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# Fourier series of sums of products of poly-Bernoulli functions and their applications

Taekyun Kim[a], Dae San Kim[b], Dmitry V. Dolgy[c], Jin-Woo Park[d,*]

[a]Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
[b]Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
[c]Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.
[d]Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.

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## Abstract

In this paper, we consider three types of sums of products of poly-Bernoulli functions and derive Fourier series expansions of them. In addition, we express those three types of functions in terms of Bernoulli functions. 

Keywords: Fourier series, Bernoulli polynomial, poly-Bernoulli polynomial, poly-Bernoulli function.
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## 1. Introduction

As is well-known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}.$$

For any integer r, the *poly-Bernoulli polynomials of index* r $\mathbb{B}_m^{(r)}(x)$ are given by the generating function, (see [1–3, 7, 11–15, 17, 20]),

$$\frac{\mathrm{Li}_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(x)\frac{t^m}{m!},$$

where $\mathrm{Li}_r(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^r}$ is the rth polylogarithmic function for $r\geqslant 1$ and a rational function for $r\leqslant 0$.

Note here that our definition of poly-Bernoulli polynomials is slightly different from the original definition of Kaneko's. Indeed, $\mathbb{B}_m^{(r)}(x)=\tilde{\mathbb{B}}_m^{(r)}(x-1)$, where $\tilde{\mathbb{B}}_m^{(r)}(x)$ denotes the Kaneko's poly-Bernoulli polynomials of index r, (see [7]). We also observe that

$$\mathbb{B}_m^{(1)}(x)=B_m(x),\ \mathbb{B}_0^{(r)}(x)=1,\ \mathbb{B}_m^{(0)}(x)=x^m,\ \mathbb{B}_m^{(0)}=\delta_{m,0}.$$

*Corresponding author
*Email addresses:* tkkim@kw.ac.kr (Taekyun Kim), dskim@sogang.ac.kr (Dae San Kim), dvdolgy@gmail.com (Dmitry V. Dolgy), a0417001@knu.ac.kr (Jin-Woo Park)

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As $\mathbb{B}_m^{(r)}(x)$ is an Appell sequence,

$$\frac{d}{dx}\mathbb{B}_m^{(r)}(x) = m\mathbb{B}_{m-1}^{(r)}(x),\ (m \geqslant 1).$$

In addition,

$$\frac{d}{dx}\left(\mathrm{Li}_{r+1}(x)\right) = \frac{1}{x}\mathrm{Li}_r(x).$$

We claim that

$$\mathbb{B}_m^{(r+1)}(1) - \mathbb{B}_m^{(r+1)}(0) = \mathbb{B}_{m-1}^{(r)}(0),\ (m \geqslant 1).$$

Indeed,

$$\sum_{m=0}^{\infty}\left(\mathbb{B}_m^{(r+1)}(1) - \mathbb{B}_m^{(r+1)}(0)\right)\frac{t^m}{m!} = \mathrm{Li}_{r+1}\left(1-e^{-t}\right),$$

and differentiation on both sides with respect to t gives

$$\sum_{m=0}^{\infty}\left(\mathbb{B}_{m+1}^{(r+1)}(1) - \mathbb{B}_{m+1}^{(r+1)}(0)\right)\frac{t^m}{m!} = \frac{\mathrm{Li}_r(1-e^{-t})}{e^t-1} = \sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(0)\frac{t^m}{m!}.$$

Recall that, for a polynomial $p(x) \in \mathbb{Q}[x]$ with $\deg p(x) \leqslant m$,

$$p(x) = \sum_{j=0}^{m} a_j B_j(x),\ a_j \in \mathbb{Q},$$

where (see [8–10])

$$a_0 = \int_0^1 p(x)dx,\ a_j = \frac{1}{j!}\left(p^{(j-1)}(1) - p^{(j-1)}(0)\right),\ \text{for } j = 1, \dots, m.$$

Now, we apply them to $p(x) = \mathbb{B}_m^{(r+1)}(x)$, and let

$$\mathbb{B}_m^{(r+1)}(x) = \sum_{j=0}^{m} a_j B_j(x).$$

Then

$$a_0 = \int_0^1 \mathbb{B}_m^{(r+1)}(x)dx = \frac{1}{m+1}\left(\mathbb{B}_{m+1}^{(r+1)}(1) - \mathbb{B}_{m+1}^{(r+1)}(0)\right) = \frac{1}{m+1}\mathbb{B}_m^{(r)}(0),$$

and, for $1 \leqslant j \leqslant m$,

$$a_j = \frac{(m)_{j-1}}{j!}\left(\mathbb{B}_{m-j+1}^{(r+1)}(1) - \mathbb{B}_{m-j+1}^{(r+1)}(0)\right) = \frac{1}{m+1}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}(0).$$

So

$$\begin{aligned}
\mathbb{B}_m^{(r+1)}(x) &= \frac{1}{m+1}\mathbb{B}_m^{(r)}(0) + \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}(0)B_j(x) \\
&= \frac{1}{m+1}\mathbb{B}_m^{(r)}(0) + \frac{1}{m+1}\sum_{j=0}^{m-1}\binom{m+1}{j+1}\mathbb{B}_j^{(r)}(0)B_{m-j}(x) \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j+1}\mathbb{B}_j^{(r)}(0)B_{m-j}(x).
\end{aligned}$$

This corresponds to Theorem 2.1 in [20]. Note that in that paper, the poly-Bernoulli polynomials are defined by yet another generating function.

