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Noether theory for Birkhoffian systems with nabla derivatives

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Abstract

There are discrete phenomena which happen only on discrete time or hold discrete space structures such as economy series, population dynamics et al.. Then there is a tool needed for these discrete issues or applications. Time scale is one of the useful tools to solve some discrete problems. In this paper, time scale is used to establish discrete Pfaff-Birkhoff principle and achieve discrete Birkhoff equations, discrete Noether identity and discrete conserved quantity for the discrete Birkhoffian system. Firstly, Birkhoff equations, Noether identity and Noether theorem with nabla derivatives on time scales are investigated by using the isochronous variational principle. Secondly, some special cases, especially the discrete Birkhoffian system are discussed. Thirdly, another method, i.e., the duality principle is introduced for the Birkhoffian system on time scales. And finally, an example is given to illustrate the results and methods.

Keywords: Noether theorem, Birkhoffian system, time scale, duality principle, isochronous variational principle.
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1. Introduction

In 1973, Turrittin in [68], pointed out: "On becoming familiar with difference equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations." Much earlier than Turrittin, Bell in 1937 in [20], once mentioned: "A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both." Under this kind of background, Hilger introduced the definition of time scale in his PhD thesis in 1988, see [41]. A time scale is an arbitrary nonempty closed subset of real numbers. From the definition, the continuous analysis can be gotten when the time scale is the set of real numbers, and the discrete analysis can be gotten when the time scale is the set of integers. However, since there are many other time scales than just the sets of real numbers and integers, one can get much more results. Therefore, unification and extension are two main features of time scales. Time scale calculus mainly refers to two kinds of calculus: time scale delta calculus and time scale nabla calculus, both of them have received a lot of attention and some results on time scales have been obtained, we refer the readers, for instance, to [5, 6, 8, 10, 15–17, 25, 26, 30, 34, 42, 52–54]. In addition, the theory of time scales has also been applied to practical problems such as the optimal control problems, see [17, 43, 78], and some problems on physics and economics, see [4, 8, 27].

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It is worth mentioning that some basic properties such as Laplace transform and existence of solutions of difference equations on time scales have been systemically discussed [3, 28]. In order to consider the memory effects, References [1, 9, 11] studied the fractional difference on time scales, which gives a useful tool for discrete fractional modelling. Some properties of fractional difference have been studied [1, 2, 9, 31, 39, 40, 44], such as Laplace transform [44], existence and stability [31], initial value problems [9], boundary value problems [39, 40] and so on. Recently, fractional differences which become more and more popular have been used to study the diffusion over random media of discrete structures [73], discrete fractional chaos [45, 69–71, 74, 75], discrete fractional control [59, 60] and so forth.

Noether theory was introduced by Noether in 1918 during the process of getting the solutions of differential equations in [61]. Noether symmetry means the invariance of Hamilton action under the infinitesimal transformations of the time and the coordinates. Actually, it is hard to obtain analytical solutions for dynamic equations of motion for constrained mechanical systems. However, we can get conserved quantities, which can help reduce the degrees of freedom of the equations of motion, through the Noether symmetry. Conserved quantities for constrained mechanical systems can not only simplify the dynamical problems, but also reveal the internal physical regularity to some extent. Hence, it is of great significance to study. Up to now many results about Noether theory have been obtained, such as Noether theory based on fractional models, see [7, 35, 80, 81]; Noether theory with time delay, see [46, 77]; Noether theory for nonlinear dynamical systems, see [36, 82]; as well as Noether theory for fractional systems of variable order, see [62, 76]. Recently, Noether theory was extended to time scales, see for instance, [18, 29, 50, 51, 55, 63, 79, 83].

However, most of the results about Noether theory on time scales just referred to Lagrangian system and Hamiltonian system. There is another more general mechanical system, i.e., Birkhoffian system.

Birkhoffian mechanics, whose quintessence is the Birkhoff equations and Pfaff-Birkhoff principle is a generalization of Hamiltonian mechanics, we refer to [24, 64]. Hamilton canonical equations keep the same forms under canonical transformations, and become the Birkhoff equations under non-canonical transformations. Therefore, Hamilton canonical equations are special cases of Birkhoff equations. Hamilton principle is also a special case of the Pfaff-Birkhoff principle. Hence, the theory of Birkhoffian mechanics can be applied to Hamiltonian mechanics, Lagrangian mechanics and Newtonian mechanics, as well as general holonomic and nonholonomic mechanics, see [57]. Besides, Birkhoffian mechanics is also applicable to statistical mechanics, quantum mechanics, biological physics, hadron physics, atomic and molecular physics and engineering, and other fields, see [64]. Galiullan in 1989 said that Birkhoffian mechanics is an important developmental direction of modern analytical mechanics in [37]. And in 1996, the theoretical framework of Birkhoffian mechanics was established by Mei et al. in [57]. Since then, scholars continue to study Birkhoffian mechanics deeply, we refer the readers for instance to [32, 33, 47–49, 56, 58, 65–67, 76, 77, 80, 81].

Hence, in this paper, we intend to study Noether theory for Birkhoffian systems on time scales with nabla derivatives. Two methods are used: the isochronous variational principle and the duality principle. In Section 2, some necessary definitions and properties about time scale calculus and the duality approach are reviewed. In Section 3, the isochronous variational principle is used to study Noether theory for Birkhoffian systems on time scales with nabla derivatives, where Pfaffian variational problem on time scales with nabla derivatives is presented for the first time, the Birkhoff equations and the Noether theorem on time scales with nabla derivatives are derived, and some special cases, especially the discrete Birkhoffian system, of the main results are also discussed. The duality principle is used in Section 4, in which Noether equations, Noether identity and conserved quantity for Birkhoffian systems on time scales with nabla derivatives are achieved again. Finally, an example is given to illustrate the results in Section 5.

2. Preliminaries

We assume that the reader is familiar with the time scale calculus in [33]. Here we mainly review some necessary results about duality from [30].

If $\mathbb{T}$ is an arbitrary time scale, we denote $\mathbb{T}^* = \{s \in \mathbb{R} | -s \in \mathbb{T}\}$ as the dual time scale of $\mathbb{T}$, and the corresponding dual objects of $\mathbb{T}^*$ are $\hat{\sigma}$, $\hat{\rho}$, $\hat{\mu}$ and $\hat{\nu}$.

Definition 2.1. Given a function $f : \mathbb{T} \to \mathbb{R}$ defined on a time scale $\mathbb{T}$, we define the dual function $f^* : \mathbb{T}^* \to \mathbb{R}$ by $f^*(s) = f(-s)$ for all $s \in \mathbb{T}^*$.

