Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2449–2456
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

A transformation algorithm for nonexpansive mappings

Xinhe Zhu[a], Shin Min Kang[b,c,*]

[a] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China.*

[b] *Center for General Education, China Medical University, Taichung 40402, Taiwan.*

[c] *Department of Mathematics and the RINS, Gyeongsang National University, Jinju 52828, Korea.*

Communicated by Y.-H. Yao

Abstract

A transformation algorithm is constructed for finding the fixed points of nonexpansive mappings. We show that the suggested algorithm converges strongly to a fixed point of nonexpansive mappings under some different control conditions.

Keywords: Iterative method, nonexpansive mapping, fixed point.
2010 MSC: 47H10, 47H09.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\emptyset \neq C \subset H$ be a closed convex set. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|,$$

for all $x, y \in C$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T.

Now it is well-known that construction of fixed points of nonexpansive mappings is an important subject in the theory of nonlinear operators and its applications in a number of applied areas, in particular, in image recovery and signal processing. There are a large number of algorithms for finding the fixed points of nonexpansive mappings in the literature: for example, Mann's method [6], Ishikawa's methd [4], Halpern's method [3], Moudafi's viscosity method [7], Yao, Chen and Yao's modified Mann's method [12] and Yao and Shahzad's method with perturbations [14].

Recently, in order to find the fixed points of nonexpansive mappings, Alghamdi et al. [1] presented the following semi-implicit midpoint rule: for given $x_0 \in H$, compute the sequence $\{x_n\}$ by the iteration

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0, \tag{1.1}$$

*Corresponding author
Email addresses: zhumath@126.com (Xinhe Zhu), smkang@gnu.ac.kr (Shin Min Kang)

doi:10.22436/jnsa.010.05.15

Received 2017-03-21

where $\alpha_n \in (0,1)$ and $T : H \to H$ is a nonexpansive mapping.

Note that the disadvantage of (1.1) is difficult to compute the next step x_{n+1}. For solving this difficulty, Yao et al. [13] converted (1.1) to the following new form

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n \frac{Ty_n + Ty_{n+1}}{2}, \quad n \geqslant 0. \tag{1.2}$$

On the other hand, in [11], Xu et al. used contractions to regularize the semi-implicit midpoint rule (1.1) and presented the following viscosity implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = \alpha_n \mathcal{S}(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0, \tag{1.3}$$

where $\alpha_n \in (0,1)$ and $\mathcal{S}$ is a contraction.

Xu et al. [11] obtained the following strong convergence theorem.

Theorem 1.1. *Let H be a Hilbert space, C a nonempty, closed, and convex subset of H, and $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\mathcal{S} : C \to C$ be a contraction with coefficient $\alpha \in [0,1)$. Assume that the sequence $\{\alpha_n\}$ satisfies the following three restrictions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$;

(C2): $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3): *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ generated by (1.3) converges in norm to a fixed point q of T, which is also the unique solution of the variational inequality

$$\langle (I-\mathcal{S})q, x-q\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T).$$

In other words, q is the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}\mathcal{S}$, that is, $P_{\mathrm{Fix}(T)}\mathcal{S}(q) = q$.

Further, Yao et al. [16] introduced the following semi-implicit midpoint method:

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0. \tag{1.4}$$

Very recently, Yao and Shahzad [15] suggested a viscosity implicit midpoint method for nonexpasnive mappings:

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n \left(\frac{Tx_n + Tx_{n+1}}{2}\right), \quad n \geqslant 0. \tag{1.5}$$

Motivated and inspired by the algorithms (1.2), (1.4) and (1.5), we will study the following implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = \alpha_n \mathcal{S}(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)(\theta_n T x_n + (1-\theta_n)Tx_{n+1}), \quad n \geqslant 0. \tag{1.6}$$

We will prove that the suggested algorithm (1.6) converges strongly to a special fixed point of nonexpansive mappings under some different conditions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $\mathcal{S} : C \to C$ is said to be contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|\mathcal{S}(x) - \mathcal{S}(y)\| \leqslant \alpha\|x-y\|,$$

for all $x, y \in C$. In this case, $\mathcal{S}$ is called α-contraction.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leqslant 0, \quad \forall x \in H, \ \ y \in C. \tag{2.1}$$

