|^2,\end{aligned}$$

where $M>0$ is an appropriate constant. This implies that

$$\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|Tz_n-z_n\|=0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point x^*. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to x^*. Then by (3.7) and (3.9), $y_n\rightharpoonup x^*$. Noticing (3.3), we can use Lemma 2.4 to get $x^*=TUTx^*$. By Proposition 2.1 (iv), we have $Tx^*=x^*$ and $Ux^*=x^*$, and hence $S(Ax^*)=Ax^*$. Thus $x^*\in \mathrm{Fix}(T)$ and $Ax^*\in\mathrm{Fix}(S)$, that is, $x^*\in\Omega$. Therefore, we can substitute x^* for $\widetilde{x}$ in (3.6) to obtain

$$\|x_n-x^*\|^2\leqslant\frac{2}{\tau-\sigma l}\langle\sigma Vx^*-\mu Fx^*,y_n-x^*\rangle.$$

Consequently, $y_n\rightharpoonup x^*$ actually implies that $x_n\to x^*$. This proves the relative norm-compactness of the net $\{x_t\}$ as $t\to 0^+$.

Letting $n\to\infty$ in (3.6), we have

$$\|x^*-\widetilde{x}\|^2\leqslant\frac{2}{\tau-\sigma l}\langle\sigma V\widetilde{x}-\mu F\widetilde{x},x^*-\widetilde{x}\rangle,\quad \widetilde{x}\in\Omega.$$

This implies that $x^*\in\Omega$ solves the variational inequality

$$\langle\sigma V\widetilde{x}-\mu F\widetilde{x},\widetilde{x}-x^*\rangle\leqslant 0,\quad \widetilde{x}\in\Omega. \tag{3.10}$$

By Lemma 2.5, (3.10) is equivalent to its dual variational inequality

$$\langle\sigma Vx^*-\mu Fx^*,\widetilde{x}-x^*\rangle\leqslant 0,\quad \widetilde{x}\in\Omega.$$

This is exactly (3.2). By uniqueness of the solution of the variational inequality (3.2), we deduce that each cluster point of $\{x_t\}$ as $t\to 0^+$ equals to x^*. Therefore $x_t\to x^*$ as $t\to 0^+$. This completes the proof. □

Taking $F=I$ and $\mu=1$ in Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let the net $\{x_t\}$ be defined by*

$$x_t=T[I-\gamma A^*(I-S)A]T[t\sigma Vx_t+(1-t)x_t],\quad t\in\left(0,\frac{1}{1-\sigma l}\right). \tag{3.11}$$

Then $\{x_t\}$ converges strongly as $t\to 0$ to a point x^* which is the unique solution of variational inequality

$$x^*\in\Omega\quad\textit{such that}\quad\langle\sigma Vx^*-x^*,\widetilde{x}-x^*\rangle\leqslant 0,\quad\forall\widetilde{x}\in\Omega.$$

Taking $V=0$ in (3.11), we get the following corollary.

Corollary 3.4. *Let the net* $\{x_t\}$ *be defined by*

$$x_t = T[I-\gamma A^*(I-S)A]T[(1-t)x_t], \quad t \in (0,1). \tag{3.12}$$

Then $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* x^* *which is the minimum norm solution of the SCFPP* (1.2).

Proof. If we take $V = 0$, then (3.11) reduces to (3.12). Thus, $x_t \to x^* \in \Omega$ which satisfies

$$\langle -x^*, \widetilde{x} - x^* \rangle \leqslant 0, \quad \forall \widetilde{x} \in \Omega.$$

Thus

$$\|x^*\|^2 \leqslant \langle x^*, \widetilde{x} \rangle \leqslant \|x^*\|\|\widetilde{x}\|, \quad \forall \widetilde{x} \in \Omega,$$

which implies $\|x^*\| \leqslant \|\widetilde{x}\|$ for all $\widetilde{x} \in \Omega$. That is, x^* is the minimum norm solution of the SCFPP (1.2). This completes the proof. □

Next, we propose the following iterative algorithm which generates a sequence in an explicit way:

$$x_{n+1} = T[I-\gamma A^*(I-S)A]T[\alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n], \quad n \geqslant 0, \tag{3.13}$$

where $\{\alpha_n\} \subset [0,1]$ and $x_0 \in H_1$ is an arbitrary initial guess, and establishes strong convergence of this sequence to a point x^*, which is also a solution of the variational inequality (3.2).