Proposition 2.2. *Let* $f : \mathbb{T} \to \mathbb{R}$, $a, b \in \mathbb{T}$, $a < b$, *then*

(1) $(\mathbb{T}^\kappa)^* = (\mathbb{T}^*)_\kappa$ *and* $(\mathbb{T}_\kappa)^* = (\mathbb{T}^*)^\kappa$;
(2) $([a, b])^* = [-b, -a]$ *and* $([a, b]^\kappa)^* = [-b, -a]_\kappa \subseteq \mathbb{T}^*$;
(3) $\hat{\sigma}(s) = -\rho(-s) = -\rho^*(s)$ *and* $\hat{\rho}(s) = -\sigma(-s) = -\sigma^*(s)$ *for all* $s \in \mathbb{T}^*$;
(4) $\hat{\mu}(s) = \nu^*(s)$ *and* $\hat{\nu}(s) = \mu^*(s)$ *for all* $s \in \mathbb{T}^*$;
(5) $f \in C_{rd}$ (*resp.* $f \in C_{ld}$) $\Leftrightarrow f^* \in C_{ld}$ (*resp.* $f^* \in C_{rd}$), $f \in C^1_{rd}$ (*resp.* $f \in C^1_{ld}$) $\Leftrightarrow f^* \in C^1_{ld}$ (*resp.* $f^* \in C^1_{rd}$);
(6) *if* f *is delta (resp. nabla) differentiable at* $t_0 \in \mathbb{T}^\kappa$ *(resp.* $t_0 \in \mathbb{T}_\kappa$*), then* f^* *is nabla (resp. delta) differentiable at* $-t_0 \in (\mathbb{T}^*)_\kappa$ *(resp.* $-t_0 \in (\mathbb{T}^*)^\kappa$*), and*

$$\begin{aligned}
f^{\triangle}(t_0) &= -(f^*)^{\hat{\nabla}}(-t_0) \quad (\textit{resp. } f^{\nabla}(t_0) = -(f^*)^{\hat{\triangle}}(-t_0)),\\
f^{\triangle}(t_0) &= -((f^*)^{\hat{\nabla}})^*(t_0) \quad (\textit{resp. } f^{\nabla}(t_0) = -((f^*)^{\hat{\triangle}})^*(t_0)),\\
(f^{\triangle})^*(-t_0) &= -(f^*)^{\hat{\nabla}}(-t_0) \quad (\textit{resp. } (f^{\nabla})^*(-t_0) = -(f^*)^{\hat{\triangle}}(-t_0));
\end{aligned}$$

(7) $\int_a^b f(t)\triangle t = \int_{-b}^{-a} f^*(s)\hat{\nabla}s$, $\int_a^b f(t)\nabla t = \int_{-b}^{-a} f^*(s)\hat{\triangle}s$.

Lemma 2.3 ([54]). *Let* $g \in C_{ld}$, $g : [a, b] \to \mathbb{R}$, *if*

$$\int_a^b g(t)\eta^{\nabla}(t)\nabla t = 0,$$

holds for all $\eta \in C^1_{ld}$ *with* $\eta(a) = \eta(b) = 0$, *then*

$$g(t) = c, \quad t \in [a, b]_\kappa,$$

for some $c \in \mathbb{R}$.

3. Main results obtained by the isochronous variational principle

In this section, we intend to study Birkhoff equations, Noether identity and Noether theorem for Birkhoffian systems on time scales with nabla derivatives.

3.1. *Birkhoff equations with nabla derivatives*

We consider the following problem

$$\bar{S}(a_i(\cdot)) = \int_c^d [R_j(s, a_i^{\rho}(s)) \cdot a_j^{\nabla}(s) - B(s, a_i^{\rho}(s))]\nabla s \to \min, \tag{3.1}$$

for all $a_i(\cdot) \in C^1_{ld}$ with $a_i(c) = A_i$, $a_i(d) = B_i$, where $a_i^{\rho}(s) = (a_i \circ \rho)(s)$, $a_j^{\nabla}(s)$ is the nabla derivatives, $B : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, $B \in C^1_{ld}$ is called Birkhoffian, $R_j : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, $R_j \in C^1_{ld}$ are called Birkhoff functions, $i, j = 1, 2, \cdots, 2n$. We denote the partial derivatives of R_j and B with respect to their first variable by $\partial_0 R_j$ and $\partial_0 B$, and denote the partial derivatives of R_j, B with respect to their $l + 1 -$th variable by $\partial_l R_j$, $\partial_l B$, $l = 1, 2, \cdots, 2n$.

The isochronous variational principle

$$\delta\bar{S} = 0, \tag{3.2}$$

with the exchange relationships

$$(\delta a_i)^\nabla = \delta a_i^\nabla, \quad (\delta a_i)^\rho = \delta a_i^\rho, \tag{3.3}$$

and the boundary value conditions

$$\delta a_i|_{s=c} = \delta a_i|_{s=d} = 0,$$

is called Pfaff-Birkhoff principle on time scales with nabla derivatives, where formula (3.3) can be proved as $(\delta a_i)^\triangle = \delta a_i^\triangle$, $(\delta a_i)^\sigma = \delta a_i^\sigma$ in [29].

Expanding formula (3.2), we obtain

$$\begin{aligned}
\delta\bar{S} &= \int_c^d [\delta R_j(s, a_i^\rho)\cdot a_j^\nabla + R_j(s,a_i^\rho)\cdot \delta a_j^\nabla - \delta B(s,a_i^\rho)]\nabla s \\
&= \int_c^d [\partial_l R_j(s,a_i^\rho)\delta a_l^\rho \cdot a_j^\nabla + R_j(s,a_i^\rho)\cdot \delta a_j^\nabla - \partial_l B(s,a_i^\rho)\delta a_l^\rho]\nabla s \\
&= \int_c^d \{[-\int_c^s (\partial_l R_j(\tau, a_i^\rho(\tau))\cdot a_j^\nabla(\tau))\nabla\tau + R_l + \int_c^s \partial_l B(\tau, a_i^\rho(\tau))\nabla\tau](\delta a_l)^\nabla\}\nabla s \\
&= 0.
\end{aligned}$$

Hence, from Lemma 2.3, we get

$$\partial_l R_j(s,a_i^\rho)\cdot a_j^\nabla - \partial_l B(s,a_i^\rho) - R_l^\nabla = 0. \tag{3.4}$$

Equations (3.4) are called Birkhoff equations on time scales with nabla derivatives.

3.2. *Noether theorem with nabla derivatives*

Firstly, we consider the infinitesimal transformations without transforming the time.