Lemma 2.1 ([10]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{S} : C \to C$ be an α-contraction and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For given $y_0 \in C$ arbitrarily, let the sequence $\{y_n\}$ be defined iteratively by the manner*

$$y_n = \alpha_n \mathcal{S}(y_n) + (1 - \alpha_n)Ty_n, \quad n \geqslant 0, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. The sequence $\{y_n\}$ generated by (2.2) converges strongly to $q = P_{\mathrm{Fix}(T)}\mathcal{S}(q)$ provided $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2 ([2]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that $\{y_n\}$ is a sequence in C such that $y_n \rightharpoonup x^\dagger$ and $(I - T)y_n \to 0$. Then $x^\dagger \in \mathrm{Fix}(T)$.*

Lemma 2.3 ([8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geqslant 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.4 ([9]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leqslant (1 - \alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \quad n \geqslant 0,$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leqslant 0$;
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we present our algorithm and demonstrate its convergence analysis.

Algorithm 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $\mathcal{S} : C \to C$ an α-contraction. Let $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{\theta_n\} \subset (0, 1)$ be three sequences. For given $x_0 \in C$ arbitrarily, let the sequence x_{n+1} be generated iteratively by the manner

$$x_{n+1} = \alpha_n \mathcal{S}(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)[\theta_n Tx_n + (1 - \theta_n)Tx_{n+1}], \quad n \geqslant 0. \tag{3.1}$$

Remark 3.2. Algorithm (3.1) is well-defined. As a matter of fact, for fixed $u \in C$, we can define a mapping

$$x \mapsto T_u x := \alpha\mathcal{S}(u) + \beta u + (1 - \alpha - \beta)[\theta Tu + (1 - \theta)Tx].$$

Then, we have

$$\begin{aligned}\|T_u x - T_u y\| &= (1 - \alpha - \beta)(1 - \theta)\|Tx - Ty\| \\ &\leqslant (1 - \alpha - \beta)(1 - \theta)\|x - y\|.\end{aligned}$$

This means T_u is a contraction with coefficient $(1 - \alpha - \beta)(1 - \theta) \in (0, 1)$. Hence, algorithm (3.1) is well-defined.

Next, we give several useful results.

Conclusion 3.3. The sequence $\{x_n\}$ generated by (3.1) is bounded.

Proof. Let $x^\dagger \in \mathrm{Fix}(T)$. From (3.1), we have

$$\begin{aligned}\|x_{n+1}-x^\dagger\| &= \|\alpha_n(\mathcal{S}(x_n)-\mathcal{S}(x^\dagger))+\alpha_n(\mathcal{S}(x^\dagger)-x^\dagger)+\beta_n(x_n-x^\dagger)\\ &\quad +(1-\alpha_n-\beta_n)[\theta_n Tx_n+(1-\theta_n)Tx_{n+1}-x^\dagger]\|\\ &\leqslant \alpha_n\|\mathcal{S}(x_n)-\mathcal{S}(x^\dagger)\|+\alpha_n\|\mathcal{S}(x^\dagger)-x^\dagger\|+\beta_n\|x_n-x^\dagger\|\\ &\quad +(1-\alpha_n-\beta_n)\|\theta_n(Tx_n-x^\dagger)+(1-\theta_n)(Tx_{n+1}-x^\dagger)\|\\ &\leqslant \alpha_n\alpha\|x_n-x^\dagger\|+\alpha_n\|\mathcal{S}(x^\dagger)-x^\dagger\|+\beta_n\|x_n-x^\dagger\|\\ &\quad +(1-\alpha_n-\beta_n)\theta_n\|x_n-x^\dagger\|+(1-\alpha_n-\beta_n)(1-\theta_n)\|x_{n+1}-x^\dagger\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^\dagger\| &\leqslant \frac{\alpha_n\alpha+\beta_n+(1-\alpha_n-\beta_n)\theta_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\|x_n-x^\dagger\|+\frac{\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\|\mathcal{S}(x^\dagger)-x^\dagger\|\\ &=\left[1-\frac{(1-\alpha)\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\right]\|x_n-x^\dagger\|+\frac{(1-\alpha)\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\frac{1}{1-\alpha}\|\mathcal{S}(x^\dagger)-x^\dagger\|\\ &\leqslant \max\left\{\|x_n-x^\dagger\|,\frac{1}{1-\alpha}\|\mathcal{S}(x^\dagger)-x^\dagger\|\right\}.\end{aligned}$$