For any real number $x$, we let

$$\langle x\rangle = x - \lfloor x \rfloor \in [0,1)$$

denote the fractional part of $x$.

Fourier series expansion of higher-order Bernoulli functions were treated in the recent paper [16]. Here we consider the following three types of sums of products of poly-Bernoulli functions and derive Fourier series expansions of them. In addition, we express those three types of functions in terms of Bernoulli functions.

(1) $\alpha_m(\langle x\rangle) = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle), (m \geqslant 1)$;

(2) $\beta_m(\langle x\rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle), (m \geqslant 1)$;

(3) $\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle), (m \geqslant 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [18, 21]).

As to $\gamma_m(< x >)$, we note that the following polynomial identity follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$:

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x) = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) + \frac{1}{m}\sum_{l=1}^{m}\binom{m}{l}\left(\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)(H_{m-1} - H_{m-l})\right)B_l(x),$$

where $H_l = \sum_{j=1}^{l} \frac{1}{j}$ are the harmonic numbers and

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)}\left(\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)} + \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

The obvious polynomial identities can be derived also for $\alpha_m(\langle x\rangle)$ and $\beta_m(\langle x\rangle)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(\langle x\rangle)B_{m-k}(\langle x\rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [5]) and the Miki's identity (see [4, 6, 19]).

## 2. Sums of products of poly-Bernoulli functions of the first type

For integers $r, s, m$ with $m \geqslant 1$, we let

$$\begin{aligned}
\alpha_m(x) &= \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x),\\
\alpha_m'(x) &= \sum_{k=0}^{m}\left\{k\mathbb{B}_{k-1}^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x) + (m-k)\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k-1}^{(s+1)}(x)\right\}\\
&= \sum_{k=1}^{m} k\mathbb{B}_{k-1}^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x) + \sum_{k=0}^{m-1}(m-k)\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k-1}^{(s+1)}(x)\\
&= \sum_{k=0}^{m-1}(k+1)\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) + \sum_{k=0}^{m-1}(m-k)\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x)\\
&=(m+1)\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) = (m+1)\alpha_{m-1}(x), (m \geqslant 1).
\end{aligned}$$

From this, we see that

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right).$$

We observe that

$$\begin{aligned}\alpha_m(1)-\alpha_m(0) &= \sum_{k=0}^{m}\left(\mathbb{B}_k^{(r+1)}(1)\mathbb{B}_{m-k}^{(s+1)}(1)-\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)\\ &= \sum_{k=1}^{m-1}\left(\mathbb{B}_k^{(r+1)}(1)\mathbb{B}_{m-k}^{(s+1)}(1)-\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)\\ &\quad+\left(\mathbb{B}_m^{(s+1)}(1)-\mathbb{B}_m^{(s+1)}\right)+\left(\mathbb{B}_m^{(r+1)}(1)-\mathbb{B}_m^{(r+1)}\right)\\ &= \sum_{k=1}^{m-1}\left(\left(\mathbb{B}_k^{(r+1)}+\mathbb{B}_{k-1}^{(r)}\right)\left(\mathbb{B}_{m-k}^{(s+1)}+\mathbb{B}_{m-k-1}^{(s)}\right)-\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)+\mathbb{B}_{m-1}^{(s)}+\mathbb{B}_{m-1}^{(r)}\\ &= \sum_{k=1}^{m-1}\left(\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k}^{(s+1)}+\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right)+\mathbb{B}_{m-1}^{(s)}+\mathbb{B}_{m-1}^{(r)}\\ &= \sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\sum_{k=0}^{m-1}\mathbb{B}_k^{(s+1)}\mathbb{B}_{m-k-1}^{(r)}+\sum_{k=1}^{m-1}\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}.\end{aligned}$$

For $m \geqslant 1$, let $\triangle_m = \triangle_m(r,s) = \alpha_m(1)-\alpha_m(0)$. Then

$$\alpha_m(0)=\alpha_m(1) \Longleftrightarrow \triangle_m = 0, \quad \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\triangle_{m+1}.$$

Now, we are going to consider the function

$$\alpha_m(\langle x\rangle) = \sum_{k=0}^{m}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle),\ (m \geqslant 1),$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi i n x},$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x\rangle)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx.$$

We are now ready to determine the Fourier coefficients $A_m^{(m)}$.