Definition 3.1. Under the following transformations

$$\tilde{s} = s, \quad \tilde{a_i}(s) = a_i(s) + \varepsilon\xi_i(s,a_j) + o(\varepsilon), \tag{3.5}$$

where ε is an infinitesimal parameter, ξ_i, $i = 1,2,\cdots,2n$ are called the infinitesimal generators of the transformations, formula (3.1) is said to be invariant if and only if

$$\int_{s_c}^{s_d} [R_j(s,a_i^\rho)a_j^\nabla - B(s,a_i^\rho)]\nabla s = \int_{s_c}^{s_d} [R_j(s,\tilde{a}_i^\rho)\tilde{a}_j^\nabla - B(s,\tilde{a}_i^\rho)]\nabla s,$$

holds for any $[s_c, s_d] \subseteq [c, d]$.

Definition 3.2. A quantity $I(s, a_i, a_i^\rho, a_i^\nabla)$ (resp. $I(s, a_i, a_i^\sigma, a_i^\triangle)$) is said to be a conserved quantity of the Birkhoffian system with nabla derivatives (resp. the Birkhoffian system with delta derivatives) if and only if $\frac{\nabla}{\nabla s} I(s,a_i,a_i^\rho,a_i^\nabla) = 0$ (resp. $\frac{\triangle}{\triangle s} I(s,a_i,a_i^\sigma,a_i^\triangle) = 0$) holds along the equations of motion for the system.

Theorem 3.3. *Under the transformations* (3.5)*, if formula* (3.1) *is invariant, then*

$$\partial_l R_j(s,a_i^\rho)\xi_l^\rho a_j^\nabla + R_j(s,a_i^\rho)\xi_j^\nabla - \partial_l B(s,a_i^\rho)\xi_l^\rho = 0, \tag{3.6}$$

where $\xi_i^\rho(s,a_j) = \xi_i(\rho(s), a_j(\rho(s)))$, $\xi_i^\nabla(s,a_j) = \frac{\nabla}{\nabla s}\xi_i(s,a_j)$.

Proof. From Definition 3.1, we get

$$R_j(s,a_i^\rho)a_j^\nabla - B(s,a_i^\rho) = R_j(s,a_i^\rho + \varepsilon\xi_i^\rho)(a_j^\nabla + \varepsilon\xi_j^\nabla) - B(s,a_i^\rho+\varepsilon\xi_i^\rho). \tag{3.7}$$

Differentiating both sides of formula (3.7) with respect to ε and letting $\varepsilon = 0$, we can get the intended result. □

Theorem 3.4. *If formula (3.1) is invariant under Definition 3.1, then*

$$I(s, a_i, a_i^\rho, a_i^\nabla) = R_j(s, a_i^\rho)\xi_j(s, a_i),$$

is a conserved quantity for the Birkhoffian system (3.4).

Proof. From (3.4) and formula (3.6), we have

$$\begin{aligned}\frac{\nabla}{\nabla s} I(s, a_i, a_i^\rho, a_i^\nabla) &= \frac{\nabla}{\nabla s}[R_j(s, a_i^\rho)\xi_j(s, a_i)] = R_j\xi_j^\nabla + R_j^\nabla \xi_j^\rho \\ &= R_j\xi_j^\nabla + [\partial_l R_i(s, a_j^\rho) \cdot a_i^\nabla - \partial_l B(s, a_j^\rho)]\xi_l^\rho = 0.\end{aligned}$$

□

Secondly, we consider the general infinitesimal transformations:

$$\tilde{s} = P(s, a_j, \varepsilon) = s + \varepsilon\xi_0(s, a_j) + o(\varepsilon), \quad \tilde{a_i}(\tilde{s}) = Q_i(s, a_j, \varepsilon) = a_i(s) + \varepsilon\xi_i(s, a_j) + o(\varepsilon), \tag{3.8}$$

where ε is an infinitesimal parameter, $\xi_0 : [c, d] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $\xi_i : [c, d] \times \mathbb{R}^{2n} \to \mathbb{R}$ are the infinitesimal generators, and both of them are nabla differential functions.

Let $U = \{a_i | a_i : [c, d] \to \mathbb{R}, a_i \in C^1_{ld}\} \subseteq C^1_{ld}$, suppose that the map $s \in [c, d] \mapsto \alpha(s) := P(s, a_j, \varepsilon) \in \mathbb{R}$ is an increasing C^1_{ld} function and its image is a new time scale, whose backward jump operator and nabla operator are $\tilde{\rho}$ and $\tilde{\nabla}$, respectively. And we can easily get

$$\tilde{\rho} \circ \alpha = \alpha \circ \rho.$$

Definition 3.5. Under the transformations (3.8), formula (3.1) is said to be invariant if and only if

$$\int_{s_c}^{s_d} [R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho)]\nabla s = \int_{\tilde{s_c}}^{\tilde{s_d}} [R_j(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))\tilde{a_j}^{\tilde{\nabla}}(\tilde{s}) - B(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))]\tilde{\nabla}\tilde{s},$$

holds for any $[s_c, s_d] \subseteq [c, d]$.

Theorem 3.6. *Under the infinitesimal transformations (3.8), formula (3.1) is invariant if and only if*

$$(\partial_0 R_j \cdot a_j^\nabla - \partial_0 B)\xi_0 + (\partial_i R_j \cdot a_j^\nabla - \partial_i B)\xi_i^\rho + R_j\xi_j^\nabla - B\xi_0^\nabla = 0. \tag{3.9}$$

Proof. Since

$$\begin{aligned}&\int_{s_c}^{s_d} [R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho)]\nabla s \\ &= \int_{\alpha(s_c)}^{\alpha(s_d)} [R_j(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))\tilde{a_j}^{\tilde{\nabla}}((\tilde{s})) - B(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))]\tilde{\nabla}\tilde{s} \\ &= \int_{s_c}^{s_d} [R_j(\alpha(s), (\tilde{a_i} \circ \tilde{\rho} \circ \alpha)(s))\tilde{a_j}^{\tilde{\nabla}}(\alpha(s)) - B(\alpha(s), (\tilde{a_i} \circ \tilde{\rho} \circ \alpha)(s))]\alpha^\nabla(s)\nabla s \\ &= \int_{s_c}^{s_d} [R_j(\alpha(s), (\tilde{a_i} \circ \alpha \circ \rho)(s))\frac{(\tilde{a_j} \circ \alpha)^\nabla(s)}{\alpha^\nabla(s)} - B(\alpha(s), (\tilde{a_i} \circ \alpha \circ \rho)(s))]\alpha^\nabla(s)\nabla s \\ &= \int_{s_c}^{s_d} [R_j(P, Q_i^\rho)\frac{Q_j^\nabla}{P^\nabla} - B(P, Q_i^\rho)]P^\nabla\nabla s,\end{aligned}$$

holds for any $[s_c, s_d] \subseteq [c, d]$, we have

$$R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho) = [R_j(P, Q_i^\rho)\frac{Q_j^\nabla}{P^\nabla} - B(P, Q_i^\rho)]P^\nabla. \tag{3.10}$$

Then differentiating both sides of formula (3.10) with respect to ε and letting $\varepsilon = 0$, we can get the intended result. □

Formula (3.9) is called Noether identity for Birkhoffian systems on time scales with nabla derivatives.

