



## Approximate solutions of fuzzy differential equations of fractional order using modified reproducing kernel Hilbert space method

Asia Khalaf Albzeirat<sup>a,\*</sup>, Muhammad Zaini Ahmad<sup>a</sup>, Shaher Momani<sup>b,c</sup>, Norazrizal Aswad Abdul Rahman<sup>a</sup>

<sup>a</sup>*Institute of Engineering Mathematics, Universiti Malaysia Perlis, Pauh Putra Main Campus, 02600 Arau, Perlis, Malaysia.*

<sup>b</sup>*Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan.*

<sup>c</sup>*Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.*

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### Abstract

In this paper, we use the modified reproducing kernel Hilbert space method to approximate the solution of fuzzy differential equations of fractional order. Using this method, we construct a new algorithm to approximate the solution of such differential equations. The proposed algorithm produces solutions in terms of interval-valued fuzzy numbers. Two numerical examples are tested and the results showed that the proposed algorithm is able to produce solutions that approach to the exact solutions. It concludes that the proposed algorithm can be considered as a modern algorithm that complements to the existing ones. ©2017 All rights reserved.

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### 1. Introduction

For the last decade, fuzzy differential equations (FDEs) have been studied by many researchers in different approaches [3, 12, 29, 31, 35]. All of these studies aimed to find the solutions of fuzzy differential equations of fractional order. This involves the process of introducing new methods as well as improving the classical methods to solve such differential equations. It is a fact that the improvement of problem solving tools effects directly how we handle the world around us. For instance, in [9] the authors proposed a new algorithm to find multiple solutions of the fractional differential equations. Thus, a further research has to be made to construct new methods, or by improving the existing classical methods direct or indirectly [33, 34].

This paper considers the general form of fuzzy differential equation of fractional order possessing Caputo fractional derivative:

$$\begin{aligned} D_{c,a,z}^{\alpha} M(z) &= g(z, M(z)), \\ M(z_0) &= z_F = (A, B, C), \end{aligned} \tag{1.1}$$

\*Corresponding author

Email address: [asiaajor@yahoo.com](mailto:asiaajor@yahoo.com) (Asia Khalaf Albzeirat)

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where  $D_{c,a,z}^\alpha M(z)$  is the fractional derivative in the sense of Caputo,  $0 < \alpha \leq 1$ ,  $z > z_0$  and  $M(z_0) = (A, B, C)$  is a fuzzy initial value.

At instance, fuzzy differential equations of fractional order might be very complicated to be solved. Unlike ordinary differential equations, there are a few methods that have been proposed for solving fuzzy differential equations. Many of them are extension from the classical methods (see in [5, 17, 19, 24, 38]). There are several methods proposed by the authors to solve Eq. (1.1) (see in [4, 6–8, 26]).

From the previous studies, it was shown that the reproducing kernel theory (RKT) is popular in solving ordinary differential equations. The RKT was first developed in 1908 by Zaremba [14]. Since then, many modifications have been made. For example, Higgins [22] provided the crucial modification for this method. Furthermore, Mercer [30] improved this theory and demonstrated it on several real life applications.

In the last decades, the RKT has been used by many researchers in solving problems in signal processing, estimation theory, stochastic process, and hybrid wavelet (see in [10, 21, 23, 36]). Currently, some researchers expanded the application of RKT for finding the solution of fractional problems [1, 2, 11, 28, 37]. From these studies, it is necessary to focus on the application of this method in solving fuzzy differential equation of fractional order. Therefore, in this study we propose a new procedure on solving fuzzy differential equation of fractional order based on RKT.

The rest of this paper is organized as follows. In Section 2, we provide basic concepts of fractional calculus, fuzzy set and the RKT. In Section 3, we propose a new algorithm for approximating the solution of fuzzy differential equation of fractional order. While in Section 4, we present two numerical examples to show the effectiveness of the proposed algorithm. Finally, conclusion will be drawn in the last section.

## 2. Basic concepts

This section briefly elaborates the basic concepts of fractional calculus, fuzzy differential equations of fractional order, and reproducing kernel Hilbert space.

**Definition 2.1** ([20]). The left fractional integral and the right fractional integral in the sense of Riemann-Liouville for  $\alpha \in (0, 1)$  are defined as

$$J_{RL,a,z}^\alpha M(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z - \zeta)^{\alpha-1} M(\zeta) d\zeta,$$

and

$$J_{RL,z,b}^\alpha M(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (\zeta - z)^{\alpha-1} M(\zeta) d\zeta,$$

respectively.

**Definition 2.2** ([32]). The left Caputo fractional derivative is defined as follows.

$$D_{c,a,z}^\alpha M(z) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_a^z (z - \zeta)^{[\alpha] - \alpha - 1} M^{[\alpha]}(\zeta) d\zeta,$$

where  $M^{[\alpha]} = \frac{dM^{[\alpha]}(\zeta)}{d\zeta^{[\alpha]}}$  and  $[\alpha] \leq \alpha < [\alpha]$ ,  $\alpha \in \mathbb{Z}^+$ , where the symbol  $[\cdot]$  denotes the nearest integer number greater than  $\alpha$  and  $\lfloor \cdot \rfloor$  denotes the nearest integer number less than  $\alpha$ . While the right Caputo fractional derivative is defined as

$$D_{c,b,z}^\alpha M(z) = \frac{(-1)^{[\alpha]}}{\Gamma([\alpha] - \alpha)} \int_z^b (\zeta - z)^{[\alpha] - \alpha - 1} M^{[\alpha]}(\zeta) d\zeta.$$

The properties of the Caputo fractional derivative [27] is listed below.

- (i)  $D_{c,a,z}^\alpha J_{RL,a,z}^\alpha M(z) = M(z)$ ;

$$(ii) J_{RL,a,z}^\alpha D_{c,a,z}^\alpha M(z) = M(z) - \sum_{k=0}^{\lfloor \alpha \rfloor} \left( M^{(k)}(0) \cdot \frac{z^k}{k!} \right), z > 0, \lfloor \alpha \rfloor \leq \alpha < \lceil \alpha \rceil \in Z^+.$$

We denote the set of all real numbers by  $\mathbb{R}$  and the space of  $n$ -dimensional fuzzy numbers by  $\mathbb{R}_F^n$  where  $u_F(z) : \mathbb{R}^n \rightarrow [0, 1]$ .

**Definition 2.3.** Let a fuzzy number  $u_F(z) \in \mathbb{R}_F$  and  $r \in [0, 1]$ . The  $r$ -cut of  $u_F(z)$  is the crisp set that contains all elements with degree in  $u_F(z)$  either greater than or equal to  $r$  such that

$$[u_F(z)]^r = \{z \in \mathbb{R} : u_F(z) \geq r\}.$$

For fuzzy number  $u_F(z)$ , its  $r$ -cut are closed and bounded interval in  $\mathbb{R}$  and we denoted them by

$$[u_F(z)]^r = [u_{1,1r}(z), u_{1,2r}(z)],$$

where  $u_{1,1r}(z) = \min\{z \mid z \in [u_F(z)]^r\}$  and  $u_{1,2r}(z) = \max\{z \mid z \in [u_F(z)]^r\}$  for every  $r \in [0, 1]$ .