**Case 1** : $n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx\\ &= -\frac{1}{2\pi i n}\left[\alpha_m(x)e^{-2\pi i n x}\right]_0^1+\frac{1}{2\pi i n}\int_0^1\alpha_m'(x)e^{-2\pi i n x}dx\\ &= \frac{m+1}{2\pi i n}A_n^{(m-1)}-\frac{1}{2\pi i n}\triangle_m\\ &= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n}A_n^{(m-2)}-\frac{1}{2\pi i n}\triangle_{m-1}\right)-\frac{1}{2\pi i n}\triangle_m\\ &= \frac{(m+1)m}{(2\pi i n)^2}A_n^{(m-2)}-\frac{m+1}{(2\pi i n)^2}\triangle_{m-1}-\frac{1}{2\pi i n}\triangle_m\end{aligned}$$

$$
\begin{aligned}
&= \frac{(m+1)m}{(2\pi i n)^2}\left(\frac{m-1}{2\pi i n}A_n^{(m-3)} - \frac{1}{2\pi i n}\triangle_{m-2}\right) - \frac{m+1}{(2\pi i n)^2}\triangle_{m-1} - \frac{1}{2\pi i n}\triangle_m \\
&= \frac{(m+1)_3}{(2\pi i n)^3}A_n^{(m-3)} - \sum_{j=1}^{3}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\triangle_{m-j+1} \\
&\vdots \\
&= \frac{(m+1)!}{(2\pi i n)^m}A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\triangle_{m-j+1},
\end{aligned}
$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi i n x}dx = 0$. Thus

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\triangle_{m-j+1}.$$

**Case 2** : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\triangle_{m+1}.$$

We recall the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(a) for $m \geqslant 2$,

$$B_m(\langle x\rangle) = -m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}$$

$\alpha(\langle x\rangle)$ $(m \geqslant 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers $m$ with $\triangle_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\triangle_m \neq 0$.

Assume first that $m$ is a positive integer with $\triangle_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$
\begin{aligned}
\alpha_m(\langle x\rangle) &= \frac{1}{m+2}\triangle_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\triangle_{m-j+1}\right)e^{2\pi i n x} \\
&= \frac{1}{m+2}\triangle_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\triangle_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\
&= \frac{1}{m+2}\triangle_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\triangle_{m-j+1}B_j(\langle x\rangle) + \triangle_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}
$$

Now, we can state our first theorem.

**Theorem 2.1.** *For each positive integer $l$, we let*

$$\triangle_l = \triangle_l(r,s) = \sum_{k=0}^{l-1}\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \sum_{k=0}^{l-1}\mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)} + \sum_{k=1}^{l-1}\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}.$$

*Assume that $\triangle_m = 0$ for positive integer $m$. Then we have the following.*

(a) $\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x \rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x \rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle) = \frac{1}{m+2}\triangle_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \triangle_{m-j+1} \right) e^{2\pi i n x},$$

*for all* $x \in (-\infty, \infty)$. *Here the convergence is uniform.*

(b)

$$\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x \rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle) = \frac{1}{m+2}\triangle_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \triangle_{m-j+1} B_j(\langle x \rangle),$$

*for all* $x \in (-\infty, \infty)$.

*Here* $B_j(\langle x \rangle)$ *is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\triangle_m \neq 0$. Then $\alpha_m(0) \neq \alpha_m(1)$. Then $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\triangle_m = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)} + \frac{1}{2}\triangle_m,$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

**Theorem 2.2.** *For each positive integer* $l$, *we let*

$$\triangle_l = \triangle_l(r,s) = \sum_{k=0}^{l-1} \mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \sum_{k=0}^{l-1} \mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)} + \sum_{k=1}^{l-1} \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}.$$

*Assume that* $\triangle_m \neq 0$, *for a positive integer* $m$. *Then we have the following.*

(a)

$$\frac{1}{m+2}\triangle_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \triangle_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x \rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)} + \frac{1}{2}\triangle_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

*Here the convergence is pointwise.*

(b)

$$\frac{1}{m+2}\triangle_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \triangle_{m-j+1} B_j(\langle x \rangle) = \frac{1}{m+2} \sum_{j=0}^{m} \binom{m+2}{j} \triangle_{m-j+1} B_j(\langle x \rangle)$$
$$= \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x \rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle), \text{ for } x \notin \mathbb{Z},$$

$$\frac{1}{m+2}\triangle_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \triangle_{m-j+1} B_j(\langle x \rangle) = \frac{1}{m+2} \sum_{\substack{j=0 \\ j\neq 1}}^{m} \binom{m+2}{j} \triangle_{m-j+1} B_j(\langle x \rangle)$$
$$= \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)} + \frac{1}{2}\triangle_m, \text{ for } x \in \mathbb{Z}.$$

In the special case of $s = 0$, we obtain the following results about the Fourier series of sums of products of poly-Bernoulli and Bernoulli functions.