Theorem 3.7. *If formula* (3.1) *is invariant under Definition* 3.5, *then*

$$I(s, a_j, a_j^\rho, a_j^\nabla) = R_i(s, a_j^\rho)\xi_i(s, a_j) + [\nu(s)(\partial_0 R_j \cdot a_j^\nabla - \partial_0 B) - B]\xi_0,$$

is a conserved quantity for the Birkhoffian system on time scales with nabla derivatives.

Proof. The idea of this proof is to transform the invariance of formula (3.1) under Definition 3.5 to the invariance of formula (3.1) under Definition 3.1, and then make use of Theorem 3.4.

We denote

$$\bar{S}(a_i(\cdot)) = \int_{s_c}^{s_d} [R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho)]\nabla s \doteq \int_{s_c}^{s_d} G(s, a_i^\rho, a_i^\nabla)\nabla s.$$

$$\begin{aligned}\bar{\bar{S}}(\theta(\cdot), a_i(\cdot)) &= \int_{s_c}^{s_d} [R_j(\theta^\rho + \nu(s)\theta^\nabla, a_i^\rho)\frac{a_j^\nabla}{\theta^\nabla} - B(\theta^\rho + \nu(s)\theta^\nabla, a_i^\rho)]\theta^\nabla \nabla s \\ &\doteq \int_{s_c}^{s_d} \bar{G}(s, \theta^\rho, a_i^\rho, \theta^\nabla, a_i^\nabla)\nabla s.\end{aligned}$$

When $\theta(s) = s$, we have

$$\int_{s_c}^{s_d} [R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho)]\nabla s = \int_{s_c}^{s_d} [R_j(\theta^\rho + \nu(s)\theta^\nabla, a_i^\rho)\frac{a_j^\nabla}{\theta^\nabla} - B(\theta^\rho + \nu(s)\theta^\nabla, a_i^\rho)]\theta^\nabla \nabla s.$$

That is,

$$\bar{S}(a_i(\cdot)) = \bar{\bar{S}}(\theta(\cdot), a_i(\cdot)).$$

Hence, when $\theta(s) = s$, we have

$$\begin{aligned}\bar{\bar{S}}(\theta(\cdot), a_i(\cdot)) = \bar{S}(a_i(\cdot)) &= \int_{s_c}^{s_d} [R_j(s, a_i^\rho)a_j^\nabla - B(s, a_i^\rho)]\nabla s \\ &= \int_{\alpha(s_c)}^{\alpha(s_d)} [R_j(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))\tilde{a_j}^{\tilde{\nabla}}(\tilde{(}s)) - B(\tilde{s}, \tilde{a_i}^{\tilde{\rho}}(\tilde{s}))]\tilde{\nabla}\tilde{s} \\ &= \int_{s_c}^{s_d} [R_j(\alpha(s), (\tilde{a_i} \circ \tilde{\rho} \circ \alpha)(s))\tilde{a_j}^{\tilde{\nabla}}(\alpha(s)) - B(\alpha(s), (\tilde{a_i} \circ \tilde{\rho} \circ \alpha)(s))]\alpha^\nabla(s)\nabla s \\ &= \int_{s_c}^{s_d} [R_j(\alpha(s), (\tilde{a_i} \circ \alpha \circ \rho)(s))\frac{(\tilde{a_j} \circ \alpha)^\nabla(s)}{\alpha^\nabla(s)} - B(\alpha(s), (\tilde{a_i} \circ \alpha \circ \rho)(s))]\alpha^\nabla(s)\nabla s \\ &= \int_{s_c}^{s_d} [R_j(\alpha^\rho + \nu(s)\alpha^\nabla, (\tilde{a_i} \circ \alpha)^\rho(s))\frac{(\tilde{a_j} \circ \alpha)^\nabla}{\alpha^\nabla} - B(\alpha^\rho + \nu(s)\alpha^\nabla, (\tilde{a_i} \circ \alpha)^\rho(s))]\alpha^\nabla \nabla s \\ &\doteq \int_{s_c}^{s_d} \bar{G}(s, \alpha^\rho, (\tilde{a_i} \circ \alpha)^\rho, \alpha^\nabla, (\tilde{a_i} \circ \alpha)^\nabla)\nabla s = \bar{\bar{S}}(\alpha(\cdot), (\tilde{a_i} \circ \alpha)(\cdot)).\end{aligned}$$

For $\theta(s) = s$, we have

$$(\alpha(s), (\tilde{a_i} \circ \alpha)(s)) = (P(s, a_j, \varepsilon), Q_i(s, a_j, \varepsilon)) = (P(\theta(s), a_j, \varepsilon), Q_i(\theta(s), a_j, \varepsilon)).$$

That is, $\bar{\bar{S}}$ is invariant on $\bar{\bar{U}} = \{(\theta, a_i) | \theta(s) = s, a_i \in U\}$ under the following transformations

$$(\tilde{\theta}, \tilde{a_i}) = (P(\theta, a_j, \varepsilon), Q_i(\theta, a_j, \varepsilon)),$$

in the sense of Definition 3.1. Therefore, when $\theta(s) = s$, we get from Theorem 3.4 that

$$I = \frac{\partial \bar{G}(s, \theta^\rho, a_i^\rho, \theta^\nabla, a_i^\nabla)}{\partial \theta^\nabla}\xi_0 + \frac{\partial \bar{G}(s, \theta^\rho, a_i^\rho, \theta^\nabla, a_i^\nabla)}{\partial a_j^\nabla}\xi_j, \tag{3.11}$$

where

$$\frac{\partial \bar{G}(s,\theta^{\rho},a_i^{\rho},\theta^{\nabla},a_i^{\nabla})}{\partial a_j^{\nabla}} = R_j(\theta^{\rho}+\nu(s)\theta^{\nabla},a_i^{\rho}) = R_j(s,a_i^{\rho}), \tag{3.12}$$

$$\begin{aligned}\frac{\partial \bar{G}(s,\theta^{\rho},a_i^{\rho},\theta^{\nabla},a_i^{\nabla})}{\partial \theta^{\nabla}} &= \nu(s)\partial_0 R_j(\theta^{\rho}+\nu(s)\theta^{\nabla},a_i^{\rho})\cdot a_j^{\nabla} \\ &\quad -\nu(s)\partial_0 B(\theta^{\rho}+\nu(s)\theta^{\nabla},a_i^{\rho})\cdot\theta^{\nabla} - B(\theta^{\rho}+\nu(s)\theta^{\nabla},a_i^{\rho}) \\ &= \nu(s)[\partial_0 R_j(s,a_i^{\rho})\cdot a_j^{\nabla} - \partial_0 B(s,a_i^{\rho})] - B(s,a_i^{\rho}).\end{aligned} \tag{3.13}$$

The intended result can be obtained by substituting formula (3.12) and formula (3.13) into formula (3.11). □

3.3. Special cases

In this section, we discuss some special cases such as the continuous Birkhoffian system, the discrete Birkhoffian system, the Hamiltonian system on time scales with nabla derivatives, as well as the Lagrangian system on time scales with nabla derivatives.