Theorem 3.5. *Let* $\Omega \neq \emptyset$, *and let* $\{x_n\}$ *be the sequence generated by the explicit algorithm* (3.13), *where* $\{\alpha_n\}$ *satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$ *as* $n \to \infty$, *which solves the variational inequality* (3.2).

Proof. Let $\widetilde{x} \in \Omega$ and let $U = I - \gamma A^*(I-S)A$. Then (3.13) becomes

$$x_{n+1} = TUT[\alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n], \quad n \geqslant 0.$$

We divide the proof into the following steps:

Step 1. We show that $\{x_n\}$ is bounded. In fact, from (3.13), we deduce

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\| &= \|TUT[\alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n] - TUT\widetilde{x}\| \\
&\leqslant \|\alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n - \widetilde{x}\| \\
&\leqslant \alpha_n \sigma \|V x_n - V\widetilde{x}\| + \|(I-\alpha_n \mu F)x_n - (I-\alpha_n \mu F)\widetilde{x}\| + \alpha_n \|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&\leqslant \alpha_n l \|x_n - \widetilde{x}\| + (1-\alpha_n \tau)\|x_n - \widetilde{x}\| + \alpha_n \|\sigma V\widetilde{x} - \mu F\widetilde{x}\| \\
&= [1-(\tau - \sigma l)\alpha_n]\|x_n - \widetilde{x}\| + (\tau - \sigma l)\alpha_n \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}.
\end{aligned}$$

It follows by induction that

$$\|x_{n+1} - \widetilde{x}\| \leqslant \max\left\{\|x_n - \widetilde{x}\|, \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}\right\} \leqslant \cdots \leqslant \max\left\{\|x_0 - \widetilde{x}\|, \frac{\|\sigma V\widetilde{x} - \mu F\widetilde{x}\|}{\tau - \sigma l}\right\}.$$

This means that $\{x_n\}$ is bounded. It is easy to deduce that $\{Vx_n\}$, $\{Ux_n\}$ and $Fx_n\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty} \|TUTz_n - z_n\| = 0$. To this end, set

$$y_n := T[\alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n],$$

and

$$z_n := \alpha_n \sigma V x_n + (I-\alpha_n \mu F)x_n, \quad n \geqslant 0.$$

Since U is averaged by Proposition 2.2 (ii) and also, every firmly nonexpansive mapping is averaged, thus

T is averaged. Since the composite of finitely many averaged mappings is averaged by Proposition 2.1 (c), TU is averaged. Hence, there exists a positive constant $\lambda_1 \in (0,1)$ such that $TU = (1-\lambda_1)I + \lambda_1 G_1$, where G_1 is a nonexpansive mapping. Since T is averaged, there exists $\lambda_2 \in (0,1)$ such that $T = (1-\lambda_2)I + \lambda_2 G_2$, where G_2 is a nonexpansive mapping. It follows that

$$\begin{aligned} y_n &= Tz_n \\ &= ((1-\lambda_2)I + \lambda_2 G_2)z_n \\ &= ((1-\lambda_2)I + \lambda_2 G_2)(\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n) \\ &= (1-\lambda_2)(\alpha_n \sigma V x_n + (I-\alpha_n\mu F)x_n) + \lambda_z G_2 z_n \\ &= (1-\lambda_2)(x_n + \alpha_n(\sigma V x_n - \mu F x_n)) + \lambda_2 G_2 z_n \\ &= (1-\lambda_2)x_n + \lambda_2\left[\frac{(1-\lambda_2)}{\lambda_2}\alpha_n(\sigma V x_n - \mu F x_n) + G_2 z_n\right] = (1-\lambda_2)x_n + \lambda_2 q_n, \end{aligned} \tag{3.14}$$

where

$$q_n = \frac{(1-\lambda_2)}{\lambda_2}\alpha_n(\sigma V x_n - \mu F x_n) + G_2 z_n.$$

