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Application of fixed point theory for approximating of a positive-additive functional equation in intuitionistic random C^*-algebras

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Abstract

We apply a fixed point theorem for approximating of a positive-additive functional equation in intuitionistic random C^*-algebras.

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1. Introduction and preliminaries

The concept of distribution function and survival functions was introduced by Abdu et al. [1] and Saadati et al. [18]. Using these functions the authors defined intuitionistic random C^*-algebras and gave some properties and example of these spaces also for more results on stability please see [4, 5, 9, 10, 12–15, 17, 19, 20].

A metric d on non empty set Ω with rage $[0, \infty]$ is called a *generalized metric*.

Theorem 1.1 ([6]). *Assume that $J : \Omega \to \Omega$ be a contractive mapping with Lipschitz constant $L < 1$ on generalized metric space (Ω, d). Then for $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geqslant n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all $y \in \Gamma$.*

Definition 1.2 ([7]). Let $(A, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$ be an intuitionistic random Banach algebra C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

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Note that A^+ is a closed convex cone (see [7]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [8]).

In this paper, we introduce the following functional equation

$$T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) = \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m, \tag{1.1}$$

for all $x, y \in A^+$ and a fixed integer m greater than 1, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*, in which the function $f(x) = cx, \quad c \geqslant 0$, in the set of non-negative real numbers is a solution of the functional equation (1.1).

Throughout this paper, let A^+ and B^+ be the sets of positive elements in intuitionistic random C^*-algebras (A, N) and (B, N), respectively. Assume that m is a fixed integer greater than 1.

2. Stability of the positive-additive functional equation (1.1): fixed point approach

Lemma 2.1 ([16])**.** *Let* $T : A^+ \to B^+$ *be a positive-additive mapping satisfying* (1.1)*. Then* T *satisfies*

$$T(2^{mn}x) = 2^{mn}T(x),$$

for all $x \in A^+$ *and all* $n \in \mathbb{Z}$*.*

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1.1) in intuitionistic random C^*-algebras.

Note that the fundamental ideas in the proofs of the main results in this section are contained in [2, 3].

Theorem 2.2. *Let* $\varphi : A^+ \times A^+ \times (0, \infty) \to L^*$ *be a function such that there exists an* $E < 1$ *with*

$$\varphi(x, y, t) \geqslant_L \varphi\left(2^m x, 2^m y, \frac{2^m t}{E}\right), \tag{2.1}$$

for all $x, y \in A^+$ *and* $t > 0$*. Let* $f : A^+ \to B^+$ *be a mapping satisfying*

$$\mathcal{P}_{\mu,\nu}\left(f\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\right)^m, t\right) \geqslant_L \varphi(x, y, t), \tag{2.2}$$

for all $x, y \in A^+$ *and* $t > 0$*. Then there exists a unique positive-additive mapping* $T : A^+ \to A^+$ *satisfying* (1.1) *and*

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geqslant_L \varphi\left(x, x, \frac{(2^m - 2^m L)t}{E}\right), \tag{2.3}$$

for all $x \in A^+$ *and* $t > 0$*.*

Proof. Letting $y = x$ in (2.2), we get

$$\mathcal{P}_{\mu,\nu}(f(2^m x) - 2^m f(x), t) \geqslant_L \varphi(x, x, t), \tag{2.4}$$

for all $x \in A^+$ and $t > 0$.

Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \mathcal{P}_{\mu,\nu}(g(x) - h(x), t) \geqslant_L \varphi\left(x, x, \frac{t}{\mu}\right), \ \ \forall x \in A^+, \quad t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [11]).

Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2^m g\left(\frac{x}{2^m}\right),$$

for all $x \in A^+$.

Let $g, h \in X$ be given such that $d(g, h) = \varepsilon$. Then

$$\mathcal{P}_{\mu,\nu}(g(x) - h(x), t) \geqslant_L \varphi(x, x, t),$$

for all $x \in A^+$ and $t > 0$. Hence

$$\mathcal{P}_{\mu,\nu}(Jg(x) - Jh(x), t) = \mathcal{P}_{\mu,\nu}(2^m g\left(\frac{x}{2^m}\right) - 2^m h\left(\frac{x}{2^m}\right), t) \geqslant_L \varphi\left(x, x, \frac{t}{E}\right),$$

for all $x \in A^+$ and $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leqslant E\varepsilon$. This means that

$$d(Jg, Jh) \leqslant Ed(g, h),$$

for all $g, h \in X$.

It follows from (2.4) that

$$\mathcal{P}_{\mu,\nu}(f(x) - 2^m f\left(\frac{x}{2^m}\right), t) \geqslant_L \varphi\left(x, x, \frac{2^m t}{E}\right),$$

for all $x \in A^+$ and $t > 0$. So $d(f, Jf) \leqslant \frac{L}{2^m}$.