Corollary 3.8. *If $\mathbb{T}=\mathbb{R}$, we have*

$$\rho(s)=s, \quad \nu(s)=0, \quad f^{\nabla}(s)=\dot{f}(s).$$

In this case, (3.4) reduces to the classical Birkhoff equations

$$\left(\frac{\partial R_j}{\partial a_i}-\frac{\partial R_i}{\partial a_j}\right)\dot{a}_j - \frac{\partial B(s,a_j)}{\partial a_i} - \frac{\partial R_i(s,a_j)}{\partial s} = 0. \tag{3.14}$$

Formula (3.9) gives the classical Noether identity for the Birkhoffian system (3.14)

$$\left(\frac{\partial R_j}{\partial s}-\frac{\partial B}{\partial s}\right)\xi_0 + \left(\frac{\partial R_j}{\partial a_i}\dot{a}_j - \frac{\partial B}{\partial a_i}\right)\xi_i + R_j\dot{\xi}_j - B\dot{\xi}_0 = 0.$$

And Theorem 3.7 gives the classical conserved quantity for the Birkhoffian system (3.14)

$$I = R_i(s,a_j)\xi_i(s,a_j) - B\xi_0 = \text{const.}\,.$$

The results in Corollary 3.8 coincide with the results in [37].

Corollary 3.9. *Consider the following transformations:*

$$a_i^{\rho} = \begin{cases} q_i^{\rho}, & i=1,2,\cdots,n \\ p_{i-n}, & i=n+1,n+2,\cdots,2n \end{cases}, \quad R_i = \begin{cases} p_i, & i=1,2,\cdots,n \\ 0, & i=n+1,n+2,\cdots,2n \end{cases}, \quad B=H, \tag{3.15}$$

where q_m denotes the generalized coordinate, p_m denotes the generalized momentum, $m=1,2,\cdots,2n$, H is the Hamiltonian. From formula (3.15), we can get the following Hamilton action with nabla derivatives

$$\bar{S}_H = \int_c^d (p_m q_m^{\nabla} - H)\nabla s, \quad m=1,2,\cdots,n.$$

Then (3.4) reduces to the Hamilton canonical equations on time scales with nabla derivatives

$$q_{\chi}^{\nabla} = \frac{\partial H(s,q_m^{\rho},p_m)}{\partial p_{\chi}}, \quad p_{\chi}^{\nabla} = -\frac{\partial H(s,q_m^{\rho},p_m)}{\partial q_{\chi}^{\rho}}, \quad \chi=1,2,\cdots,n. \tag{3.16}$$

Formula (3.9) gives the Noether identity for the Hamiltonian system (3.16)

$$p_m\xi_m^{\nabla} - \frac{\partial H(s,q_m^{\rho},p_m)}{\partial s}\xi_0 - \frac{\partial H}{\partial q_m^{\rho}}\xi_m^{\rho} - H\xi_0^{\nabla} = 0.$$

And Theorem 3.7 gives the conserved quantity for the Hamiltonian system (3.16)

$$I_H = p_m\xi_m - [\nu(s)\frac{\partial H}{\partial s} + H]\xi_0 = \text{const.}\,.$$

Corollary 3.10. *Consider*

$$p_\chi = \frac{\partial L(s, q_m^\rho, q_m^\nabla)}{\partial q_\chi^\nabla}, \quad H = p_\chi q_\chi^\nabla - L,$$

where L *is the Lagrangian. Then Euler-Lagrange equations, Noether identity and conserved quantity for Lagrangian systems on time scales with nabla derivatives can be achieved as follows*

$$\frac{\nabla}{\nabla s}\frac{\partial L(s, q_m^\rho, q_m^\nabla)}{\partial q_\chi^\nabla} = \frac{\partial L(s, q_m^\rho, q_m^\nabla)}{\partial q_\chi^\rho},$$

$$\frac{\partial L(s, q_m^\rho, q_m^\nabla)}{\partial s}\xi_0 + \frac{\partial L}{\partial q_\chi^\rho}\xi_\chi^\rho + \frac{\partial L}{\partial q_\chi^\nabla}\xi_\chi^\nabla + L\xi_0^\nabla - \frac{\partial L}{\partial q_\chi^\nabla}q_\chi^\nabla \xi_0^\nabla = 0,$$

$$I_L = \frac{\partial L(s, q_m^\rho, q_m^\nabla)}{\partial q_\chi^\nabla}\xi_\chi + [L - \frac{\partial L}{\partial q_m^\nabla}q_m^\nabla + \nu(s)\frac{\partial L}{\partial s}]\xi_0 = \text{const.}\,.$$

The results in Corollary 3.10 coincide with the results in [30, 55].

In the remainder of this section, we consider the discrete Birkhoffian system in detail. In this section, we restrict ourselves to the time scale $\mathbb{T} = \{c, c+1, \cdots, d\}$, where $c, d \in \mathbb{T}$. And we assume that $\mathbb{T}$ has enough points to ensure the following calculations. It is obvious that $[c, d]_\kappa = [\sigma(c), d]$. Since $\mathbb{T} = \mathbb{Z}$, we have

$$\rho(s) = s - 1, \quad \nu(s) = 1, \quad f^\nabla(s) = f(s) - f^\rho(s) \doteq \nabla_n f(s).$$

Then formula (3.1) gives the following discrete Pfaff action

$$\bar{S}_D = \sum_{s=c+1}^{d}[R_j(s, a_i^\rho)\nabla_n a_j - B(s, a_i^\rho)].$$

The isochronous variational principle

$$\delta\bar{S}_D = 0,$$

with the exchange relationships

$$\nabla_n(\delta a_i) = \delta(\nabla_n a_i), \quad \delta a_i^\rho = (\delta a_i)^\rho,$$

and boundary value conditions

$$\delta a_i|_{s=c} = \delta a_i|_{s=d} = 0,$$

is called discrete Pfaff-Birkhoff principle.