For more details on  $r$ -cut of fuzzy numbers, please see in [13, 15, 16, 25, 39]. Fuzzy numbers are often classified into membership function. One of the most commonly used membership functions is the triangular membership function, or often referred as triangular fuzzy number.

**Definition 2.4** ([4]). A fuzzy number  $u_F$  is called triangular fuzzy number if its membership function has the following form:

$$u_F(z) = \begin{cases} 0, & \text{if } z < A, \\ \frac{z-A}{B-A}, & \text{if } A \leq z < B, \\ \frac{C-z}{C-B}, & \text{if } B \leq z \leq C, \\ 0, & \text{if } z > C, \end{cases}$$

and its  $r$ -cut is computed as follows:

$$[u]^r = [A + r(B - A), C - r(C - B)], \quad r \in [0, 1].$$

**Lemma 2.5.** Let  $M(z)$  be a fuzzy-valued function such that  $M(z) : [a, b] \rightarrow \mathbb{R}_F$  where  $M(z)$  is a bounded and continuous function in open interval  $(a, b)$ . The fractional derivative of  $M_F(z)$  in the sense of Caputo can be expressed using  $r$ -cut representation as follows:

$$[D_{c,a,z}^\alpha M(z)]^r = [D_{c,a,z}^\alpha M(z)]^r = [D_{c,a,z}^\alpha M_{1,1r}(z), D_{c,a,z}^\alpha M_{1,2r}(z)],$$

where  $D_c^\alpha$  is the operation that its effective representation is limited to  $M_F(z)$ .

By using Definition 2.3 and the properties of Caputo fractional derivative, we have

$$\begin{aligned} [J_{RL,a,z}^\alpha D_{c,a,z}^\alpha M(z)]^r &= [J_{RL,a,z}^\alpha D_{c,a,z}^\alpha M_{1,1r}(z), J_{RL,a,z}^\alpha D_{c,a,z}^\alpha M_{1,2r}(z)] \\ &= \left[ M_{1,1r}(z) - \sum_{k=0}^{m-1} M_{1,1r}^k(0^+) \frac{z^k}{k!}, M_{1,2r}(z) - \sum_{k=0}^{m-1} M_{1,2r}^k(0^+) \frac{z^k}{k!} \right]. \end{aligned}$$

**Definition 2.6** ([18]). Let HS be a Hilbert space. Suppose that  $U_y(z) \in HS$  and  $M(y) \in HS$  satisfy the following condition:

$$\langle M(z), U_y(z) \rangle = M(y), \forall y \in Z,$$

then,

- (i)  $U_y(z)$  is a reproducing kernel of HS;
- (ii) HS is a reproducing kernel space.

**Definition 2.7** ([18]). The function space  $FS_2^m[a, b]$  is defined as

$$FS_2^m[a, b] = \left\{ u \mid u^{(i)} \text{ is absolutely continuous, } i = 1, \dots, m-1, u^{(m)} \in L^2[a, b] \right\}.$$

The inner product in  $FS_2^m[a, b]$ , for any functions  $u(z), v(z) \in FS_2^m[a, b]$  is defined as follows:

$$\langle u, v \rangle_{FS_2^m[a, b]} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}v^{(m)} dz.$$

While the norm in  $FS_2^m[a, b]$  for any functions  $u(z), v(z) \in FS_2^m[a, b]$  is defined as follows:

$$\|u\|_{FS_2^m[a, b]} = \sqrt{\langle u, u \rangle_{FS_2^m[a, b]}}.$$

**Definition 2.8** ([18]). The inner product space of the function space  $FS_2^1[a, b]$  is defined as

$$\langle u, v \rangle_{FS_2^1[a, b]} = u(0)v(0) + \int_a^b u^{(1)}(z)v^{(1)}(z) dz, \forall u, v \in FS_2^1[a, b],$$

and the norm of  $FS_2^1[a, b]$  is defined as

$$\|u\|_{FS_2^1[a, b]} = \sqrt{\langle u(z), u(z) \rangle_{FS_2^1[a, b]}}.$$

**Theorem 2.9** ([18]). The function space  $FS_2^1[a, b]$  is a reproducing kernel Hilbert space with reproducing kernel given by

$$U_z(y) = \begin{cases} 1 + y, & \text{if } y \leq z, \\ 1 + z, & \text{if } y > z. \end{cases}$$

**Definition 2.10** ([18]). The inner product space of the function space  $FS_2^2[a, b]$  is defined as follows:

$$\langle u, v \rangle_{FS_2^2[a, b]} = \sum_{i=0}^1 u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(2)}(z)v^{(2)}(z) dz, \quad \forall u, v \in FS_2^2[a, b],$$

and the norm of  $FS_2^2[a, b]$  is defined as:

$$\|u\|_{FS_2^2[a, b]} = \sqrt{\langle u(z), u(z) \rangle_{FS_2^2[a, b]}}.$$

**Theorem 2.11** ([18]). The function space  $FS_2^2[a, b]$  is a reproducing kernel Hilbert space with reproducing kernel given by

$$K_z(y) = \begin{cases} 1 + zy + \frac{zy^2}{2} - \frac{y^3}{6}, & \text{if } y \leq z, \\ 1 + zy + \frac{z^2y}{2} - \frac{z^3}{6}, & \text{if } y > z. \end{cases}$$

### 3. New algorithm for fuzzy differential equations of fractional order

In order to use reproducing kernel theory (RKT) as the tool, we need to homogenize the initial condition in Eq. (1.1). For that, we consider

$$\begin{aligned} M^*(z) &= M(z) - z_F, \quad M^*(z_0) = 0, \\ M(z) &= M^*(z) + z_F. \end{aligned} \tag{3.1}$$

After the homogenization, Eq. (1.1) can be converted to the following form:

$$\begin{cases} (LM^*)(z) = g^*(z, M^*(z)), \\ M^*(z_0) = 0. \end{cases} \tag{3.2}$$

By applying the fuzzy set theory, Eq. (3.1) is converted into the following form:

$$\begin{aligned} D_{c,\alpha,z}^\beta M_{1,1r}^*(z) &= g_{1,1r}^*(z, M_{1,1r}^*(z), M_{1,2r}^*(z)), \\ D_{c,\alpha,z}^\beta M_{1,2r}^*(z) &= g_{1,2r}^*(z, M_{1,1r}^*(z), M_{1,2r}^*(z)), \\ M_{1,1r}^*(z) &= 0, M_{1,2r}^*(z) = 0, \end{aligned} \tag{3.3}$$

and Eq. (3.2) is converted into the following form:

$$\begin{cases} (L_{1,jr}M_{1,jr}^*)(z) = g_{1,jr}^*(z, M_{1,jr}^*(z)), \\ M_{1,jr}^*(z_0) = 0, \quad j = 1, 2. \end{cases} \tag{3.4}$$

From [18], the reproducing kernel method is employed for solving Eq. (3.4). This includes three main steps that are also used to solve ordinary differential equations. From those steps, we propose the following procedures.

