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Weak θ-ϕ-contraction and discontinuity

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Abstract

In this paper, we introduce the notion of weak θ-ϕ-contraction ensuring a convergence of successive approximations but does not force the mapping to be continuous at the fixed point. Thus, we answer one more solution to the open question raised by Rhoades in [B. E. Rhoades, Fixed point theory Appl, Berkeley, CA, (1986), Contemp. Math., Amer. Math. Soc., Providence, RI, **72** (1988), 233–245].

Keywords: Fixed point, discontinuity, weak θ-ϕ-contraction.
2010 MSC: 47H10, 54H25.

1. Introduction

A large number of contractive definitions have been put forward since Banach contraction principle was published in 1922. In a comprehensive overview of contractive definitions, Rhoades [9] compared 250 contractive definitions and showed that the most of the contractive definitions does not need to be continuous in the entire domain. However all contractive definitions force the mapping to be continuous at the fixed point. In the recent forty years, the theory of fixed point has been grown rapidly (see [2, 3, 5, 7, 8, 12, 13] and the references therein for others). In the meantime, a variety of novel concepts are proposed such as F-contraction [14], Θ-contraction [4], R-contraction [11] and so on. All of these definitions have undoubtedly enriched the theory of fixed point.

As pointed out in [1], in 1988, Rhoades [10] examined in detail the continuity of a large number of contractive mappings at their fixed points and demonstrated that though these contractive definitions do not require the map to be continuous yet the contractive definitions are strong enough to force the map to be continuous at the fixed point. So an interesting open question was raised by Rhoades [10] whether there is a contractive definition which is strong enough to generate a fixed point, but does not force the mapping to be continuous at the fixed point.

In 1999, Pant [6] obtained the first result to the open question. Just recently, Bisht and Pant [1] gave the following result as one more solution to the open question.

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Theorem 1.1 ([1])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping on X such that T^2 is continuous and satisfies the conditions:*

(i). $d(Tx, Ty) \leqslant \phi(M(x, y))$, *where* $\phi : R_+ \to R_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$;

(ii). *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < M(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leqslant \epsilon$, *where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Then T has a unique fixed point, say $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. Moreover, T is discontinuous at x^* iff $\lim_{x \to x^*} M(x, x^*) \neq 0$.

Continuity of T^2 plays a critical role to find the fixed point in Theorem 1.1, but it is difficult to satisfy for a discontinuous mapping. In this paper, a more general approach is given to answer the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the map to be continuous at the fixed point, while continuity of T^2 are not required. Compared to other definitions, our definition is very weak. As a matter of fact, many of the existing results can be derived from our results [15]. An important feature of our definition is that continuity or discontinuity at the fixed point is independent of the definition.

According to [4], we denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;

(Θ_3) θ is continuous on $(0, \infty)$.

And we denote by Φ ([15]) the set of functions $\phi : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

(Φ_1) $\phi : [1, \infty) \to [1, \infty)$ is non-decreasing;

(Φ_2) for each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$;

(Φ_3) ϕ is continuous on $[1, \infty)$.

Lemma 1.2 ([15])**.** *If $\phi \in \Phi$, then $\phi(1) = 1$, and for each $t > 1$, $\phi(t) < t$.*

Proof. Suppose to the contrary that there exists $t_0 > 1$ such that $\phi(t_0) \geqslant t_0$, by the monotonicity of $\phi(t)$, we can get $\phi^n(t_0) \geqslant t_0$ for each $n \in N$, which is a contradiction to $\lim_{n\to\infty} \phi^n(t_0) = 1$. Thus for each $t > 1$, $\phi(t) < t$. For each $t > 1$, $1 \leqslant \phi(1) < t$, passing to limit as $t \to 1$, then we get $\phi(1) = 1$. □

2. Main results

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be a weak θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X, T^2x \neq T^2y$,

$$\theta(d(T^2x, T^2y)) \leqslant \phi[\theta(N(Tx, Ty))],$$

where

$$N(Tx, Ty) = \max\{d(Tx, Ty), d(Tx, T^2x), d(Ty, T^2y), \frac{1}{2}(d(Tx, T^2y) + d(Ty, T^2x))\}.$$

Based on the definition of weak θ-ϕ-contraction, we have the following result.

Theorem 2.2. *Let (X, d) be a metric space and $T : X \to X$ be a weak θ-ϕ-contraction, i.e, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X, T^2x \neq T^2y$,*

$$\theta(d(T^2x, T^2y)) \leqslant \phi[\theta(N(Tx, Ty))], \tag{2.1}$$

where

$$N(Tx,Ty) = \max\{d(Tx,Ty), d(Tx,T^2x), d(Ty,T^2y), \frac{1}{2}(d(Tx,T^2y)+d(Ty,T^2x))\}.$$

And we suppose TX *is a complete subspace of* X. *Then* T *has a unique fixed point* $x^* \in X$ *such that the sequence* $\{T^n x\}$ *converges to* x^* *for every* $x \in X$. *Moreover,* T *is discontinuous at* x^* *iff* $\lim_{x\to x^*} N(Tx,x^*) = \lim_{x\to x^*} N(Tx,Tx^*) \neq 0$.