Using the discrete Pfaff-Birkhoff principle, we can derive

$$\frac{\partial R_j}{\partial a_i^\rho}\cdot\nabla_n a_j - \nabla_n R_i - \frac{\partial B}{\partial a_i^\rho} = 0, \quad s \in \{c+2, c+3, \cdots, d\}. \tag{3.17}$$

Equations (3.17) are the discrete Birkhoff equations.

Formula (3.9) gives the discrete Noether identity for the discrete Birkhoffian system

$$(\frac{\partial R_j}{\partial s}\nabla_n a_j - \frac{\partial B}{\partial s})\xi_0 + (\frac{\partial R_j}{\partial a_i^\rho}\nabla_n a_j - \frac{\partial B}{\partial a_i^\rho})\xi_i^\rho + R_j\cdot\nabla_n\xi_j - B\cdot\nabla_n\xi_0 = 0.$$

And Theorem 3.7 gives the discrete conserved quantity for the discrete Birkhoffian system

$$I(s, a_j, a_j^\rho, \nabla_n a_j) = R_i(s, a_j^\rho)\xi_i(s, a_j) + [(\frac{\partial R_j}{\partial s}\cdot\nabla_n a_j - \frac{\partial B}{\partial s}) - B]\xi_0 = \text{const.}\,.$$

Remark 3.11. It follows from the results of the discrete Birkhoffian system that time scale is an important tool to study the discrete problems.

4. Main results obtained by the duality principle

In this section, we use the duality principle to get the Birkhoff equations, the Noether identity and the conserved quantity for the Birkhoffian system on time scales with nabla derivatives.

4.1. Results with delta derivatives

Some results for the Birkhoffian system on time scales with delta derivatives are reviewed, see [66]. We consider the following problem

$$S(a_i(\cdot)) = \int_a^b [R_j(t, a_i^\sigma(t)) \cdot a_j^\triangle(t) - B(t, a_i^\sigma(t))]\triangle t \to \min, \tag{4.1}$$

under the given boundary conditions $a_i(a) = \alpha_i$, $a_i(b) = \beta_i$, where $a_i^\sigma(t) = (a_i \circ \sigma)(t)$, $a_j^\triangle(t) = \frac{\triangle}{\triangle t} a_j(t)$, $t \in \mathbb{T}$, $a_i(\cdot) \in C_{rd}^1$, the Birkhoff's functions $R_j : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, and the Birkhoffian $B : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ are all C_{rd}^1 functions, $i, j = 1, 2, \cdots, 2n$.

Theorem 4.1. *If problem (4.1) has a minimizer $a_{i0}(\cdot)$, then we have the following Birkhoff equations on time scales for all $t \in [a, b]^\kappa$:*

$$\partial_l R_j(t, a_{i0}^\sigma(t)) \cdot a_j^\triangle(t) - \partial_l B(t, a_{i0}^\sigma(t)) - R_l^\triangle(t, a_{i0}^\sigma(t)) = 0, \quad i,\ j,\ l = 1, 2, \cdots, 2n. \tag{4.2}$$

Equations (4.2) are the Birkhoff equations on time scales with delta derivatives.

Theorem 4.2. *If formula (4.1) is invariant under the transformations*

$$\tilde{t} = Y(t, a_i, \varepsilon) = t + \varepsilon\zeta_0(t, a_i) + o(\varepsilon), \quad \tilde{a_j}(\tilde{t}) = Z_j(t, a_i, \varepsilon) = a_j(t) + \varepsilon\zeta_j(t, a_i) + o(\varepsilon), \tag{4.3}$$

then we have

$$\begin{aligned}&[\partial_0 R_j(t, a_i^\sigma(t)) \cdot a_j^\triangle(t) - \partial_0 B(t, a_i^\sigma(t))] \cdot \zeta_0(t, a_i(t)) + R_j(t, a_i^\sigma(t))\zeta_j^\triangle(t, a_i(t)) \\ &\quad + [\partial_l R_j(t, a_i^\sigma(t)) \cdot a_j^\triangle(t) - \partial_l B(t, a_i^\sigma(t))] \cdot \zeta_l^\sigma(t, a_i(t)) - B(t, a_i^\sigma(t))\zeta_0^\triangle(t, a_i(t)) = 0.\end{aligned} \tag{4.4}$$

Formula (4.4) is the Noether identity for the Birkhoffian system on time scales with delta derivatives.

Theorem 4.3. *If formula (4.1) is invariant under the transformations (4.3), then*

$$\begin{aligned}I = {} & R_j(t, a_i^\sigma(t)) \cdot \zeta_j(t, a_i(t)) - \{\mu(t)[\partial_0 R_j(t, a_i^\sigma(t)) \cdot a_j^\triangle(t) - \partial_0 B(t, a_i^\sigma(t))] + B(t, a_i^\sigma(t))\} \cdot \zeta_0(t, a_i(t)) \\ & = \text{const.}\ .\end{aligned}$$

Theorem 4.3 is the Noether theorem for the Birkhoffian system on time scales with delta derivatives.

4.2. Results with nabla derivatives

In this section, we consider the problem (3.1) using the duality principle.

Definition 4.4. Given the Birkhoff's functions $R_j : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$ and the Birkhoffian $B : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$, we define the dual $R_j^* : \mathbb{T}^* \times \mathbb{R}^{2n} \to \mathbb{R}$ and $B^* : \mathbb{T}^* \times \mathbb{R}^{2n} \to \mathbb{R}$ by $R_j^*(x, y_i) = -R_j(-x, y_i)$ and $B^*(x, y_i) = B(-x, y_i)$, $i,\ j = 1, 2, \cdots, 2n$.