- i) First, we construct reproducing kernels  $FS_2^2[a, b]$  and  $FS_2^1[a, b]$  for Eq. (3.4), after that we introduce a linear operator as follows:

$$L_{m,jr} = FS_2^2[a, b] \rightarrow FS_2^1[a, b], \quad m = 1, \quad j = 1, 2.$$

Therefore, Eq. (3.3) is converted into the following form.

$$\begin{aligned} M_{1,1r}^*(z) &= L_{1,1r}^{ad} g_{1,1r}^*(z, M_{1,1r}^*(z), M_{1,2r}^*(z)), \\ M_{1,2r}^*(z) &= L_{1,2r}^{ad} g_{1,2r}^*(z, M_{1,1r}^*(z), M_{1,2r}^*(z)), \end{aligned}$$

where  $L_{1,1r}^{ad}$  and  $L_{1,2r}^{ad}$  are adjoint operator of  $L_{m,jr}$ ,  $m = 1$  and  $j = 1, 2$ .

- ii) Then, we let  $\{z_k\}_{k=1}^\infty$  to be a countable dense set in  $[a, b]$  and let  $e_{k,m,jr}(z) = k_{z_k}(y)$ .  $K_z(y)$  is the reproducing kernel of  $FS_2^1[a, b]$ ,  $\forall M_{m,jr}^*(z) \in FS_2^1[a, b]$ ,  $m = 1$  and  $j = 1, 2$ . Hence,

$$\langle M_{m,jr}^*(z), e_{k,m,jr}(z) \rangle_{FS_2^1} = \langle M_{m,jr}^*(z), k_z(y) \rangle_{FS_2^1} = M_{m,jr}^*(z).$$

Additionally, by allowing

$$L_{m,jr}^{ad} e_{k,m,jr}(z) = \Psi_{k,m,jr}(z),$$

we have

$$\langle M_{m,jr}^*(z), \Psi_{k,m,jr}(z) \rangle_{FS_2^2} = \langle M_{m,jr}^*(z), L_{1,1r}^{ad} e_{k,m,jr}(z) \rangle_{FS_2^2} = \langle L_{m,jr} M_{m,jr}^*(z), e_{k,m,jr}(z) \rangle_{FS_2^1}.$$

Thus,  $\Psi_{k,m,jr}(z)$  can be expressed as follows:

$$\begin{aligned} \Psi_{k,m,jr}(z) &= L_{m,jr}^{ad} e_{k,m,jr}(z) = \langle L_{m,jr}^{ad} e_{k,m,jr}(z), K_z(y) \rangle_{FS_2^2[a,b]} \\ &= \langle e_{k,m,jr}(z), L_{m,jr} K_z(y) \rangle_{FS_2^1[a,b]} = L_y K_z(y)|_{y=z_k}. \end{aligned}$$

From here, we obtain the following result.

**Theorem 3.1.** *If  $\{z_k\}_{k=1}^\infty$  is a dense set in  $[a, b]$ , then the complete system of the function space  $FS_2^2[a, b]$  under fuzzy setting is as follows:*

$$\{\Psi_{k,m,jr}(z)\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}.$$

*Proof.* For each fixed  $M_{m,jr}^*(z) \in FS_2^2[a, b]$ , and let  $\langle M_{m,jr}^*(z), \Psi_{k,m,jr}(z) \rangle = 0, k = 1, \dots$ , then

$$\begin{aligned} \langle M_{m,jr}^*(z), \Psi_{k,m,jr}(z) \rangle_{FS_2^2[a,b]} &= \langle M_{m,jr}^*(z), L_{1,1r}^{ad} e_{k,m,jr}(z) \rangle_{FS_2^2[a,b]} \\ &= \langle L_{m,jr} M_{m,jr}^*(z), e_{k,m,jr}(z) \rangle_{FS_2^2[a,b]} \\ &= L_{m,jr} M_{m,jr}^*(z_k) = 0. \end{aligned}$$

Since  $\{z_k\}_{k=1}^\infty$  is dense in  $[a, b]$ , therefore,  $L_{m,jr} M_{m,jr}^*(z_k) = 0, \forall m = 1$  and  $j = 1, 2$ . It follows that  $M_{m,jr}^*(z) = 0$  from the existence of the inverse operator of  $L_{m,jr}$  and the continuity of  $M_{m,jr}^*(z)$ .  $\square$

iii) For the third step, we construct an orthonormal system  $\{\bar{\Psi}_{k,m,jr}\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$  of  $FS_2^2[a, b]$  by using Gram-Schmidt orthogonalization process.

$$\bar{\Psi}_{k,m,jr}(z) = \sum_{l=1}^k \beta_{kl,m,jr}(z) \Psi_{k,m,jr}(z), \forall m = 1, j = 1, 2, l = 1, 2, \dots, k,$$

where  $\beta_{kl,m,jr}(z)$  is the orthogonalization coefficient.

From the third step we obtain the following results.

**Theorem 3.2.** *Let  $\{z_k\}_{k=1}^\infty$  be a dense set in  $[a, b]$ , and the solution of (3.3) is unique on  $FS_2^1[a, b]$ . Then, the solution of (3.3) can be written as follows.*

$$M_{m,jr}^*(z) = \sum_{k=1}^\infty \sum_{l=1}^k \beta_{kl,m,jr} g_{m,jr}^*(z_l, M_{m,jr}^*(z_l) \bar{\Psi}_{k,m,jr}(z)), \forall m = 1, j = 1, 2.$$

*Proof.* By applying Theorem 3.1,  $\{\bar{\Psi}_{k,m,jr}\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$  is a complete orthonormal basis for  $FS_2^2[a, b]$ . So,  $M_{m,jr}^*(z)$  can be extended into Fourier series for orthonormal system  $\{\bar{\Psi}_{k,m,jr}\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$  as follows:

$$M_{m,jr}^*(z) = \sum_{k=1}^\infty \langle M_{m,jr}^*(z), \bar{\Psi}_{k,m,jr} \rangle \Psi_{k,m,jr}(z), \forall m = 1, j = 1, 2,$$

where  $FS_2^2[a, b]$  is a reproducing kernel Hilbert space. Therefore,

$$M_{m,jr}^*(z) = \sum_{k=1}^\infty \langle M_{m,jr}^*(z), \bar{\Psi}_{k,m,jr} \rangle \Psi_{k,m,jr}(z), \forall m = 1, j = 1, 2,$$

converges in the sense of the norm in Definition 2.10. By Definition 2.6, we have