Proof. Suppose $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n \in N$.

Case 1. If $x_{n+1} = x_n$ for some $n = p \in N$, then $x^* = x_p$ is a fixed point for T.

Case 2. Suppose $x_{n+1} \neq x_n$ for each $n \in N$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in N$.

Making use of the inequality (2.1) with $x = x_{n-1}$ and $y = x_n$, we can get

$$\theta(d(x_{n+1},x_{n+2})) = \theta(d(T^2x_{n-1},T^2x_n)) \leqslant \phi[\theta(N(Tx_{n-1},Tx_n))], \tag{2.2}$$

where

$$\begin{aligned}
&N(Tx_{n-1},Tx_n)\\
&\quad= \max\{d(Tx_{n-1},Tx_n), d(Tx_{n-1},T^2x_{n-1}), d(Tx_n,T^2x_n), \frac{1}{2}(d(Tx_{n-1},T^2x_n)+d(Tx_n,T^2x_{n-1}))\}\\
&\quad= \max\{d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{1}{2}d(x_{n+1},x_{n+2})\}\\
&\quad= \max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}.
\end{aligned} \tag{2.3}$$

If $N(Tx_{n-1},Tx_n) = d(x_{n+1},x_{n+2})$, then it follows from (2.2) that

$$\theta(d(x_{n+1},x_{n+2})) = \theta(d(T^2x_{n-1},T^2x_n)) \leqslant \phi[\theta(d(x_{n+1},x_{n+2}))],$$

which is a contradiction by Lemma 1.2 since $\phi[\theta(d(x_{n+1},x_{n+2}))] < \theta(d(x_{n+1},x_{n+2}))$. Hence equality (2.3) implies for all $n \in N$, $N(Tx_{n-1},Tx_n) = d(x_n,x_{n+1})$, and from (2.2), we have

$$\theta(d(x_{n+1},x_{n+2})) \leqslant \phi[\theta(d(x_n,x_{n+1}))].$$

Repeating this step, we conclude that

$$\begin{aligned}
\theta(d(x_n,x_{n+1})) &\leqslant \phi[\theta(d(x_{n-1},x_n))]\\
&\leqslant \phi^2[\theta(d(x_{n-2},x_{n-1}))]\\
&\leqslant \phi^3[\theta(d(x_{n-3},x_{n-2}))]\\
&\;\;\vdots\\
&\leqslant \phi^n[\theta(d(x_0,x_1))].
\end{aligned}$$

By (Φ_2), we have

$$\lim_{n\to\infty} \phi^n[\theta(d(x_0,x_1))] = 1.$$

And from the definition of θ, we have

$$\lim_{n\to\infty} \theta(d(x_n,x_{n+1})) = 1.$$

Thus, by (Θ_2),

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.4}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Otherwise, there exist $\eta > 0$ and sequences $\{p(n)\}$ and $\{q(n)\}$ such that for all $n \in N$,

$$n < q(n) < p(n), d(x_{p(n)}, x_{q(n)}) \geqslant \eta, \text{ and } d(x_{p(n)-1}, x_{q(n)}) < \eta.$$

Then,

$$\eta \leqslant d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \leqslant \eta + d(x_{p(n)}, x_{p(n)-1}). \tag{2.5}$$

It follows from (2.4) and (2.5),

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \eta. \tag{2.6}$$

By triangle inequality of d, we have

$$|d(x_{p(n)+1}, x_{q(n)+1}) - d(x_{p(n)}, x_{q(n)})| \leqslant d(x_{p(n)}, x_{p(n)+1}) + d(x_{q(n)}, x_{q(n)+1}). \tag{2.7}$$

From (2.4), (2.6), and (2.7),

$$\lim_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) = \eta.$$

Making use of the inequality (2.1) with $x = x_{p(n)-1}$ and $y = x_{q(n)-1}$,

$$\theta(d(x_{p(n)+1}, x_{q(n)+1})) = \theta(d(T^2x_{p(n)-1}, T^2x_{q(n)-1})) \leqslant \phi[\theta(N(Tx_{p(n)-1}, Tx_{q(n)-1}))], \tag{2.8}$$

where

$$\begin{aligned}
&N(Tx_{p(n)-1}, Tx_{q(n)-1})\\
&\quad = \max\{d(Tx_{p(n)-1}, Tx_{q(n)-1}), d(Tx_{p(n)-1}, T^2x_{q(n)-1}), d(Tx_{q(n)-1}, T^2x_{q(n)-1}),\\
&\quad\quad \frac{1}{2}(d(Tx_{p(n)-1}, T^2x_{q(n)-1}) + d(T^2x_{p(n)-1}, Tx_{q(n)-1}))\}\\
&\quad = \max\{d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, x_{p(n)+1}), d(x_{q(n)}, x_{q(n)+1}), \frac{1}{2}(d(x_{p(n)}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)}))\}\\
&\quad \to \max\{\eta, 0, 0, \eta\} = \eta \quad (\text{as } n \to \infty).
\end{aligned}$$

So, passing to limit as $n \to \infty$ to (2.8), we obtain

$$\theta(\eta) \leqslant \phi[\theta(\eta)].$$

It follows from Lemma 1.2 that

$$\theta(\eta) \leqslant \phi[\theta(\eta)] < \theta(\eta),$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X.