**Theorem 2.3.** *For each positive integer* $l \geqslant 2$*, we let*

$$\tilde{\triangle}_l = \triangle_l(r,0) = \mathbb{B}_{l-1}^{(r+1)} + \mathbb{B}_{l-2}^{(r)} + \sum_{k=0}^{l-1} B_k \mathbb{B}_{l-k-1}^{(r)}.$$

*Assume that* $\tilde{\triangle}_m = 0$ *for an integer* $m \geqslant 2$*. Then we have the following.*

(a) $\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle) = \frac{1}{m+2}\tilde{\triangle}_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \tilde{\triangle}_{m-j+1} \right) e^{2\pi i n x},$$

*for all* $x \in (-\infty, \infty)$.

(b)

$$\sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle) = \frac{1}{m+2}\tilde{\triangle}_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\tilde{\triangle}_{m-j+1}B_j(\langle x\rangle),$$

*for all* $x \in (-\infty, \infty)$.

**Theorem 2.4.** *For each positive integer* $l \geqslant 2$*, we let*

$$\tilde{\triangle}_l = \triangle_l(r,0) = \mathbb{B}_{l-1}^{(r+1)} + \mathbb{B}_{l-2}^{(r)} + \sum_{k=0}^{l-1} B_k \mathbb{B}_{l-k-1}^{(r)}.$$

*Assume that* $\tilde{\triangle}_m \neq 0$ *for an integer* $m \geqslant 2$*. Then we have the following.*

(a)

$$\frac{1}{m+2}\tilde{\triangle}_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \tilde{\triangle}_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle), & \textit{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}B_{m-k} + \frac{1}{2}\tilde{\triangle}_m, & \textit{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\tilde{\triangle}_{m-j+1}B_j(\langle x\rangle) = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle), \textit{ for } x \notin \mathbb{Z},$$
$$\frac{1}{m+2}\sum_{\substack{j=0 \\ j\neq 1}}^{m}\binom{m+2}{j}\tilde{\triangle}_{m-j+1}B_j(\langle x\rangle) = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}B_{m-k} + \frac{1}{2}\tilde{\triangle}_m, \textit{ for } x \in \mathbb{Z}.$$

## 3. Sums of products of poly-Bernoulli functions of the second type

Let

$$\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x), \ (m \geqslant 1),$$

$$\begin{aligned}\beta'_m(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}\mathbb{B}_{k-1}^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x)+\frac{(m-k)}{k!(m-k)!}\mathbb{B}_{k}^{(r+1)}(x)\mathbb{B}_{m-k-1}^{(s+1)}(x)\right\}\\ &=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}\mathbb{B}_{k-1}^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k-1}^{(s+1)}(x)\\ &=2\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbb{B}_{k}^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x)=2\beta_{m-1}(x).\end{aligned}$$

From this we get $\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x)$, and

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right).$$

We observe that

$$\begin{aligned}\beta_m(1)-\beta_m(0) &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(\mathbb{B}_{k}^{(r+1)}(1)\mathbb{B}_{m-k}^{(s+1)}(1)-\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)\\ &=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\left(\mathbb{B}_{k}^{(r+1)}(1)\mathbb{B}_{m-k}^{(s+1)}(1)-\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)\\ &\quad+\frac{1}{m!}\left(\mathbb{B}_{m}^{(s+1)}(1)-\mathbb{B}_{m}^{(s+1)}\right)+\frac{1}{m!}\left(\mathbb{B}_{m}^{(r+1)}(1)-\mathbb{B}_{m}^{(r+1)}\right)\\ &=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\left(\left(\mathbb{B}_{k}^{(r+1)}+\mathbb{B}_{k-1}^{(r)}\right)\left(\mathbb{B}_{m-k}^{(s+1)}+\mathbb{B}_{m-k-1}^{(s)}\right)-\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right)\\ &\quad+\frac{1}{m!}\left(\mathbb{B}_{m}^{(s+1)}(1)-\mathbb{B}_{m}^{(s+1)}\right)+\frac{1}{m!}\left(\mathbb{B}_{m}^{(r+1)}(1)-\mathbb{B}_{m}^{(r+1)}\right)\\ &=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\left(\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k}^{(s+1)}+\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right)+\frac{1}{m!}\mathbb{B}_{m-1}^{(s)}+\frac{1}{m!}\mathbb{B}_{m-1}^{(r)}\\ &=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_{k}^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_{k}^{(s+1)}\mathbb{B}_{m-k-1}^{(r)}\\ &\quad+\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}.\end{aligned}$$

For $m\geqslant 1$, let $\Omega_m=\Omega_m(r,s)=\beta_m(1)-\beta_m(0)$. Then

$$\beta_m(0)=\beta_m(1)\iff\Omega_m=0.$$

In addition,

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\Omega_{m+1}.$$

Now, we are going to consider the function

$$\beta_m(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_{k}^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle),\ (m\geqslant 1),$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x\rangle) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

**Case 1** : $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right] + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\
&= \frac{2}{2\pi i n}\left(\frac{2}{2\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\right) - \frac{1}{2\pi i n}\Omega_m \\
&= \left(\frac{2}{2\pi i n}\right)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j-1} \\
&\ \ \vdots \\
&= \left(\frac{2}{2\pi i n}\right)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j=1},
\end{aligned}$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

**Case 2** : $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$ $(m \geqslant 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(\langle x\rangle)$ is continuous for those positive integers with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers with $\Omega_m \neq 0$.