$$\begin{aligned} M_{m,jr}^*(z) &= \sum_{k=1}^\infty \langle M_{m,jr}^*(z), \bar{\Psi}_{k,m,jr} \rangle \bar{\Psi}_{k,m,jr} \\ &= \sum_{k=1}^\infty \langle M_{m,jr}^*(z), \sum_{l=1}^k \beta_{kl,m,jr} \Psi_{k,m,jr}(z) \rangle_{FS_2^2[a,b]} \bar{\Psi}_{k,m,jr} \\ &= \sum_{k=1}^\infty \sum_{l=1}^k \beta_{kl,m,jr}(z) \langle M_{m,jr}^*(z), \Psi_{k,m,jr}(z) \rangle_{FS_2^2[a,b]} \bar{\Psi}_{k,m,jr} \\ &= \sum_{k=1}^\infty \sum_{l=1}^k \beta_{kl,m,jr}(z) \langle M_{m,jr}^*(z), L_{m,jr}^{ad} e_{l,m,jr}(z) \rangle_{FS_2^2[a,b]} \bar{\Psi}_{k,m,jr} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \sum_{l=1}^k \beta_{kl,m,jr}(z) \langle L_{m,jr} M_{m,jr}^*(z), e_{l,m,jr}(z) \rangle_{FS_2^1[a,b]} \bar{\Psi}_{k,m,jr} \\
 &= \sum_{k=1}^{\infty} \sum_{l=1}^k \beta_{kl,m,jr}(z) \langle g_{m,jr}^*(z, M_{m,jr}^*(z)), e_{l,m,jr}(z) \rangle_{FS_2^1[a,b]} \bar{\Psi}_{k,m,jr} \\
 &= \sum_{k=1}^{\infty} \sum_{l=1}^k \beta_{kl,m,jr}(z) g_{m,jr}^*(z_l, M_{m,jr}^*(z_l)) \bar{\Psi}_{k,m,jr}, \quad \forall m = 1, j = 1, 2.
 \end{aligned}$$

□

Based on Theorem 3.2, the approximate solution of  $M_{m,jr}(z)$  for N-terms is

$$M_{m,jr}^{*N}(z) = \sum_{k=1}^N \sum_{l=1}^k \beta_{kl,m,jr}(z) g_{m,jr}^*(z_l, M_{m,jr}^*(z_l)) \bar{\Psi}_{k,m,jr}, \quad \forall m = 1, j = 1, 2.$$

Now, by using Eq. (3.1), the new form of  $M_{m,jr}^N(z)$  is as

$$M_{m,jr}^N(z) = \left( \sum_{k=1}^N \sum_{l=1}^k \beta_{kl,m,jr}(z) g_{m,jr}(z_l, M_{m,jr}(z_l)) \bar{\Psi}_{k,m,jr} \right) + z_F, \quad \forall m = 1, j = 1, 2.$$

The third procedure is simplified in the following algorithm.

**Algorithm 3.3.** Procedure for approximating the solution of Eq. (3.1).

**Input:** In this part, we consider the interval  $[a, b] = [0, 1]$ , the integer  $N$ , the kernel  $k_z(y)$  and the differential operator  $L_{y,1,jr}$  from the definition for left Caputo fractional derivative. We also consider the function  $g_{1,jr}^*(z, M_{1,1r}^*(z)) = g_{1,jr}(z, M_{1,1r}^*(z) + z_{F,jr_d})$ , the integer  $m$ , with fuzzy initial condition  $z_F = (a, b, c)$  and the value of  $\alpha$ .

First, we approximate the solution of  $M_{1,jr}(z)$ .

**Step 1 :** Let  $k \in [0, 1]$  and  $y \in [0, 1]$ . Then,

$$K_z(y) = \begin{cases} (1 + zy + \frac{zy^2}{2} - \frac{y^3}{6}), & z \leq y, \\ (1 + zy + \frac{yz^2}{2} - \frac{z^3}{6}), & \text{otherwise.} \end{cases}$$

**Step 2 :** For the values  $j = 1, 2$ , follow Steps 3-6.

**Step 3 :** For  $k = 1, 2, \dots, N$ ,  $j = 1, 2$ ,  $d = 1, 2, \dots, m$ , set the values of  $z$ ,  $r$  and  $\psi_{k,1,jr_d}(z)$  as follows:

1. Set  $z_0 = 0$ .
2. Set  $z_k = z_{k-1} + \frac{1}{N}$ .
3. Set  $r_0 = 0$ .
4. Set  $r_d = r_{d-1} + \frac{d}{m}$ ,
5.
  - if  $j = 1$ , set  $z_{F,jr_d} = a + r_d(b - a)$ ,
  - if  $j = 2$ , set  $z_{F,jr_d} = c - r_d(c - b)$ .
6. Set  $\psi_{k,1,jr_d}(z) = L_{y,1,jr} K_z(y) |_{y=z_k}$ .

So, we get the following output

$$\{\psi_{k,1,jr_d}(z)\}_{(k,1,jr_d)}^{(N,1,2r_d)}.$$

**Step 4 :** For  $k = 2, \dots, N$ ,  $s = 1, 2, \dots, k$ , set the values of  $\beta$ ,  $R$  and  $d$  as follows.

1. Set  $\beta_{11} = \frac{1}{\|\psi_{1,1,jr_d}\|}$ .

2. Set  $\beta_{kk} = \frac{1}{\|\mathbb{R}_{k,1,jr_d}\|}$ .
3. Set  $\beta_{kp,1,jr_d} = \frac{-1}{\|\mathbb{R}_{k,1,jr}\|} (\sum_{s=p}^{k-1} d_{ks,1,jr} \beta_{sp,1,jr_d})$ ,  $p < k$ .
4. Set  $\mathbb{R}_{k,1,jr_d} = \sqrt{\|\Psi_{1,1,jr}\|^2 - \sum_{s=p}^{k-1} (d_{ks,1,jr_d})^2}$ .
5. Then,  $d_{ks,1,jr} = \Psi_{k,1,jr} \overline{\Psi_{k,1,jr_d}}$ .

Then, we get the following output

$$\beta_{ks,1,jr_d}$$

**Step 5 :** For  $k = 1, 2, \dots, N$ , we have

$$\overline{\Psi_{k,1,jr_d}(z)} = \sum_{s=1}^k \beta_{ks} \Psi_{s,1,jr_d}(z).$$

**Step 6 :** For  $k = 1, 2, \dots, N$ , we set the values for  $M$  as follows.

1. Set  $M_{1,jr_d}^*(z_0) = 0$ .
2. Set  $M_{1,jr}^*(z) = \sum_{k=1}^N \sum_{s=1}^k \beta_{ks,1,jr}(z) g_{1,jr}(z, M_{1,1r}^*(z_s) + z_{F,jr_d}) \overline{\Psi_{k,1,jr}}$ .
3. Set  $M_{1,jr_d}(z) = M_{1,jr}^*(z) + (z_{F,jr_d})$ .

**Output:** Then, we get the following output.

$$M_{1,jr_d}(z) = [M_{1,1r_d}(z) + M_{1,2r_d}(z)].$$

#### 4. Numerical examples

In this section, we provide two numerical examples.