Since TX is complete and $\{x_n\} \in TX$, so, $\{x_n\}$ converges to some point x^* in TX. Thus, there exists a point $y \in X$ such that $Ty = x^*$. We shall prove that x^* is a fixed point. If $x^* \neq Tx^*$, then

$$\theta(d(x_{n+1}, Tx^*)) = \theta(d(T^2x_{n-1}, T^2y)) \leqslant \phi[\theta(N(Tx_{n-1}, Ty))], \tag{2.9}$$

where

$$\begin{aligned}
N(Tx_{n-1}, Ty) &= \max\{d(Tx_{n-1}, Ty), d(Tx_{n-1}, T^2x_{n-1}), d(Ty, T^2y), \frac{1}{2}(d(Tx_{n-1}, T^2y) + d(Ty, T^2x_{n-1}))\}\\
&= \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{1}{2}(d(x_n, Tx^*) + d(x^*, x_{n+1}))\}\\
&\to \max\{0, 0, d(x^*, Tx^*), \frac{1}{2}d(x^*, Tx^*)\} = d(x^*, Tx^*) \quad (\text{as } n \to \infty).
\end{aligned}$$

Passing to limit as $n \to \infty$ to (2.9), we have

$$\theta(d(x^*, Tx^*)) \leqslant \phi[\theta(d(x^*, Tx^*))],$$

which is a contradiction since $\phi[\theta(d(x^*, Tx^*))] < \theta(d(x^*, Tx^*))$. Therefore, x^* is a fixed point of T.

Next, we shall show that T has only one fixed point.

Suppose there exists another fixed point y^* of T such that $Tx^* = x^* \neq Ty^* = y^*$. Then $T^2x^* = Tx^* = x^* \neq T^2y^* = Ty^* = y^*$. Making use of the inequality (2.1) with $x = x^*$ and $y = y^*$, $\theta(d(x^*, y^*)) = \theta(d(T^2x^*, T^2y^*)) \leqslant \phi[\theta(N(Tx^*, Ty^*))] = \phi[\theta(d(x^*, y^*))] < \theta(d(x^*, y^*))$, a contradiction. □

Example 2.3. Let $X = \{0\} \cup \{\pm n : n \in N\} \cup \{\pm \frac{1}{n} : n \in N\}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0; \\ -(n-1), & \text{if } x = n; \\ n-1, & \text{if } x = -n; \\ n, & \text{if } x = \frac{1}{n}; \\ -n, & \text{if } x = -\frac{1}{n}. \end{cases}$$

Now, let the function $\theta : (0, \infty) \to (1, \infty)$ defined by

$$\theta(t) = 5^t,$$

and $\phi : [1, \infty) \to [1, \infty)$ defined by

$$\phi(t) = \begin{cases} 1, & \text{if } 1 \leqslant t \leqslant 2; \\ t-1, & \text{if } t \geqslant 2. \end{cases}$$

Obviously, $\theta \in \Theta, \phi \in \Phi$. Through a series of conventional calculations, we can verify that

$$\theta(d(T^2x, T^2y)) \leqslant \phi[\theta(N(Tx, Ty))],$$

for all $x, y \in X$. Thus, T is a weak θ-ϕ-contraction. So all the hypotheses of Theorem 2.2 are satisfied, and T has a fixed point. In this example $x = 0$ is the fixed point.

Remark 2.4.

(i) All the contractive definitions mentioned above and corresponding results are not applicable to the example. In fact, T is not a nonexpansive mapping. For example, if we let $x = \frac{1}{n}, y = \frac{1}{2n}$, we have $d(Tx, Ty) = n > d(x, y) = \frac{1}{2n}$.

(i) T^2 is not continuous on X.

(i) T is not continuous at the fixed point 0. In fact, $\{\frac{1}{n}\} \to 0$, while $\{T\frac{1}{n}\} \nrightarrow T0$.

3. Conclusion

We have proved a new fixed point theorem for weak θ-ϕ-contraction in complete metric spaces. Our result answers the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the map to be continuous at the fixed point. Compared to Theorem 1.1, we do not need the continuity requirement on T^2 which plays a critical role to find the fixed point in Theorem 1.1.

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