Apply an induction to get

$$\|x_n-x^\dagger\|\leqslant \max\left\{\|x_0-x^\dagger\|,\frac{1}{1-\alpha}\|\mathcal{S}(x^\dagger)-x^\dagger\|\right\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{\mathcal{S}(x_n)\}$ and $\{Tx_n\}$. This completes the proof. □

Conclusion 3.4. Assume $\{\alpha_n\}$ satisfies conditions (C1)-(C3), $\{\beta_n\}$ satisfies

$(C4):$ either $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$ or $\lim_{n\to\infty}\frac{\beta_{n+1}-\beta_n}{\alpha_{n+1}}=0$,

and $\{\theta_n\}$ satisfies

$(C4'):$ either $\sum_{n=0}^{\infty}|\theta_{n+1}-\theta_n|<\infty$ or $\lim_{n\to\infty}\frac{\theta_{n+1}-\theta_n}{\alpha_{n+1}}=0$.

Then $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. From (3.1), we have

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\|\\ &\quad=\|\alpha_{n+1}(\mathcal{S}(x_{n+1})-\mathcal{S}(x_n))+(\alpha_{n+1}-\alpha_n)\mathcal{S}(x_n)\\ &\qquad+\beta_{n+1}(x_{n+1}-x_n)+(\beta_{n+1}-\beta_n)[x_n-(\theta_nTx_n+(1-\theta_n)Tx_{n+1})]\\ &\qquad+(1-\alpha_{n+1}-\beta_{n+1})[(\theta_{n+1}Tx_{n+1}+(1-\theta_{n+1})Tx_{n+2})-(\theta_nTx_n+(1-\theta_n)Tx_{n+1})]\\ &\qquad+(\alpha_n-\alpha_{n+1})(\theta_nTx_n+(1-\theta_n)Tx_{n+1})\|\\ &\quad\leqslant \alpha\alpha_{n+1}\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\|\mathcal{S}(x_n)\|+\|\theta_nTx_n+(1-\theta_n)Tx_{n+1}\|)\\ &\qquad+\beta_{n+1}\|x_{n+1}-x_n\|+(1-\alpha_{n+1}-\beta_{n+1})(\theta_{n+1}\|x_{n+1}-x_n\|+(1-\theta_{n+1})\|x_{n+2}-x_{n+1}\|)\\ &\qquad+(1-\alpha_{n+1}-\beta_{n+1})|\theta_{n+1}-\theta_n|(\|Tx_n\|+\|Tx_{n+1}\|)\\ &\qquad+|\beta_{n+1}-\beta_n|\|x_n-(\theta_nTx_n+(1-\theta_n)Tx_{n+1})\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+2}-x_{n+1}\|&\leqslant\left[1-\frac{(1-\alpha)\alpha_{n+1}}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}\right]\|x_{n+1}-x_n\|\\ &\quad+\frac{|\alpha_{n+1}-\alpha_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}(\|\mathcal{S}(x_n)\|+\|\theta_nTx_n+(1-\theta_n)Tx_{n+1}\|)\\ &\quad+\frac{(1-\alpha_{n+1}-\beta_{n+1})|\theta_{n+1}-\theta_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}(\|Tx_n\|+\|Tx_{n+1}\|)\\ &\quad+\frac{|\beta_{n+1}-\beta_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}\|x_n-(\theta_nTx_n+(1-\theta_n)Tx_{n+1})\|.\end{aligned}\tag{3.2}$$

Applying Lemma 2.4 to (3.2), we obtain

$$\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0. \tag{3.3}$$

By (3.1), we get

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leqslant \|x_{n+1} - x_{n+2}\| + \|x_{n+2} - Tx_{n+1}\| \\
&\leqslant \|x_{n+1} - x_{n+2}\| + \alpha_{n+1}\|\mathcal{S}(x_{n+1}) - Tx_{n+1}\| + \beta_{n+1}\|x_{n+1} - Tx_{n+1}\| \\
&\quad + (1-\alpha_{n+1}-\beta_{n+1})\|\theta_{n+1}Tx_{n+1} + (1-\theta_{n+1})Tx_{n+2} - Tx_{n+1}\| \\
&\leqslant \|x_{n+1} - x_{n+2}\| + \alpha_{n+1}\|\mathcal{S}(x_{n+1}) - Tx_{n+1}\| + \beta_{n+1}\|x_{n+1} - Tx_{n+1}\| \\
&\quad + (1-\alpha_{n+1}-\beta_{n+1})(1-\theta_{n+1})\|x_{n+1} - x_{n+2}\|.
\end{aligned}$$