**Example 4.1.** Consider the following fuzzy differential equation of order  $\alpha = 1$ .

$$D_{c,a,z}^\alpha M(z) = 2z^2 + 3z - M(z), M(z_0) = (1, 2, 3), \quad 0 \leq z \leq 1.$$

By applying fuzzy set theory, we obtain

$$\begin{aligned} D_{c,a,z}^\alpha M_{1,1r}(z) &= 2z^2 + 3z - M_{1,1r}(z), \quad M_{1,1r}(z_0) = 1 + r, r \in [0, 1], \\ D_{c,a,z}^\alpha M_{1,1r}(z) &= 2z^2 + 3z - M_{1,2r}(z), \quad M_{1,2r}(z_0) = 3 - r, r \in [0, 1]. \end{aligned}$$

To solve the equation, we used the proposed algorithm in previous section and we let  $N = 127$  and we consider the interval  $[0, 1]$ . Then, the approximate results obtained are presented in Tables 1-5. The graphical representation of the solutions can be seen in Figures 1-10. In those figures, the results obtained using our method are also compared with the exact solutions within the same interval and having the same fuzzy initial value.

For  $r = 0$ , the exact solutions are

$$M_{1,1r}(z) = 1 - 2z + z^2, M_{1,2r}(z) = e^{-z}(2 + e^z - ze^z + 2e^z z^2).$$

For  $r = 0.5$ , the exact solutions are

$$M_{1,1r}(z) = 2e^{-z} (0.25 + 0.5e^z - 0.5ze^z + z^2 e^z), \quad M_{1,2r}(z) = 2e^{-z} (0.75 + 0.5e^z - 0.5ze^z + z^2 e^z).$$

For  $r = 1$ , the exact solutions are

$$M_{1,1r}(z) = M_{1,2r}(z) = e^{-z} (1 + ze^z - 2z^2 e^z).$$

Table 1: Numerical solution for Example 4.1.

$z$	$M_{1,1r}(z)$ $r = 0$	$\hat{M}_{1,1r}(z)$ $r = 0$	Errors
0.0	1.00	1.00	0.00
0.1	0.92000	0.920001122	$1.122011299 \times 10^{-6}$
0.2	0.88000	0.8800022993	$2.299334099 \times 10^{-6}$
0.3	0.88000	0.8800034787	$3.478728604 \times 10^{-6}$
0.4	0.92000	0.9200046429	$4.642932368 \times 10^{-6}$
0.5	1.00000	1.000005923	$4.642932368 \times 10^{-6}$
0.6	1.12000	1.120007268	$7.26794065 \times 10^{-6}$
0.7	1.28000	1.280008596	$8.595896458 \times 10^{-6}$
0.8	1.48000	1.480010076	$1.007556054 \times 10^{-5}$
0.9	1.72000	1.720011705	$1.007556054 \times 10^{-5}$
1.0	2.00000	2.000013357	$1.335720074 \times 10^{-5}$

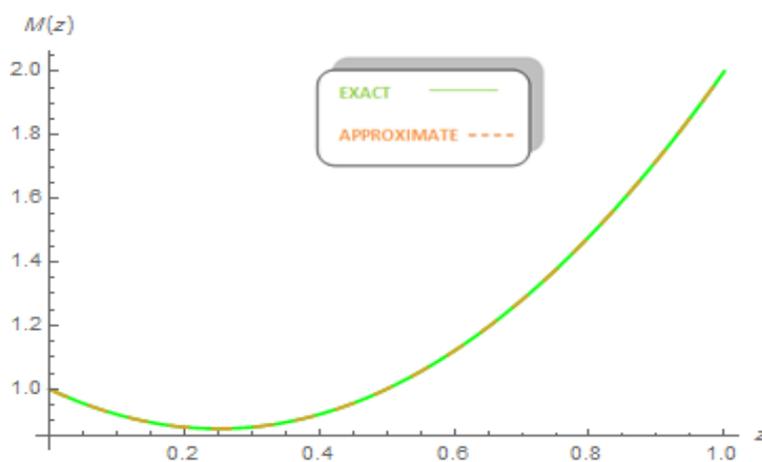


Figure 1: The exact and approximate solutions of  $M_{1,1r}(z)$ , when  $r = 0$  for Example 4.1.

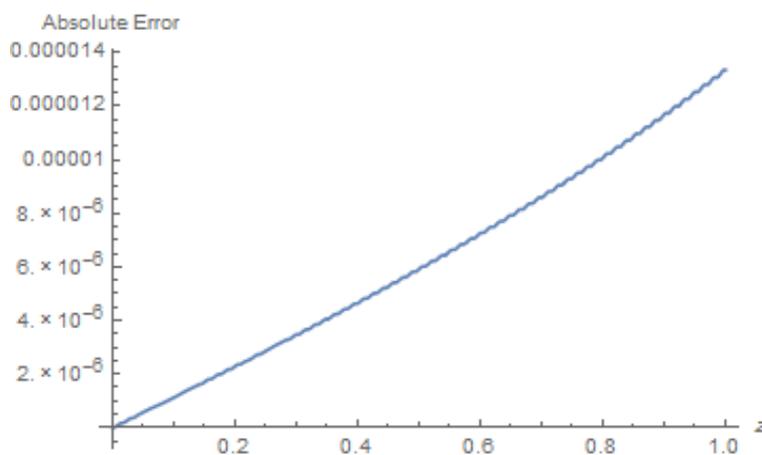


Figure 2: The errors in solutions for  $M_{1,1r}(z)$ , when  $r = 0$  for Example 4.1.

Table 2: Numerical solutions of  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.1.

$z$	$M_{1,2r}(z)$ $r = 0$	$\hat{M}_{1,2r}(z)$ $r = 0$	Errors
0.0	3.00	3.00	0.00
0.1	2.729674836071919	2.729674122	$7.143396936 \times 10^{-7}$
0.2	2.5174615061559633	2.517460432	$1.073837352 \times 10^{-6}$
0.3	2.361636441363436	2.361635354	$1.087723213 \times 10^{-6}$
0.4	2.2606400920712786	2.260639284	$8.082239611 \times 10^{-6}$
0.5	2.213061319425267	2.213061076	$2.436512458 \times 10^{-6}$
0.6	2.2176232721880527	2.217623855	$5.824507046 \times 10^{-6}$
0.7	2.273170607582819	2.273172208	$1.600046236 \times 10^{-6}$
0.8	2.378657928234443	2.378660805	$2.876281287 \times 10^{-6}$
0.9	2.5331393194811986	2.533143726	$4.406986805 \times 10^{-6}$
1.0	2.7357588823428842	2.735764967	$6.084919464 \times 10^{-6}$

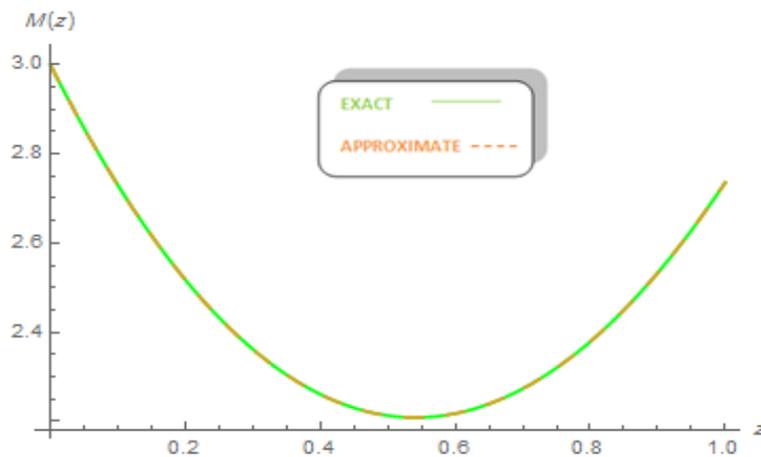


Figure 3: The exact and approximate solutions for  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.1.