By Theorem 1.1, there exists a mapping $T : A^+ \to B^+$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T\left(\frac{x}{2^m}\right) = \frac{1}{2^m} T(x), \tag{2.5}$$

for all $x \in A^+$. The mapping T is a unique fixed point of J in the set

$$M = \{g \in X : d(f, g) < \infty\}.$$

This implies that T is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geqslant_L \varphi\left(x, x, \frac{t}{\mu}\right),$$

for all $x \in A^+$ and $t > 0$;

(2) $d(J^n f, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^{mn}}\right) = T(x),$$

for all $x \in A^+$;

(3) $d(f, T) \leqslant \frac{1}{1-E} d(f, Jf)$, which implies the inequality

$$d(f, T) \leqslant \frac{E}{2^m - 2^m E}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2),

$$\begin{aligned}\mathcal{P}_{\mu,\nu}&\left(f\left(\frac{\left(x^{\frac{1}{m}}+y^{\frac{1}{m}}\right)^m}{2^{mn}}\right)-\left(\left(2^{mn}f\left(\frac{x}{2^{mn}}\right)\right)^{\frac{1}{m}}+\left(2^{mn}f\left(\frac{y}{2^{mn}}\right)\right)^{\frac{1}{m}}\right)^m,\frac{t}{2^{mn}}\right)\\&\geqslant_L \varphi\left(\frac{x}{2^{mn}},\frac{y}{2^{mn}},\frac{t}{2^{mn}}\right)\\&\geqslant_L \varphi\left(x,y,\frac{t}{L^{mn}}\right),\end{aligned}$$

for all $x, y \in A^+$, all $n \in \mathbb{N}$ and $t > 0$. So

$$\mathcal{P}_{\mu,\nu}\left(T\left(\left(x^{\frac{1}{m}}+y^{\frac{1}{m}}\right)^m\right)-\left(T(x)^{\frac{1}{m}}+T(y)^{\frac{1}{m}}\right)^m,t\right)=1_{\mathcal{L}},$$

for all $x, y \in A^+$ and $t > 0$. Thus the mapping $T : A^+ \to B^+$ is positive-additive, as desired. □

Corollary 2.3. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned}\mathcal{P}_{\mu,\nu}&\left(f\left(\left(x^{\frac{1}{m}}+y^{\frac{1}{m}}\right)^m\right)-\left(f(x)^{\frac{1}{m}}+f(y)^{\frac{1}{m}}\right)^m,t\right)\\&\geqslant_L\left(\frac{t}{t+\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}},\frac{\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}}{t+\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}}\right),\end{aligned}\tag{2.6}$$

for all $x, y \in A^+$ and $t > 0$. Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x)-T(x),t)\geqslant_L\left(\frac{t}{t+\frac{2\theta_1+\theta_2}{2^{mp}-2^m}\|x\|^p},\frac{\frac{2\theta_1+\theta_2}{2^{mp}-2^m}\|x\|^p}{t+\frac{2\theta_1+\theta_2}{2^{mp}-2^m}\|x\|^p}\right),$$

for all $x \in A^+$ and $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y,t)=\left(\frac{t}{t+\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}},\frac{\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}}{t+\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}}\right),$$

for all $x, y \in A^+$ and $t > 0$. Then we can choose $E = 2^{m-mp}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : A^+ \times A^+ \times (0,\infty)] \to L^*$ be a function such that there exists an $E < 1$ with*

$$\varphi(x,y,t)\geqslant_L \varphi\left(\frac{x}{2^m},\frac{y}{2^m},\frac{t}{2^mE}\right),$$

for all $x, y \in A^+$ and $t > 0$. Let $f : A^+ \to B^+$ be a mapping satisfying (2.2). Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x)-T(x),t)\geqslant_L \varphi(x,x,(2^m-2^mE)t),$$

for all $x \in A^+$ and $t > 0$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2^m}g\left(2^mx\right),$$

for all $x \in A^+$.

It follows from (2.4) that

$$\mathcal{P}_{\mu,\nu}\left(f(x) - \frac{1}{2^m} f(2^m x), t\right) \geqslant_L \varphi(x, x, 2^m t),$$

for all $x \in A^+$ and $t > 0$. So $d(f, Jf) \leqslant \frac{1}{2^m}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying (2.6). Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) and*

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geqslant_L \left(\frac{t}{t + \frac{2\theta_1+\theta_2}{2^m - 2^{mp}}} \|x\|^p, \frac{\frac{2\theta_1+\theta_2}{2^m - 2^{mp}}}{t + \frac{2\theta_1+\theta_2}{2^m - 2^{mp}}} \|x\|^p\right),$$

for all $x \in A^+$ and $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y, t) = \left(\frac{t}{t + \frac{2\theta_1+\theta_2}{2^m - 2^{mp}}} \|x\|^p, \frac{\frac{2\theta_1+\theta_2}{2^m - 2^{mp}}}{t + \frac{2\theta_1+\theta_2}{2^m - 2^{mp}}} \|x\|^p\right),$$

for all $x, y \in A^+$ and $t > 0$. Then we can choose $E = 2^{mp-m}$ and we get the desired result. □

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