Moreover, we get

$$\begin{aligned} &\|q_{n+1} - q_n\| \\ &= \|\frac{(1-\lambda_2)}{\lambda_2}\alpha_{n+1}(\sigma V x_{n+1} - \mu F x_{n+1}) + G_2 z_{n+1}) - \frac{(1-\lambda_2)}{\lambda_2}\alpha_n(\sigma V x_n - \mu F x_n) - G_2 z_n\| \\ &\leqslant \|G_2 z_{n+1} - G_2 z_n\| + \frac{(1-\lambda_2)}{\lambda_2}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|\sigma V x_n - \mu F x_n\|] \\ &\leqslant \|z_{n+1} - z_n\| + \frac{(1-\lambda_2)}{\lambda_2}[\alpha_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \alpha_n\|(\sigma V x_n - \mu F x_n)\|]. \end{aligned} \tag{3.15}$$

In view of (3.13) and (3.14), we have

$$\begin{aligned} x_{n+1} &= TUy_n \\ &= ((1-\lambda_1)I + \lambda - 1G_1)y_n \\ &= (1-\lambda_1)y_n + \lambda_1 G_1 y_n \\ &= (1-\lambda_1)[(1-\lambda_2)x_n + \lambda_2 q_n] + \lambda_1 G_1 y_n \\ &= (1-\lambda_1)(1-\lambda_2)x_n + (1-\lambda_1)\lambda_2 q_n + \lambda_1 G_1 y_n \\ &= (1-(\lambda_1+\lambda_2-\lambda_1\lambda_2))x_n + (1-\lambda_1)\lambda_2 q_n + \lambda_1 G_1 y_n \\ &= (1-\lambda_3)x_n + \lambda_3\left[\frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n + \frac{\lambda_1}{\lambda_3}G_1 y_n\right] \\ &= (1-\lambda_3)x_n + \lambda_3 p_n, \end{aligned} \tag{3.16}$$

where $\lambda_3 = \lambda_1 + \lambda_2 - \lambda_1\lambda_2$ and $p_n = \frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n + \frac{\lambda_1}{\lambda_3}G_1 y_n$. Thus, from (3.15), we derive

$$\begin{aligned} \|p_{n+1} - p_n\| &= \|\frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_{n+1} + \frac{\lambda_1}{\lambda_3}G_1 y_{n+1} - \frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n - \frac{\lambda_1}{\lambda_3}G_1 y_n\| \\ &= \|\frac{(1-\lambda_1)\lambda_2}{\lambda_3}(q_{n+1} - q_n) + \frac{\lambda_1}{\lambda_3}(G_1 y_{n+1} - G_1 y_n)\| \\ &\leqslant \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1} - q_n\| + \frac{\lambda_1}{\lambda_3}\|G_1 y_{n+1} - G_1 y_n\| \\ &\leqslant \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1} - q_n\| + \frac{\lambda_1}{\lambda_3}\|y_{n+1} - y_n\| \end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1}-q_n\| + \frac{\lambda_1}{\lambda_3}\|Tz_{n+1}-Tz_n\| \\
&\leqslant \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1}-q_n\| + \frac{\lambda_1}{\lambda_3}\|z_{n+1}-z_n\| \\
&\leqslant \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|z_{n+1}-z_n\| \\
&\quad + \frac{(1-\lambda_1)(1-\lambda_2)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|] + \frac{\lambda_1}{\lambda_3}\|z_{n+1}-z_n\| \\
&= \frac{\lambda_1+\lambda_2-\lambda_1\lambda_2}{\lambda_3}\|z_{n+1}-z_n\| \\
&\quad + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|] \\
&= \|z_{n+1}-z_n\| + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|] \\
&= \|(\alpha_{n+1}\sigma Vx_{n+1} + (I-\alpha_{n+1}\mu F)x_{n+1} - (\alpha_n\sigma Vx_n + (I-\alpha_n\mu Fx_n)n\| \\
&\quad + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|(\sigma Vx_{n+1}-\mu Fx_{n+1})\| + \alpha_n\|(\sigma Vx_n-\mu Fx_n)\|] \\
&= \|(x_{n+1}-x_n) + \alpha_{n+1}(\sigma Vx_{n+1}-\mu Fx_{n+1}) - \alpha_n(\sigma Vx_n-\mu Fx_n)\| \\
&\quad + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|] \\
&\leqslant \|x_{n+1}-x_n\| + \alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\| \\
&\quad + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|p_{n+1}-p_n\| - \|x_{n+1}-x_n\| &\leqslant \alpha_{n+1}\|\sigma Vx_{n+1}-\mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\| \\
&\quad + \frac{(1-\lambda_3)}{\lambda_3}[\alpha_{n+1}\|\sigma Vx_{n+1}\| - \mu Fx_{n+1}\| + \alpha_n\|\sigma Vx_n-\mu Fx_n\|]
\end{aligned}
$$

and

$$\limsup_{n\to\infty}(\|p_{n+1}-p_n\| - \|x_{n+1}-x_n\|) \leqslant 0. \tag{3.17}$$