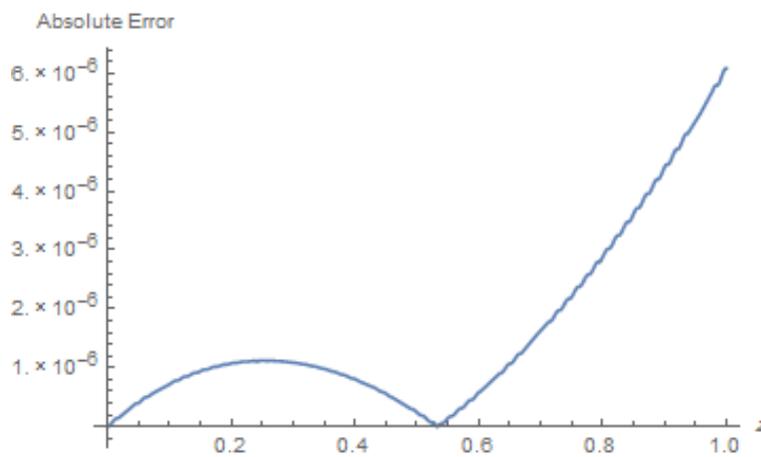


Figure 4: The errors in solutions for  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.1.

Table 3: Numerical solutions of  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.1.

$z$	$M_{1,1r}(z)$ $r = 0.5$	$\hat{M}_{1,1r}(z)$ $r = 0.5$	Errors
0.0	1.50	1.50	0.00
0.1	1.3724187090179798	1.372419372	$6.629235505 \times 10^{-7}$
0.2	1.2893653765389907	1.289366833	$1.456041236 \times 10^{-6}$
0.3	1.250409110340859	1.250411447	$2.337115649 \times 10^{-6}$
0.4	1.2551600230178197	1.255163303	$3.280143285 \times 10^{-6}$
0.5	1.3032653298563168	1.303269711	$4.381550751 \times 10^{-6}$
0.6	1.3944058180470131	1.394411415	$5.596568163 \times 10^{-6}$
0.7	1.7046644820586112	1.528299499	$6.846933903 \times 10^{-6}$
0.8	1.9232848298702998	1.704672758	$8.275740723 \times 10^{-6}$
0.9	1.9232848298702998	1.92329471	$9.880615046 \times 10^{-6}$
1.0	2.183939720585721	2.18395126	$1.153913042 \times 10^{-5}$

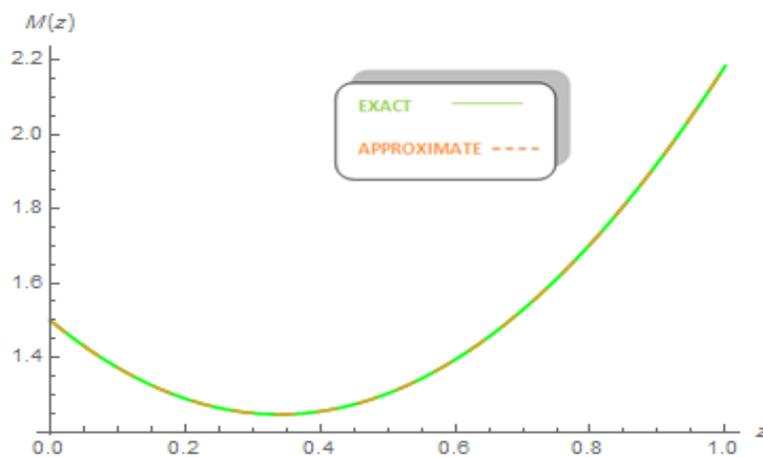


Figure 5: The exact and approximate of  $M_{1,1r}(z)$  when  $r = 0.5$  for Example 4.1.

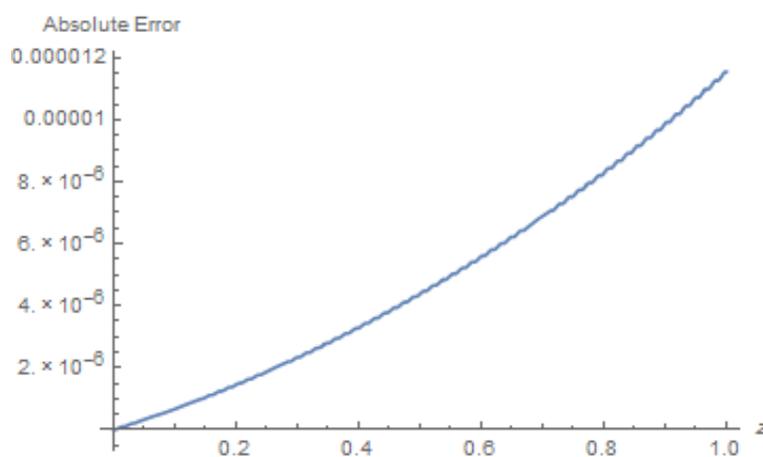


Figure 6: The errors in solutions for  $M_{1,1r}(z)$  when  $r = 0.5$  for Example 4.1.

Table 4: Numerical solutions of  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.1.