Remark 4.5. It is easy to check that

$$\begin{aligned}\partial_0 R_j^*(x, y_i) &= \partial_0 R_j(-x, y_i), \\ \partial_l R_j^*(x, y_i) &= -\partial_l R_j(-x, y_i), \\ \partial_0 B^*(x, y_i) &= -\partial_0 B(-x, y_i),\end{aligned}$$

and

$$\partial_l B^*(x, y_i) = \partial_l B(-x, y_i), \quad i,\ j,\ l = 1, 2, \cdots, 2n.$$

Proposition 4.6. *Given the Birkhoff's functions* $R_j : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$ *and the Birkhoffian* $B : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$, *then for all* $a_i \in C^1_{ld}([c, d])$, *we have*

$$\int_c^d [R_j(s, a_i^\rho(s)) \cdot a_j^\nabla(s) - B(s, a_i^\rho(s))]\nabla s = \int_{-d}^{-c} [R_j^*(t, (a_i^*)^{\hat\sigma}(t)) \cdot (a_j^*)^{\hat\triangle}(t) - B^*(t, (a_i^*)^{\hat\sigma}(t))]\hat\triangle t.$$

Proof. From Proposition 2.2 and Definition 4.4, we have

$$\begin{aligned} R_j^*(t, (a_i^*)^{\hat\sigma}(t)) \cdot (a_j^*)^{\hat\triangle}(t) - B^*(t, (a_i^*)^{\hat\sigma}(t)) &= -R_j^*(t, a_i^*(-\rho(-t))) \cdot a_j^\nabla(-t) - B^*(t, a_i^*(-\rho(-t))) \\ &= R_j(-t, a_i^\rho(-t)) \cdot a_j^\nabla(-t) - B(-t, a_i^\rho(-t)). \end{aligned}$$

Hence, we can get the intended result using Proposition 2.2. □

Theorem 4.7. *If problem* (3.1) *has a minimizer* $\bar a_{i0}$, *then we have the following Birkhoff equations on time scales for all* $s \in [c, d]_\kappa$:

$$\partial_l R_j(s, \bar a_{i0}^\rho(s)) \cdot a_j^\nabla(s) - \partial_l B(s, \bar a_{i0}^\rho(s)) - R_l^\nabla(s, \bar a_{i0}^\rho(s)) = 0, \quad i, j, l = 1, 2, \cdots, 2n.$$

Proof. Since $\bar a_{i0}$ is a minimizer of problem (3.1), then it follows from Proposition 4.6 that $\bar a_{i0}^*$ is a minimizer for the following variational problem

$$\bar S^*(g_i) = \int_{-d}^{-c} [R_j^*(t, g_i^{\hat\sigma}(t)) \cdot (a_j^*)^{\hat\triangle}(t) - B^*(t, g_i^{\hat\sigma}(t))]\hat\triangle t, \quad g_i(-c) = A_i, \quad g_i(-d) = B_i,$$

where $g_i \in C^1_{rd}$. Therefore, we can apply Theorem 4.1 to get the Birkhoff equations for all $t \in [-d, -c]^\kappa$ as follows:

$$\partial_l R_j^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) \cdot (a_j^*)^{\hat\triangle}(t) - \partial_l B^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) - (R_l^*)^{\hat\triangle}(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = 0. \tag{4.5}$$

The following work is to rewrite formula (4.5).

Since

$$R_j^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = -R_j(-t, \bar a_{i0}^\rho(-t)), \quad B^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = B(-t, \bar a_{i0}^\rho(-t)), \tag{4.6}$$

from Proposition 2.2 and Remark 4.5, we can obtain

$$\begin{aligned} &\partial_l R_j^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = -\partial_l R_j(-t, \bar a_{i0}^\rho(-t)), \quad \partial_l B^*(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = \partial_l B(-t, \bar a_{i0}^\rho(-t)), \\ &(R_l^*)^{\hat\triangle}(t, (\bar a_{i0}^*)^{\hat\sigma}(t)) = R_l^\nabla(-t, \bar a_{i0}^\rho(-t)), \quad (a_j^*)^{\hat\triangle}(t) = -a_j^\nabla(-t). \end{aligned} \tag{4.7}$$

Substituting formula (4.7) into formula (4.5), we have

$$\partial_l R_j(-t, \bar a_{i0}^\rho(-t)) \cdot a_j^\nabla(-t) - \partial_l B(-t, \bar a_{i0}^\rho(-t)) - R_l^\nabla(-t, \bar a_{i0}^\rho(-t)) = 0.$$

Since $t \in [-d, -c]^\kappa$, let $s = -t \in [c, d]_\kappa$, we can get

$$\partial_l R_j(s, \bar a_{i0}^\rho(s)) \cdot a_j^\nabla(s) - \partial_l B(s, \bar a_{i0}^\rho(s)) - R_l^\nabla(s, \bar a_{i0}^\rho(s)) = 0.$$

□

Theorem 4.8. *If formula* (3.1) *is invariant under the infinitesimal transformations* (3.8), *then we have*

$$\begin{aligned} &[\partial_0 R_j(s, \bar a_{i0}^\rho(s)) \cdot a_j^\nabla(s) - \partial_0 B(s, \bar a_{i0}^\rho(s))] \cdot \xi_0(s, \bar a_{i0}(s)) + [\partial_l R_j(s, \bar a_{i0}^\rho(s)) \cdot a_j^\nabla(s) - \partial_l B(s, \bar a_{i0}^\rho(s))] \\ &\quad \times \xi_l^\rho(s, \bar a_{i0}(s)) + R_j(s, \bar a_{i0}^\rho(s)) \cdot \xi_j^\nabla(s, \bar a_{i0}(s)) - B(s, \bar a_{i0}^\rho(s)) \cdot \xi_0^\nabla(s, \bar a_{i0}(s)) = 0, \end{aligned} \tag{4.8}$$

for all $s \in [c, d]_\kappa$.

Proof. If $\bar{S}$ is invariant under the infinitesimal transformations (3.8) on U, then $\bar{S}^*$ is invariant on $\bar{U} = \{g_i | g_i : [-d, -c] \to \mathbb{R}, g_i \in C^1_{rd}\}$ under the infinitesimal transformations

$$\tilde{t} = t - \varepsilon\xi_0^*(t, g_i) + o(\varepsilon), \quad \tilde{g_j}(\tilde{t}) = g_j(t) + \varepsilon\xi_j^*(t, g_i) + o(\varepsilon),$$

where $\xi_0^*(x, y) = \xi_0(-x, y)$, $\xi_j^*(x, y) = \xi_j(-x, y)$. Hence, from Theorem 4.2, for all $t \in [-d, -c]^\kappa$, we get

$$\begin{aligned}
&[\partial_0 R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) \cdot (a_j^*)^{\hat{\Delta}}(t) - \partial_0 B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t))] \cdot [-\xi_0^*(t, \bar{a}_{i0}^*(t))] \\
&\quad + [\partial_l R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) \cdot (a_j^*)^{\hat{\Delta}}(t) - \partial_l B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t))] \cdot (\xi_l^*)^{\hat{\sigma}}(t, \bar{a}_{i0}^*(t)) \\
&\quad + R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t))(\xi_j^*)^{\hat{\Delta}}(t, \bar{a}_{i0}^*(t)) - B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) \cdot [-(\xi_0^*)^{\hat{\Delta}}(t, \bar{a}_{i0}^*(t))] = 0.
\end{aligned} \tag{4.9}$$