Thus, from (3.16), (3.17), and Lemma 2.6, we obtain

$$\lim_{n\to\infty}\|p_n-x_n\| = 0. \tag{3.18}$$

Also, by (3.16) and (3.18), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lambda_3\lim_{n\to\infty}\|p_n-x_n\| = 0, \tag{3.19}$$

and

$$\lim_{n\to\infty}\|z_n-x_n\| = \lim_{n\to\infty}\alpha_n\|\sigma Vx_n-\mu Fx_n\| = 0. \tag{3.20}$$

Therefore, from (3.19) and (3.20), we have

$$
\begin{aligned}
\|TUTz_n-z_n\| &= \|TUTz_n-x_n+x_n-z_n\| \\
&\leqslant \|TUTz_n-x_n\| + \|x_n-z_n\| \\
&= \|x_{n+1}-x_n\| + \|x_n-z_n\| \to 0 \ \text{ as } n\to\infty.
\end{aligned}
$$

Step 3. We show that $\lim_{n\to\infty}\|Tz_n-z_n\| = \lim_{n\to\infty}\|y_n-z_n\| = 0$. To this end, let $\widetilde{x}\in\Omega$. Then we have

$$|\|TUTz_n-\widetilde{x}\| - \|z_n-\widetilde{x}\|| \leqslant \|TUTz_n-z_n\|.$$

Taking limit on the both sides and using Step 2, we have

$$\begin{aligned} \lim_{n\to\infty} |(\|TUTz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\|)| = 0, \\ |\lim_{n\to\infty} (\|TUTz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\|)| = 0, \\ \lim_{n\to\infty} (\|TUTz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\|) = 0. \end{aligned} \tag{3.21}$$

By nonexpansiveness of TU and T, we get

$$\|TUTz_n - \widetilde{x}\| \leqslant \|Tz_n - \widetilde{x}\| \leqslant \|z_n - \widetilde{x}\|,$$

and so,

$$\|TUTz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\| \leqslant \|Tz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\| \leqslant 0.$$

Thus, from (3.21), we induce

$$\lim_{n\to\infty} (\|Tz_n - \widetilde{x}\| - \|z_n - \widetilde{x}\|) = 0.$$

Since T is firmly nonexpansive and hence strongly nonexpansive [6], we have

$$\lim_{n\to\infty} \|Tz_n - z_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0.$$

Step 4. We show that $\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \leqslant 0$, where x^* is the unique solution of the variational inequality (3.2). Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_{n_i} - x^*\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $\widetilde{x}$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $\widetilde{x}$. Therefore, from Step 2, (3.20), and Lemma 2.4, we have $x_{n_i} \to \widetilde{x} \in \mathrm{Fix}(TUT)$. Since T and U are averaged, by Proposition 2.1 (iv), we have $z \in \mathrm{Fix}(T)$ and $z \in \mathrm{Fix}(U)$, and hence $Az \in \mathrm{Fix}(S)$ by Proposition 2.2 (iii). Thus $z \in \Omega$. Therefore we derive

$$\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle \sigma Vx^* - \mu Fx^*, x_{n_i} - x^*\rangle = \langle \sigma Vx^* - \mu Fx^*, \widetilde{x} - x^*\rangle \leqslant 0.$$

This together with (3.20) and Step 3 imply that

$$\limsup_{n\to\infty}\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \leqslant 0.$$

Step 5. We show that $\lim_{n\to\infty} x_n = x^*$, where x^* is the unique solution of the variational inequality (3.2). We observe that

$$\|Tz_n - x^*\|^2 = \langle Tz_n - y_n, Tz_n - x^*\rangle + \langle z_n - x^*, Tz_n - x^*\rangle.$$