Assume first that $m$ is a positive integer with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}
\beta_m(\langle x\rangle) =& \frac{1}{2}\Omega_{m+1} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x} \\
=& \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j}\right)
\end{aligned}$$

$$=\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)+\Omega_m\times\begin{cases}B_j(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}$$

$$=\sum_{\substack{j=0\\ j\neq1}}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)+\Omega_m\times\begin{cases}B_j(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}$$

Now, we can state our first theorem.

**Theorem 3.1.** *For each positive integer* $l$, *we let*

$$\Omega_l=\Omega_l(r,s)=\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)}+\sum_{k=1}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}.$$

*Assume that* $\Omega_m=0$ *for a positive integer* $m$. *Then we have the following.*

(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle)$ *has the Fourier expansion*

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle)=\frac{1}{2}\Omega_{m+1}-\sum_{\substack{n=-\infty\\ n\neq0}}^{\infty}\left(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},$$

*for all* $x\in(-\infty,\infty)$. *Here the convergence is uniform.*

(b)

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle)=\sum_{\substack{j=0\\ j\neq1}}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle),$$

*for all* $x\in(-\infty,\infty)$. *Here* $B_j(\langle x\rangle)$ *is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m\neq0$. Then $\beta_m(0)\neq\beta_m(1)$. $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0)+\beta_m(1)\right)=\beta_m(0)+\frac{1}{2}\Omega_m=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m,$$

for $x\in\mathbb{Z}$.

Next, we can state our second theorem.

**Theorem 3.2.** *For each positive integer* $l$, *we let*

$$\Omega_l=\Omega_l(r,s)=\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)}+\sum_{k=1}^{l-1}\frac{1}{k!(l-k)!}\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}.$$

*Assume that* $\Omega_m\neq0$ *for a positive integer* $m$. *Then we have the following.*

(a)

$$\frac{1}{2}\Omega_{m+1}-\sum_{\substack{n=-\infty\\ n\neq0}}^{\infty}\left(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m, & \text{for } x\in\mathbb{Z},\end{cases}$$

*for all* $x\in(-\infty,\infty)$. *Here the convergence is pointwise.*

(b)

$$\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle),\ \textit{for } x\notin\mathbb{Z},$$

$$\sum_{\substack{j=0\\j\neq1}}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m,\ \textit{for } x\in\mathbb{Z}.$$

In the special case of $s=0$, we obtain the following results about the Fourier series of sums of products of poly-Bernoulli and Bernoulli functions.

**Theorem 3.3.** *For each positive integer* $l\geqslant 2$, *we let*

$$\tilde{\Omega}_l=\Omega_l(r,0)=\frac{1}{(l-1)!}\mathbb{B}_{l-1}^{(r+1)}+\frac{1}{(l-1)!}\mathbb{B}_{l-2}^{(r)}+\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}B_k\mathbb{B}_{l-k-1}^{(r)}.$$

*Assume that* $\tilde{\Omega}_m=0$ *for an integer* $m\geqslant 2$. *Then we have the following.*

(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle)$ *has the Fourier expansion*

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle)=\frac{1}{2}\tilde{\Omega}_{m+1}-\sum_{\substack{n=-\infty\\n\neq0}}^{\infty}\left(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\tilde{\Omega}_{m-j+1}\right)e^{2\pi inx},$$

*for all* $x\in(-\infty,\infty)$.

*(b)*

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle)=\sum_{\substack{j=0\\j\neq1}}^{m}\frac{2^{j-1}}{j!}\tilde{\Omega}_{m-j+1}B_j(\langle x\rangle),$$

*for all* $x\in(-\infty,\infty)$.

**Theorem 3.4.** *For each positive integer* $l\geqslant 2$, *we let*

$$\tilde{\Omega}_l=\Omega_l(r,0)=\frac{1}{(l-1)!}\mathbb{B}_{l-1}^{(r+1)}+\frac{1}{(l-1)!}\mathbb{B}_{l-2}^{(r)}+\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}B_k\mathbb{B}_{l-k-1}^{(r)}.$$

*Assume that* $\tilde{\Omega}_m\neq 0$ *for a positive integer* $m$. *Then we have the following.*

(a)

$$\frac{1}{2}\tilde{\Omega}_{m+1}-\sum_{\substack{n=-\infty\\n\neq0}}^{\infty}\left(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\tilde{\Omega}_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}B_{m-k}+\frac{1}{2}\tilde{\Omega}_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\tilde{\Omega}_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle),\ \textit{for } x\notin\mathbb{Z},$$

$$\sum_{\substack{j=0\\j\neq1}}^{m}\frac{2^{j-1}}{j!}\tilde{\Omega}_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}B_{m-k}+\frac{1}{2}\tilde{\Omega}_m,\ \textit{for } x\in\mathbb{Z}.$$