Thus,

$$\|x_{n+1} - Tx_{n+1}\| \leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\mathcal{S}(x_{n+1}) - Tx_{n+1}\| + \frac{2}{1-\beta_{n+1}}\|x_{n+2} - x_{n+1}\|. \tag{3.4}$$

This together with (C1) and (3.3) implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Conclusion 3.5. Assume $\{\alpha_n\}$ satisfies (C1)–(C2), $\{\beta_n\}$ satisfies

(C5): $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$,

and $\{\theta_n\}$ satisfies

(C5′): $\lim_{n\to\infty}(\theta_{n+1} - \theta_n) = 0$.

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Write $y_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for all $n \geqslant 0$. Then we have

$$\begin{aligned}
y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\mathcal{S}(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})(\theta_{n+1}Tx_{n+1} + (1-\theta_{n+1})Tx_{n+2})}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\mathcal{S}(x_n) + (1-\alpha_n-\beta_n)(\theta_nTx_n + (1-\theta_n)Tx_{n+1})}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\mathcal{S}(x_{n+1}) - \mathcal{S}(x_n)) + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}[(\theta_{n+1}Tx_{n+1} + (1-\theta_{n+1})Tx_{n+2}) \\
&\quad - (\theta_nTx_n + (1-\theta_n)Tx_{n+1})] + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(\mathcal{S}(x_n) - (\theta_nTx_n + (1-\theta_n)Tx_{n+1})).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leqslant \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)\|\mathcal{S}(x_n) - (\theta_nTx_n + (1-\theta_n)Tx_{n+1})\| + \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| \\
&\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}[\theta_{n+1}\|x_{n+1} - x_n\| + (1-\theta_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad + |\theta_{n+1} - \theta_n|(\|Tx_n\| + \|Tx_{n+1}\|)].
\end{aligned} \tag{3.5}$$

Substitute (3.2) into (3.5) to get

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\quad\leqslant \left[1-\frac{(1-\alpha)\alpha_{n+1}}{1-\beta_{n+1}}\right]\|x_{n+1}-x_n\| + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n}{1-\beta_n}\right)\|\mathcal{S}(x_n)-(\theta_n Tx_n+(1-\theta_n)Tx_{n+1})\| \\
&\qquad + \frac{|\alpha_{n+1}-\alpha_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}(\|\mathcal{S}(x_n)\|+\|\theta_n Tx_n+(1-\theta_n)Tx_{n+1}\|) \\
&\qquad + \frac{2|\theta_{n+1}-\theta_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}(\|Tx_n\|+\|Tx_{n+1}\|) \\
&\qquad + \frac{|\beta_{n+1}-\beta_n|}{1-(1-\theta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}\|x_n-(\theta_n Tx_n+(1-\theta_n)Tx_{n+1})\|.
\end{aligned}$$

According to conditions (C1) and (C5), we derive that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leqslant 0.$$

This together with Lemma 2.3 implies that

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

Note that $y_n-x_n=\frac{x_{n+1}-x_n}{1-\beta_n}$. Consequently, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Noting that (3.4), we have

$$\|x_n-Tx_n\|\leqslant\frac{\alpha_n}{1-\beta_n}\|\mathcal{S}(x_n)-Tx_n\|+\frac{2}{1-\beta_n}\|x_{n+1}-x_n\|. \tag{3.7}$$

Combining (3.6), (3.7) and condition (C1), we derive

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

This completes the proof. □

Theorem 3.6. *Assume $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ satisfy the one of the following conditions:*

(i) (C1)-(C4) *and* $(C4)'$;
(ii) (C1), (C2), (C5) *and* $(C5)'$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $q=P_{Fix(T)}\mathcal{S}(q)$.