Since T is a firmly nonexpansive mapping with a fixed point x^*, we have $\langle Tz_n - z_n, Tz_n - x^*\rangle \leqslant 0$. Thus we derive

$$\begin{aligned} \|Tz_n - x^*\|^2 &\leqslant \langle z_n - x^*, Tz_n - x^*\rangle \\ &= \langle \alpha_n\sigma(Vx_n - Vx^*) + (I - \alpha_n\mu F)x_n - (I - \alpha_n\mu F)x^*, Tz_n - x^*\rangle + \alpha_n\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \\ &\leqslant (\alpha_n\sigma l\|x_n - x^*\| + (1 - \alpha_n\tau)\|x_n - x^*\|)\|Tz_n - x^*\| + \alpha_n\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \\ &= (1 - \alpha_n(\tau - \sigma l))\|x_n - x^*\|\|Tz_n - x^*\| + \alpha_n\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \\ &\leqslant \frac{1 - \alpha_n(\tau - \sigma l)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|Tz_n - x^*\|^2 + \alpha_n\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle. \end{aligned}$$

It follows that

$$\|Tz_n - x^*\|^2 \leqslant [1 - \alpha_n(\tau - \sigma l)]\|x_n - x^*\|^2 + 2\alpha_n\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle. \tag{3.22}$$

From (3.13) and (3.22), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|TUTz_n - x^*\|^2\\ &\leqslant \|Tz_n - x^*\|^2\\ &\leqslant [1-\alpha_n(\tau-\sigma l)]\|x_n - x^*\|^2 + \alpha_n(\tau-\sigma l)\frac{2}{\tau-\sigma l}\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle.\end{aligned} \tag{3.23}$$

Put $\lambda_n = \alpha_n(\tau-\sigma l)$ and

$$\delta_n = \frac{2}{\tau-\sigma l}\langle \sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle.$$

It can be easily seen from Step 4 and conditions (C1) and (C2) that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leqslant 0$. Since (3.23) reduces to

$$\|x_{n+1}-x^*\|^2 \leqslant (1-\lambda_n)\|x_n - x^*\|^2 + \lambda_n\delta_n,$$

by Lemma 2.7, we conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Putting $\mu = 1$ and $F = I$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = T[I-\gamma A^*(I-S)A]T[\alpha_n\sigma Vx_n + (1-\alpha_n)x_n], \quad n \geqslant 0. \tag{3.24}$$

Assume that the sequence $\{\alpha_n\} \in [0,1]$ satisfies the conditions (C1) *and* (C2) *in Theorem* 3.5*. Then $\{x_n\}$ converges strongly to a point $x^* \in \Omega$ which is the unique solution of the variational inequality* (3.9)*.*

Putting $V = 0$ in (3.24), we get the following corollary.

Corollary 3.7. *Let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = T[I-\gamma A^*(I-S)A]T[(1-\alpha_n)x_n], \quad n \geqslant 0.$$

Assume that the sequence $\{\alpha_n\}$ satisfies the conditions (C1) *and* (C2) *in Theorem* 3.5*. Then $\{x_n\}$ converges strongly to a point x^* which is the minimum norm solution of the SCFPP* (1.2)*.*

Remark 3.8.

1) It is well-known that the metric projection is firmly nonexpansive and hence nonexpansive. Thus iterative algorithms (3.1) and (3.13) are more general than [16, iterative algorithms (3.1) and (3.11)], respectively. Indeed, if we consider $T = P_C$, $S = P_Q$, $\mathrm{Fix}(T) = C$ and $\mathrm{Fix}(S) = Q$, then Theorem 3.2 and Theorem 3.5 generalize [16, Theorem 3.2] and [16, Theorem 3.5], respectively.

2) Theorem 3.2 and Theorem 3.5 also improve [2, Theorem 3.5 and Theorem 3.7] and [31, Theorem 3.1 and Theorem 3.5], respectively.

3) Corollary 3.3 and Corollary 3.6 generalize [16, Corollary 3.3 and Corollary 3.6] and [31, Corollary 3.2 and Corollary 3.7], respectively.

4) Corollary 3.4 and Corollary 3.7 improve [31, Corollary 3.3 and Corollary 3.9], respectively.

4. Applications

Now, as in [2], we apply our iterative algorithms to study some problems from nonlinear and convex analysis.