## 4. Sums of products of poly-Bernoulli functions of the third type

Let

$$\begin{aligned}
\gamma_m(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x), \ (m \geqslant 2), \\
\gamma'_m(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left\{ k\mathbb{B}_{k-1}^{(r+1)}(x)\mathbb{B}_{m-k}^{(s+1)}(x) + (m-k)\,\mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-k-1}^{(s+1)}(x) \right\} \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) + \sum_{k=1}^{m-1} \frac{1}{k} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1}\mathbb{B}_{m-1}^{(s+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) \\
&\quad + \frac{1}{m-1}\mathbb{B}_{m-1}^{(r+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{k} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1}\mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}\mathbb{B}_{m-1}^{(s+1)}(x) + (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} \mathbb{B}_k^{(r+1)}(x)\mathbb{B}_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1}\mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}\mathbb{B}_{m-1}^{(s+1)}(x) + (m-1)\gamma_{m-1}(x).
\end{aligned}$$

So,

$$\gamma'_m(x) = \frac{1}{m-1}\left(\mathbb{B}_{m-1}^{(r+1)}(x) + \mathbb{B}_{m-1}^{(s+1)}(x)\right) + (m-1)\gamma_{m-1}(x).$$

From this, we have

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(s+1)}(x)\right)\right)' = \gamma_m(x),$$

$$\begin{aligned}
\int_0^1 \gamma_m(x)dx &= \frac{1}{m}\left[\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(s+1)}(x)\right]_0^1 \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\left(\mathbb{B}_{m+1}^{(r+1)}(1) - \mathbb{B}_{m+1}^{(r+1)}(0)\right)\right. \\
&\quad \left. - \frac{1}{m(m+1)}\left(\mathbb{B}_{m+1}^{(s+1)}(1) - \mathbb{B}_{m+1}^{(s+1)}(0)\right)\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{B}_k^{(r+1)}(1)\mathbb{B}_{m-k}^{(s+1)}(1) - \mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)\left(\mathbb{B}_{m-k}^{(s+1)} + \mathbb{B}_{m-k-1}^{(s)}\right) - \mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k}^{(s+1)} + \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right).
\end{aligned}$$

For $m \geqslant 2$, we let

$$\Lambda_m = \Lambda_m(r,s) = \gamma_m(1) - \gamma_m(0) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left( \mathbb{B}_k^{(r+1)} \mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_k^{(s+1)} \mathbb{B}_{m-k-1}^{(r)} + \mathbb{B}_{k-1}^{(r)} \mathbb{B}_{m-k-1}^{(s)} \right).$$

Then $\gamma_m(0) = \gamma_m(1)$ if and only if $\Lambda_m = 0$. Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m} \left( \Lambda_{m+1} - \frac{1}{m(m+1)} \mathbb{B}_m^{(r)} - \frac{1}{m(m+1)} \mathbb{B}_m^{(s)} \right).$$

We are now going to consider

$$\gamma_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(\langle x \rangle) \mathbb{B}_{m-k}^{(s+1)}(\langle x \rangle), \ (m \geqslant 2),$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(\langle x \rangle) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx.$$

We are now ready to determine the Fourier coefficients $C_n^{(m)}$.

**Case 1** : $n \neq 0$.

We can show that

$$\int_0^1 \mathbb{B}_l^{(r+1)}(x) e^{-2\pi i n x} dx = - \sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{l-k}^{(r)},$$

and we let

$$\begin{aligned}
\Phi_{m,r} &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{m-k-1}^{(r)}, \\
C_n^{(m)} &= - \frac{1}{2\pi i n} \left[ \gamma_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= - \frac{1}{2\pi i n} \left( \gamma_m(1) - \gamma_m(0) \right) \\
&\quad + \frac{1}{2\pi i n} \int_0^1 \left( (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} \mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1} \mathbb{B}_{m-1}^{(s+1)}(x) \right) e^{-2\pi i n x} dx \\
&= - \frac{1}{2\pi i n} \Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)} \int_0^1 \mathbb{B}_{m-1}^{(r+1)}(x) e^{-2\pi i n x} dx \\
&\quad + \frac{1}{2\pi i n(m-1)} \int_0^1 \mathbb{B}_{m-1}^{(s+1)}(x) e^{-2\pi i n x} dx \\
&= - \frac{1}{2\pi i n} \Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n(m-1)} \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{m-k-1}^{(r)} \\
&\quad - \frac{1}{2\pi i n(m-1)} \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{m-k-1}^{(s)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_{m,r} - \frac{1}{2\pi in(m-1)}\Phi_{m,s} \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in} C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Phi_{m-1,r}\right. \\
&\quad \left. - \frac{1}{2\pi in(m-2)}\Phi_{m-1,s}\right) - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_{m,r} - \frac{1}{2\pi in(m-1)}\Phi_{m,s} \\
&= \frac{(m-1)_2}{(2\pi in)^2} C_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1,r} \\
&\quad - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1,s} \\
&\ \vdots \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1,r} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1,s},
\end{aligned}
$$

where we note that $C_n^{(1)} = 0$. We observe here that

$$
\begin{aligned}
\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1,r} &= \sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}\mathbb{B}_{m-j-k}^{(r)} \\
&= \frac{1}{m}\sum_{l=1}^{m}\frac{(m)_l}{(2\pi in)^l}\frac{\mathbb{B}_{m-l}^{(r)}}{m-l+1}\left(H_{m-1}-H_{m-l}\right).
\end{aligned}
$$