$z$	Exact solution $M_{1,2r}(z)$ $r = 0.5$	$\hat{M}_{1,2r}(z)$ $r = 0.5$	Errors
0.0	2.50	2.50	0.00
0.1	2.2772561270539393	2.277255872	$2.5525194585 \times 10^{-7}$
0.2	2.1080961296169725	2.108095899	$2.305444893 \times 10^{-6}$
0.3	1.991227331022577	1.991227385	$5.388974111 \times 10^{-6}$
0.4	1.925480069053459	1.925480624	$5.545651214 \times 10^{-6}$
0.5	1.9097959895689502	1.909797288	$1.298082753 \times 10^{-6}$
0.6	1.9432174541410394	1.943219708	$2.253823192 \times 10^{-6}$
0.7	2.0248779556871144	2.024881305	$3.349008792 \times 10^{-6}$
0.8	2.1539934461758325	2.153998122	$4.6761011 \times 10^{-6}$
0.9	2.329854489610899	2.329860721	$6.231529552 \times 10^{-6}$
1.0	2.5518191617571633	2.551827065	$7.902989782 \times 10^{-5}$

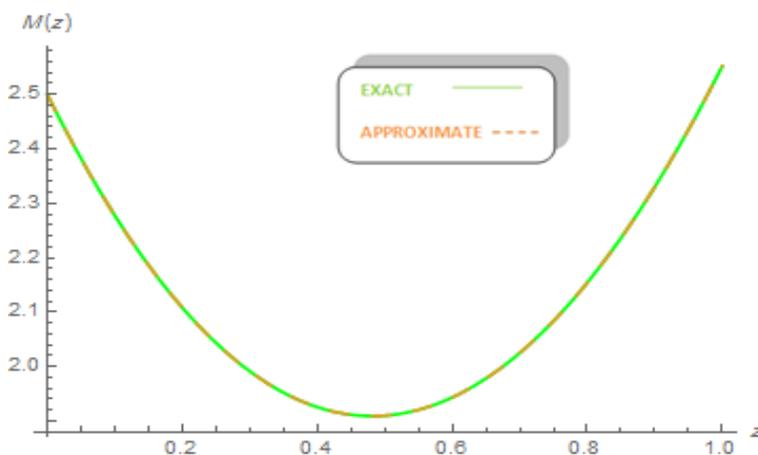


Figure 7: The exact and approximate solutions of  $M_{1,2r}(z)$  when  $r = 0.5$  for Example 4.1.

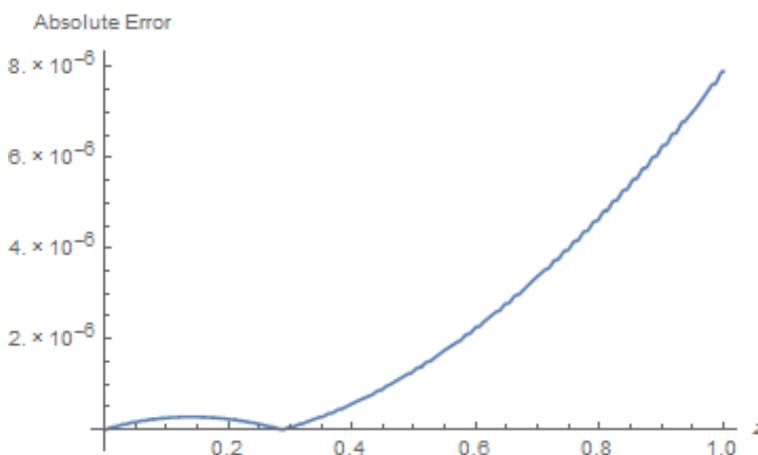


Figure 8: The errors in solutions for  $M_{1,2r}(z)$  when  $r = 0.5$  for Example 4.1.

Table 5: Numerical solutions of  $M_{1,1r}(z)$  and  $M_{1,2r}(z)$  for different values of  $r$  and  $\alpha = 0.8$ .

$z$	$M_{1,1r}(z) = M_{1,2r}(z), r = 1.0$ $r = 1.0$	$\hat{M}_{1,1r}(z) = M_{1,2r}(z), r = 1.0$ $r = 1.0$	Errors
0.0	2.00	2.00	0.00
0.1	1.8248374180359594	1.824837622	$2.038358025 \times 10^{-7}$
0.2	1.6987307530779816	1.698731366	$6.127483738 \times 10^{-7}$
0.3	1.620818220681718	1.620819416	$1.195502695 \times 10^{-6}$
0.4	1.5903200460356395	1.590321963	$1.917354203 \times 10^{-6}$
0.5	1.6065306597126334	1.6065335	$2.839816752 \times 10^{-6}$
0.6	1.6688116360940264	1.668815561	$3.925195678 \times 10^{-6}$
0.7	1.7765853037914097	1.776590402	$5.097971347 \times 10^{-6}$
0.8	1.929328964117222	1.92933544	$6.475920912 \times 10^{-6}$
0.9	2.1265696597405994	2.126577716	$8.05607230 \times 10^{-6}$
1.0	2.3678794411714423	2.367889162	$9.72106010 \times 10^{-5}$

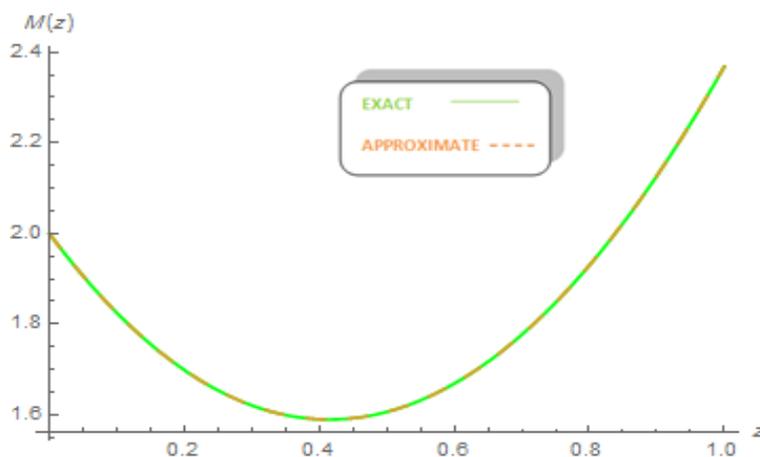


Figure 9: The exact and approximate solutions of  $M_{1,1r}(z) = M_{1,2r}(z)$  when  $r = 1.0$  for Example 4.1.

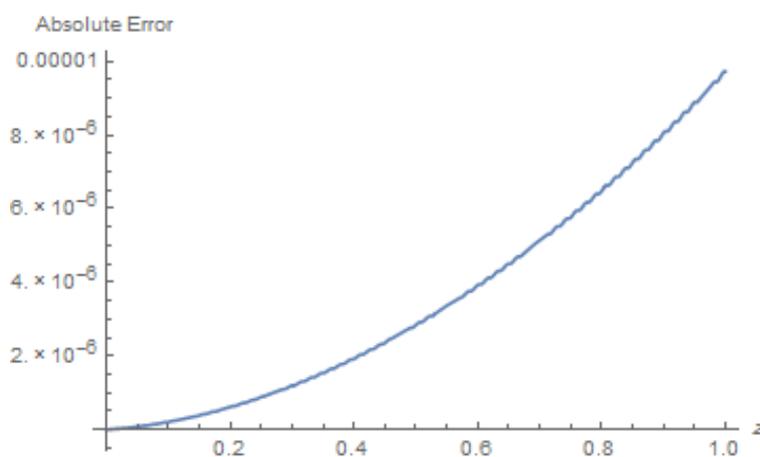


Figure 10: The errors in solutions for  $M_{1,1r}(z) = M_{1,2r}(z)$  when  $r = 1.0$  for Example 4.1.