4.1. Variational problems via resolvent operators

For a given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well-known that its associated resolvent operator $J_r^M = (I + rM)^{-1}$ is firmly nonexpansive and $0 \in M(x) \iff J_r^M(x) = x$ for $r > 0$; see, for instance, [1, 24]. This means zeros of M are exactly fixed points of its resolvent operator. Let $T = J_r^M$ and $S = J_r^N$, where $N : H_2 \to 2^{H_2}$ is a maximal monotone operator. We consider the problem of finding $x^* \in \Omega_1$ such that

$$\langle \sigma V x^* - \mu F x^*, \tilde{x} - x^* \rangle \leqslant 0, \quad \forall \tilde{x} \in \Omega_1,$$

where $\Omega_1 = M^{-1}(0) \cap A^{-1}(N^{-1}(0))$. Under these restrictions, iterative algorithms (3.1) and (3.13) reduce the following iterative algorithms, respectively.

Algorithm 4.1. For any $t \in (0, \frac{1}{\tau - \sigma l})$, define a net $\{x_t\} \subset H_1$ in an implicit way:

$$x_t = J_r^M[I - \gamma A^*(I - J_r^N)A]J_r^M[t\sigma V x_t + (I - t\mu F)x_t].$$

Algorithm 4.2. For an arbitrarily chosen $x_0 \in H_1$, compute in an explicit way:

$$x_{n+1} = J_r^M[I - \gamma A^*(I - J_r^N)A]J_r^M[\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n], \quad n \geqslant 0,$$

where $\{\alpha_n\} \subset [0, 1]$.

Since the resolvent operators are firmly nonexpansive, the strong convergence of the net $\{x_t\}$ (respectively, the sequence $\{x_n\}$) generated by Algorithm 4.1 (respectively, Algorithm 4.2) can be derived from Theorem 3.2 (respectively, Theorem 3.5).

4.2. Equilibrium problems via resolvent operators

Let C be a nonempty closed convex subset of a Hilbert space, and let $\Theta : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: find $z \in C$ such that

$$\Theta(z, y) \geqslant 0, \quad \forall y \in C. \tag{4.1}$$

The set of all $z \in C$ which satisfies (4.1) is denoted by $EP(C, \Theta)$, i.e.,

$$EF(C, \Theta) = \{z \in C : \Theta(z, y) \geqslant 0, \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction Θ satisfies the following conditions:

(H1) $\Theta(x, x) = 0$, $\forall x - nC$;
(H2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leqslant 0$, $\forall x,\ y \in C$;
(H3) $\lim_{t\downarrow 0} \Theta(tz + (1 - t)x, y) \leqslant \Theta(x, y)$, $\forall x,\ y,\ z \in C$;
(H4) for each $x \in H$, $y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

It is well-known ([3, 12]) that the associated resolvent operator $T_r^\Theta : H \to C$ defined by

$$T_r^\Theta x = \{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0, \ \forall y \in C\},$$

is firmly nonexpansive and $\mathrm{Fix}(T_r^\Theta) = EP(C, \Theta)$. Let $T = T_r^\Theta$ and $S = S_\nu^\Phi$, where $\Phi : Q \times Q \to \mathbb{R}$ is another function. We consider the problem of finding $x^* \in \Omega_2$ such that

$$\langle \sigma V x^* - \mu F x^*, \tilde{x} - x^* \rangle \leqslant 0, \quad \forall \tilde{x} \in \Omega_2,$$

where $\Omega_2 = EP(C, \Theta) \cap A^{-1}(EP(Q, \Phi))$. Under these restrictions, iterative algorithms (3.1) and (3.13) reduces the following iterative algorithms, respectively.

Algorithm 4.3. For any $t \in (0, \frac{1}{\tau - \sigma l})$, define a net $\{x_t\} \subset H_1$ in an implicit way:

$$x_t = T_r^{\Theta}[I - \gamma A^*(I - S_v^{\Phi})A]T_r^{\Theta}[t\sigma V x_t + (I - t\mu F)x_t].$$

Algorithm 4.4. For an arbitrarily chosen $x_0 \in H_1$, compute in an explicit way:

$$x_{n+1} = T_r^{\Theta}[I - \gamma A^*(I - S_v^{\Phi})A]T_r^{\Theta}[\alpha_n \sigma V x_n + (I - \alpha_n \mu F)x_n], \quad n \geqslant 0,$$

where $\{\alpha_n\} \subset [0, 1]$.

Since the resolvent operators are firmly nonexpansive, the strong convergence of the net $\{x_t\}$ (respectively, the sequence $\{x_n\}$) generated by Algorithm 4.3 (respectively, Algorithm 4.4) can be derived from Theorem 3.2 (respectively, Theorem 3.5).

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