Thus,

$$
C_n^{(m)} = -\frac{1}{m}\sum_{l=1}^{m}\frac{(m)_l}{(2\pi in)^l}\left\{\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)\left(H_{m-1}-H_{m-l}\right)\right\}.
$$

**Case 2** : $n = 0$.

$$
C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right).
$$

$\gamma_m(\langle x\rangle)$ $(m \geqslant 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(\langle x\rangle)$ is continuous for those integers $m \geqslant 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geqslant 2$ with $\Lambda_m \neq 0$.

Assume first that $m$ is an integer $\geqslant 2$ with $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. $\gamma_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to $\gamma_m(\langle x\rangle)$, and

$$
\begin{aligned}
\gamma_m(\langle x\rangle) =& \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) - \frac{1}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{\sum_{l=0}^{m}\frac{(m)_l}{(2\pi in)^l}\Big(\Lambda_{m-l+1}\right. \\
&\left. + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)\left(H_{m-1}-H_{m-l}\right)\Big)\right\} e^{2\pi inx} \\
=& \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) \\
&+ \frac{1}{m}\sum_{l=1}^{m}\binom{m}{l}\left(\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)\left(H_{m-1}-H_{m-l}\right)\right) \\
&\times\left(-l!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^l}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) \\
&\quad + \frac{1}{m}\sum_{l=2}^{m}\binom{m}{l}\left(\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)(H_{m-1} - H_{m-l})\right)B_l(\langle x\rangle) \\
&\quad + \Lambda_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}
$$

Now, we can state our first theorem.

**Theorem 4.1.** *For each positive integer $l \geqslant 2$, we let*

$$
\Lambda_l = \Lambda_l(r,s) = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)} + \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right)
$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$ for a positive integer $m \geqslant 2$. Then we have the following.*

(a)

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) \\
&\quad - \frac{1}{m}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{\sum_{l=1}^{m}\frac{(m)_l}{(2\pi i n)^l}\left(\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)(H_{m-1} - H_{m-l})\right)\right\}e^{2\pi i n x},
\end{aligned}
$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

(b)

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right) \\
&\quad + \frac{1}{m}\sum_{l=2}^{m}\binom{m}{l}\left(\Lambda_{m-l+1} + \frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)} + \mathbb{B}_{m-l}^{(s)}\right)(H_{m-1} - H_{m-l})\right)B_l(\langle x\rangle),
\end{aligned}
$$

*for $x \in (-\infty, \infty)$. Here $B_l(\langle x\rangle)$ is the Bernoulli function.*

Assume next that $m$ is an integer $\geqslant 2$ with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. $\gamma_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. So the Fourier series of $\gamma_m(\langle x\rangle)$ converges pointwise to $\gamma_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$
\frac{1}{2}\left(\gamma_m(0) + \gamma_m(1)\right) = \gamma_m(0) + \frac{1}{2}\Lambda_m = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)} + \frac{1}{2}\Lambda_m,
$$

for $x \in \mathbb{Z}$. Now, we can state our second theorem.

**Theorem 4.2.** *For each positive integer $l \geqslant 2$, we let*

$$
\Lambda_l = \Lambda_l(r,s) = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(\mathbb{B}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_k^{(s+1)}\mathbb{B}_{l-k-1}^{(r)} + \mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right),
$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for a positive integer $m \geqslant 2$. Then we have the following.*

(a)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}-\frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right)$$
$$-\frac{1}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{\sum_{l=1}^{m}\frac{(m)_l}{(2\pi i n)^l}\left(\Lambda_{m-l+1}+\frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)}+\mathbb{B}_{m-l}^{(s)}\right)(H_{m-1}-H_{m-l})\right)\right\}e^{2\pi i n x}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m, & \text{for } x\in\mathbb{Z}.\end{cases}$$

*Here the convergence is pointwise.*

(b)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}-\frac{1}{m(m+1)}\mathbb{B}_m^{(s)}\right)+\frac{1}{m}\sum_{l=1}^{m}\binom{m}{l}\Big(\Lambda_{m-l+1}$$
$$+\frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)}+\mathbb{B}_{m-l}^{(s)}\right)(H_{m-1}-H_{m-l})\Big)B_l(\langle x\rangle)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)\mathbb{B}_{m-k}^{(s+1)}(\langle x\rangle),\ \textit{for } x\notin\mathbb{Z},$$
$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}-\frac{1}{m(m+1)}+\mathbb{B}_m^{(s)}\right)+\frac{1}{m}\sum_{l=2}^{m}\binom{m}{l}\Big(\Lambda_{m-l+1}$$
$$+\frac{1}{m-l+1}\left(\mathbb{B}_{m-l}^{(r)}+\mathbb{B}_{m-l}^{(s)}\right)(H_{m-1}-H_{m-l})\Big)B_l(\langle x\rangle)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}\mathbb{B}_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m.$$

In the special case of $s=0$, we obtain the following results about the Fourier series of sums of products of poly-Bernoulli and Bernoulli functions.