Proof. By Conclusion 3.4 and Conclusion 3.5, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.8}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle q-\mathcal{S}(q),q-x_n\rangle\leqslant 0, \tag{3.9}$$

where $q\in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)}\mathcal{S}$, that is, $q=P_{Fix(T)}\mathcal{S}(q)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $\breve{x}$ and

$$\limsup_{n\to\infty}\langle P_{Fix(T)}\mathcal{S}(q)-\mathcal{S}(q),P_{Fix(T)}\mathcal{S}(q)-x_n\rangle=\lim_{i\to\infty}\langle P_{Fix(T)}\mathcal{S}(q)-\mathcal{S}(q),P_{Fix(T)}\mathcal{S}(q)-x_{n_i}\rangle.$$

By Lemma 2.2 and (3.8), we deduce $\check{x} \in \mathrm{Fix}(T)$. This together with (2.1) implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle P_{\mathrm{Fix}(T)}\mathcal{S}(q)-\mathcal{S}(q), P_{\mathrm{Fix}(T)}\mathcal{S}(q)-x_n\rangle &= \lim_{i\to\infty}\langle P_{\mathrm{Fix}(T)}\mathcal{S}(q)-\mathcal{S}(q), P_{\mathrm{Fix}(T)}\mathcal{S}(q)-x_{n_i}\rangle \\ &= \langle P_{\mathrm{Fix}(T)}\mathcal{S}(q)-\mathcal{S}(q), P_{\mathrm{Fix}(T)}\mathcal{S}(q)-\check{x}\rangle \\ &\leqslant 0.\end{aligned}$$

Finally, we prove that $x_n \to q$. From (3.1), we have

$$\begin{aligned}&\|x_{n+1}-q\|^2 \\ &\quad= \alpha_n\langle\mathcal{S}(x_n)-\mathcal{S}(q), x_{n+1}-q\rangle + \alpha_n\langle\mathcal{S}(q)-q, x_{n+1}-q\rangle \\ &\qquad+(1-\alpha_n-\beta_n)\langle\theta_nTx_n+(1-\theta_n)Tx_{n+1}-q, x_{n+1}-q\rangle + \beta_n\langle x_n-q, x_{n+1}-q\rangle \\ &\quad\leqslant \alpha_n\alpha\|x_n-q\|\|x_{n+1}-q\| + \alpha_n\langle\mathcal{S}(q)-q, x_{n+1}-q\rangle \\ &\qquad+(1-\alpha_n-\beta_n)(\theta_n\|x_n-q\|+(1-\theta_n)\|x_{n+1}-q\|)\|x_{n+1}-q\| + \beta_n\|x_n-q\|\|x_{n+1}-q\| \\ &\quad\leqslant \frac{\alpha\alpha_n+\beta_n+(1-\alpha_n-\beta_n)\theta_n}{2}\|x_n-q\|^2 + \frac{1-(1-\alpha)\alpha_n+(1-\alpha_n-\beta_n)(1-\theta_n)}{2}\|x_{n+1}-q\|^2 \\ &\qquad+\alpha_n\langle\mathcal{S}(q)-q, x_{n+1}-q\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 \leqslant &\left[1-\frac{2(1-\alpha)\alpha_n}{1+(1-\alpha)\alpha_n-(1-\alpha_n-\beta_n)(1-\theta_n)}\right]\|x_n-q\|^2 \\ &+\frac{2\alpha_n}{1+(1-\alpha)\alpha_n-(1-\alpha_n-\beta_n)(1-\theta_n)}\langle\mathcal{S}(q)-q, x_{n+1}-q\rangle.\end{aligned} \tag{3.10}$$

Apply Lemma 2.4 and (3.9) to (3.10) to deduce that $x_n \to q$. This completes the proof. □

Next, we can define the following algorithm.

Algorithm 3.7. For given $y_0 \in C$ arbitrarily, let the sequence $\{y_n\}$ be defined iteratively by the manner

$$y_n = \alpha_n\mathcal{S}(y_n)+\beta_ny_n+(1-\alpha_n-\beta_n)Ty_n, \quad n \geqslant 0, \tag{3.11}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$ are two sequences.

Conclusion 3.8. The sequence $\{y_n\}$ generated by (3.11) converges strongly to $q = P_{\mathrm{Fix}(T)}\mathcal{S}(q)$ provided $\lim_{n\to\infty}\alpha_n = 0$.