**Example 4.2.** Consider the following fuzzy differential equation of order  $\alpha = 0.8$ .

$$D_{c,\alpha,z}^\alpha M(z) = 2z^2 + 3z - M(z), \quad M(z_0) = (2, 3, 4), 0 \leq z \leq 1.$$

By applying fuzzy set theory, we obtain

$$D_{c,\alpha}^\alpha M_{1,1r}(z) = 2z^2 + 3z - M_{1,1r}(z), \quad M_{1,1r}(z_0) = 2 + r, r \in [0, 1],$$

$$D_{c,\alpha}^\alpha M_{1,2r}(z) = 2z^2 + 3z - M_{1,2r}(z), \quad M_{1,2r}(z_0) = 4 - r, r \in [0, 1].$$

Table 6 shows the results for Example 4.2 at  $\alpha = 0.8$  with the values of  $r$  set to be  $r = 0, r = 0.5$  and  $r = 1$ . The illustration of the results can be referred in Figures 11, 12, and 13. It is clear that the solutions create a form of interval which indicates that the solutions are fuzzy numbers.

Table 6: Numerical solutions of  $M_{1,1r}(z)$  and  $M_{1,2r}(z)$  for different values of  $r$  and  $\alpha = 0.8$ .

$z$	$M_{1,1r}(z)$ $r = 0$	$M_{1,2r}(z)$ $r = 0$	$M_{1,1r}(z)$ $r = .5$	$M_{1,2r}(z)$ $r = .5$	$M_{1,r}(z) = M_{1,2r}(z)$ $r = 1$
0.0	2.00	4.00	2.5	3.5	3.0
0.1	1.736841535	3.444681614	2.163801555	3.017721594	2.590761574
0.2	1.609843308	3.122571631	1.988025389	2.74438955	2.36620747
0.3	1.561542031	2.922538361	1.901791114	2.582289278	2.242040196
0.4	1.573674645	2.810071823	1.88277394	2.500972529	2.191873234
0.5	1.637606954	2.768810968	1.920407958	2.486009965	2.203208961
0.6	1.748321	2.789137328	2.008525082	2.528933246	2.268729164
0.7	1.902594453	2.864761074	2.143136108	2.624219419	2.383677764
0.8	2.09823305	2.991302225	2.321500344	2.768034932	2.544767638
0.9	2.33369741	3.165591107	2.541670834	2.957617683	2.749644258
1.0	2.608035252	3.385443423	2.802387295	3.19109138	2.996739338

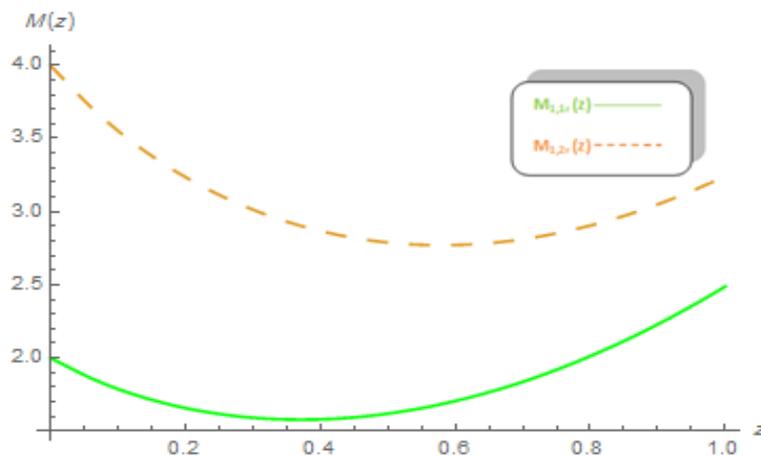


Figure 11: Approximate solutions of  $M_{1,1r}(z)$  and  $M_{1,2r}(z)$  when  $r = 0$  for Example 4.2.

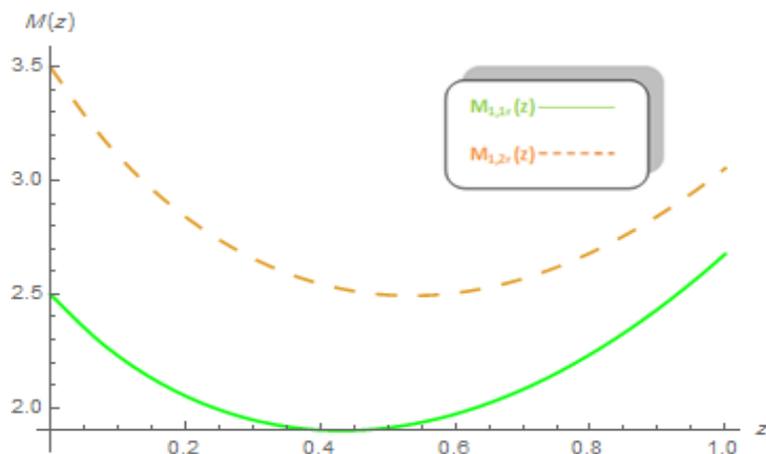


Figure 12: Approximate solutions of  $M_{1,1r}(z)$  and  $M_{1,2r}(z)$  when  $r = 0.5$  for Example 4.2.

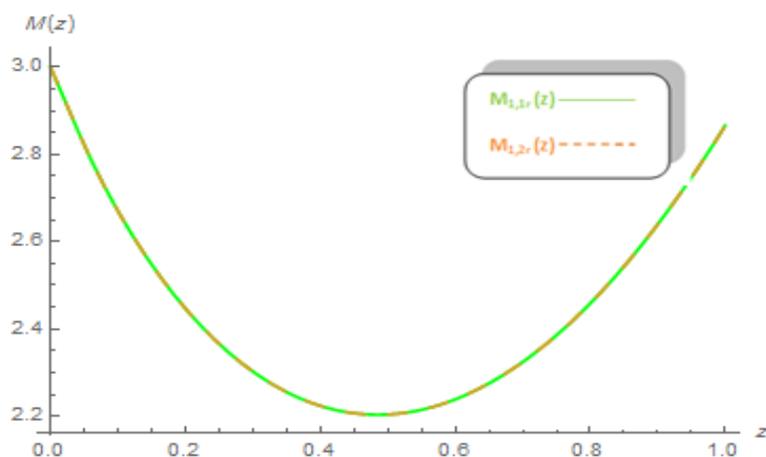


Figure 13: Approximate solutions of  $M_{1,1r}(z)$  and  $M_{1,2r}(z)$  when  $r = 1.0$  for Example 4.2.

## 5. Conclusion

In this paper, we have studied the solutions of fuzzy initial value problem for differential equations of integer and fractional order with Caputo derivatives. This has been done by modifying the reproducing kernel Hilbert space. N-term approximation solution is obtained from the method, and it can be further applied widely and more efficiently compared to traditional methods to solve fuzzy initial value problems for differential equations of integer order. This improved method contributes in many ways, such as solving fuzzy system effectively, reduce time for solving problems related to real phenomena and engineering applications.

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