**Theorem 4.3.** *For each positive integer* $l\geqslant 2$*, we let*

$$\tilde{\Lambda}_l=\Lambda_l(r,0)=\frac{1}{l-1}\mathbb{B}_{l-1}^{(r+1)}+\frac{1}{l-1}\mathbb{B}_{l-2}^{(r)}+\sum_{k=1}^{l-1}\frac{1}{k(l-k)}B_k\mathbb{B}_{l-k-1}^{(r)}$$

*with* $\tilde{\Lambda}_1=0$*. Assume that* $\tilde{\Lambda}_m=0$ *for a positive integer* $m\geqslant 2$*. Then we have the following.*

(a)

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x\rangle)B_{m-k}(\langle x\rangle)$$
$$=\frac{1}{m}\left(\tilde{\Lambda}_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}\right)-\frac{1}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{\sum_{l=1}^{m-1}\frac{(m)_l}{(2\pi i n)^l}\Big(\tilde{\Lambda}_{m-l+1}\right.$$
$$\left.+\frac{\mathbb{B}_{m-l}^{(r)}}{m-l+1}(H_{m-1}-H_{m-l})\Big)+\frac{2m!}{(2\pi i n)^m}H_{m-1}\right\}e^{2\pi i n x},$$

*for all* $x\in(-\infty,\infty)$*.*

(b)

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(\langle x \rangle) B_{m-k}(\langle x \rangle)$$
$$= \frac{1}{m}\left(\tilde{\Lambda}_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_k^{(r)}\right) + \frac{1}{m}\sum_{l=2}^{m-1}\binom{m}{l}\left(\tilde{\Lambda}_{m-l+1} + \frac{1}{m-l+1}\mathbb{B}_{m-l}^{(r)}(H_{m-1} - H_{m-l})\right) B_l(\langle x \rangle) + \frac{2}{m} H_{m-1} B_m(\langle x \rangle),$$

*for all* $x \in (-\infty, \infty)$.

**Theorem 4.4.** *For each positive integer* $l \geqslant 2$, *let*

$$\tilde{\Lambda}_l = \Lambda_l(r, 0) = \frac{1}{l-1}\mathbb{B}_{l-1}^{(r+1)} + \frac{1}{l-1}\mathbb{B}_{l-2}^{(r)} + \sum_{k=1}^{l-1}\frac{1}{k(l-k)} B_k \mathbb{B}_{l-k-1}^{(r)}$$

*with* $\tilde{\Lambda}_1 \neq 0$. *Assume that* $\tilde{\Lambda}_m \neq 0$ *for a positive integer* $m \geqslant 2$. *Then we have the following.*

(a)

$$\frac{1}{m}\left(\tilde{\Lambda}_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)}\right) - \frac{1}{m}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{\sum_{l=1}^{m-1}\frac{(m)_l}{(2\pi i n)^l}\left(\tilde{\Lambda}_{m-l+1} + \frac{\mathbb{B}_{m-l}^{(r)}}{m-l+1}(H_{m-1} - H_{m-l})\right) + \frac{2m!}{(2\pi i n)^m} H_{m-1}\right\} e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(\langle x \rangle) B_{m-k}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)} B_{m-k} + \frac{1}{2}\tilde{\Lambda}_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)}\right) + \frac{1}{m}\sum_{l=1}^{m-1}\binom{m}{l}\left(\tilde{\Lambda}_{m-l+1} + \frac{1}{m-l+1}\mathbb{B}_{m-l}^{(r)}(H_{m-1} - H_{m-l})\right) B_l(\langle x \rangle) + \frac{2}{m} H_{m-1} B_m(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(\langle x \rangle) B_{m-k}(\langle x \rangle), \text{ for } x \notin \mathbb{Z},$$

$$\frac{1}{m}\left(\tilde{\Lambda}_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)}\right) + \frac{1}{m}\sum_{l=2}^{m}\binom{m}{l}\left(\tilde{\Lambda}_{m-l+1} + \frac{1}{m-l+1}\mathbb{B}_{m-l}^{(r)}(H_{m-1} - H_{m-l})\right) B_l(\langle x \rangle) + \frac{2}{m} H_{m-1} B_m(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)} B_{m-k} + \frac{1}{2}\tilde{\Lambda}_m, \text{ for } x \in \mathbb{Z}.$$

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