From Proposition 2.2 and Remark 4.5, we get

$$\begin{aligned}
&\partial_0 R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) = \partial_0 R_j(-t, \bar{a}_{i0}^\rho(-t)), \quad \partial_0 B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) = -\partial_0 B(-t, \bar{a}_{i0}^\rho(-t)), \\
&(\xi_l^*)^{\hat{\sigma}}(t, \bar{a}_{i0}^*(t)) = \xi_l^\rho(-t, \bar{a}_{i0}(-t)), \quad (\xi_j^*)^{\hat{\Delta}}(t, \bar{a}_{i0}^*(t)) = -\xi_j^\nabla(-t, \bar{a}_{i0}(-t)), \\
&\xi_0^*(t, \bar{a}_{i0}^*(t)) = \xi_0(-t, \bar{a}_{i0}(-t)), \quad (\xi_0^*)^{\hat{\Delta}}(t, \bar{a}_{i0}^*(t)) = -\xi_0^\nabla(-t, \bar{a}_{i0}(-t)).
\end{aligned} \tag{4.10}$$

Using formulae (4.6), (4.7) and (4.10), we can rewrite formula (4.9), and get the intended result. □

Formula (4.8) is the Noether identity for the Birkhoffian system on time scales with nabla derivatives.

Theorem 4.9. *If formula* (3.1) *is invariant under the infinitesimal transformations* (3.8)*, then we obtain the following conserved quantity for the Birkhoffian system on time scales with nabla derivatives for all* $s \in [c, d]_\kappa$

$$\begin{aligned}
I = {} & R_j(s, \bar{a}_{i0}^\rho(s)) \cdot \xi_j(s, \bar{a}_{i0}(s)) + \{\nu(s)[\partial_0 R_j(s, \bar{a}_{i0}^\rho(s)) \cdot a_j^\nabla(s) \\
& - \partial_0 B(s, \bar{a}_{i0}^\rho(s))] - B(s, \bar{a}_{i0}^\rho(s))\} \cdot \xi_0(s, \bar{a}_{i0}(s)) = \text{const.}\,.
\end{aligned} \tag{4.11}$$

Proof. From Theorem 4.3, for all $t \in [-d, -c]^\kappa$, we can conclude that

$$\begin{aligned}
I = {} & R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) \cdot \xi_j^*(t, \bar{a}_{i0}^*(t)) - \{\hat{\mu}(t)[\partial_0 R_j^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t)) \cdot (a_j^*)^{\hat{\Delta}}(t) \\
& - \partial_0 B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t))] + B^*(t, (\bar{a}_{i0}^*)^{\hat{\sigma}}(t))\} \cdot [-\xi_0^*(t, \bar{a}_{i0}^*(t))],
\end{aligned}$$

is a constant. It follows from formulae (4.6), (4.7), (4.10) and Proposition 2.2 that formula (4.11) holds. □

Theorem 4.9 is the Noether theorem for the Birkhoffian system on time scales with nabla derivatives.

Remark 4.10. Both methods produce the same results of Noether equations, Noether identity and conserved quantity for Birkhoffian system on time scales with nabla derivatives.

5. An example

Try to find the conserved quantities for the following Birkhoffian system

$$B = (a_2^\rho)^2 + 2t \cdot a_2^\rho, \quad R_1 = a_2^\rho, \quad R_2 = 0.$$

From Theorem 4.8, we can get the following Noether identity

$$-2a_2^\rho \cdot \xi_0 + (a_1^\nabla - 2a_2^\rho - 2t) \cdot \xi_2^\rho + a_2^\rho \cdot \xi_1^\nabla - [(a_2^\rho)^2 + 2t \cdot a_2^\rho] \cdot \xi_0^\nabla = 0.$$

By calculation, we have

$$\xi_0 = 1, \quad \xi_1 = 2t, \quad \xi_2 = 0.$$

If we consider this system on the time scale $\mathbb{T} = \{3^n | n \in \mathbb{N} \bigcup \{0\}\}$, we have

$$\rho(t) = \frac{1}{3}t, \quad \nu(t) = t - \rho(t) = \frac{2}{3}t.$$

From Theorem 4.9, we get the conserved quantity

$$I = -\frac{4}{3}t \cdot a_2^{\rho} - (a_2^{\rho})^2 = \text{const.}\ .$$

If we consider this system on the time scale $\mathbb{T} = h\mathbb{Z} = \{hk | k \in \mathbb{Z}\}$, $h > 0$, we have

$$\rho(t) = t - h, \quad \nu(t) = t - \rho(t) = h.$$

In this case, we get the conserved quantity

$$I = -2h \cdot a_2^{\rho} - (a_2^{\rho})^2 = \text{const.}\ .$$

And if we consider this system on the time scale $\mathbb{T} = \mathbb{R}$, we have

$$\rho(t) = t, \quad \nu(t) = 0.$$

In this case, we get the conserved quantity

$$I = -t \cdot a_2 - (a_2)^2 = \text{const.}\ . \tag{5.1}$$

Formula (5.1) can be found in [58].

6. Conclusion

Birkhoff equations, Noether identity and Noether theorem on time scales with nabla derivatives for the Birkhoffian system are obtained in this paper, where Theorem 3.3–3.7, Theorem 4.7–4.9, and Corollary 3.9 are new works. Corollary 3.8 and Corollary 3.10 coincide with the original results.

The main results are achieved through two methods: the isochronous variational principle and the duality principle. It is obtained that only when the dual results have been achieved, can the duality principle be used, though it is elegant. Therefore, the isochronous variational principle is more general.

On the basis of those results obtained in this paper, perhaps further research such as perturbation to Noether symmetry and adiabatic invariants on time scales for constrained mechanical systems can be studied. Apart from Noether symmetry method, Lie symmetry method and Mei symmetry method are also important modern integral methods. It is hoped that they can also be used to study the constrained mechanical systems on time scales.

In addition, fractional calculus has been applied to various fields of science and engineering, such as quantum mechanics, chaotic mechanics, anomalous diffusion, plasma physics and so on. Fractional mechanics can describe both conservative and non-conservative systems. Recently, fractional Euler-Lagrange equations [14], fractional Hamilton equations [12], Hamiltonian structure of fractional first order Lagrangian [38], constant of motion in fractional multi time Hamiltonian and the dual action for fractional mechanics [13], etc. have been obtained. Besides, the research on fractional calculus on time scales [19, 21–23] has also just started. Therefore, studying fractional mechanical systems on time scales will be a good topic in future.

What is more, lattice fractional diffusion equations obtained through fractional differences [72, 73] have also been studied to some extent. Hence, the discrete fractional variational problem on lattices on time scales is also an aspect to study in future.

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