In fact, we can rewrite (3.11) as $y_n = \frac{\alpha_n}{1-\beta_n}\mathcal{S}(y_n)+(1-\frac{\alpha_n}{1-\beta_n})Ty_n$ for all n. Thus, Conclusion 3.8 can be deduced from Lemma 2.1.

Next we use Conclusion 3.8 to show the convergence analysis of algorithm (3.1) under other control conditions.

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.1) and (3.11), respectively. Note that the sequences $\{x_n\}$ and $\{y_n\}$ are all bounded. First, we have the following estimate.

$$\begin{aligned}\|x_{n+1}-y_n\| &= \|\alpha_n(\mathcal{S}(x_n)-\mathcal{S}(y_n))+\beta_n(x_n-y_n)+(1-\alpha_n-\beta_n)(\theta_nTx_n+(1-\theta_n)Tx_{n+1}-Ty_n)\| \\ &\leqslant \alpha_n\alpha\|x_n-y_n\|+\beta_n\|x_n-y_n\|+(1-\alpha_n-\beta_n)\theta_n\|x_n-y_n\| \\ &\quad+(1-\alpha_n-\beta_n)(1-\theta_n)\|x_{n+1}-y_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-y_n\| &\leqslant \left[1-\frac{(1-\alpha)\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\right]\|x_n-y_n\| \\ &\leqslant \left[1-\frac{(1-\alpha)\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\theta_n)}\right]\|x_n-y_{n-1}\|+\|y_n-y_{n-1}\|.\end{aligned}$$

It is easily seen that if $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\frac{\|y_n-y_{n-1}\|}{\alpha_n} = 0$ and $\liminf_{n\to\infty}\theta_n > 0$, then we get $\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0$ by Lemma 2.4. Consequently, $x_n \to q = P_{Fix(T)}\mathcal{S}(q)$ provided $\lim_{n\to\infty}\alpha_n = 0$.

Next, we estimate $\|y_n - y_{n-1}\|$. From (3.11), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\alpha_n(\mathcal{S}(y_n) - \mathcal{S}(y_{n-1})) + (\alpha_n - \alpha_{n-1})\mathcal{S}(y_{n-1}) + \beta_n(y_n - y_{n-1})\\ &\quad + (\beta_n - \beta_{n-1})y_{n-1} + (1-\alpha_n-\beta_n)(Ty_n - Ty_{n-1}) + (\alpha_{n-1}-\alpha_n+\beta_{n-1}-\beta_n)Ty_{n-1}\|\\ &\leqslant (1-\alpha_n+\alpha\alpha_n)\|y_n-y_{n-1}\| + |\alpha_n-\alpha_{n-1}|\|\mathcal{S}(y_{n-1})\|\\ &\quad + |\beta_n-\beta_{n-1}|\|y_{n-1}\| + (|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|)\|Ty_{n-1}\|.\end{aligned}$$

Hence,

$$\frac{\|y_n-y_{n-1}\|}{\alpha_n} \leqslant \frac{|\alpha_n-\alpha_{n-1}|}{(1-\alpha)\alpha_n^2}(\|\mathcal{S}(y_{n-1})\| + \|Ty_{n-1}\|) + \frac{|\beta_n-\beta_{n-1}|}{(1-\alpha)\alpha_n^2}(\|y_{n-1}\| + \|Ty_{n-1}\|).$$

If $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n^2} = \lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n^2} = 0$, we derive that $\lim_{n\to\infty}\frac{\|y_n-y_{n-1}\|}{\alpha_n} = 0$. So, we obtain immediately the following theorem.

Theorem 3.9. *Assume* $\{\alpha_n\}$ *satisfies* (C1), (C2) *and*

(C6): $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n^2} = 0$,

$\{\beta_n\}$ *satisfies*

(C7): $\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n^2} = 0$,

and $\{\theta_n\}$ *satisfies*

(C8): $\liminf_{n\to\infty}\theta_n > 0$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $q = P_{Fix(T)}\mathcal{S}(q)$.

Remark 3.10. Note that the conditions (C1), (C2) and (C6) was presented by Lions in [5]. At the same time, (C4) is different from (C5). In fact, we can choose $\beta_n = \beta \in (0,1)$ in (C5).

References

[1] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad, H.-K. Xu, *The implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory Appl., **2014** (2014), 9 pages. 1

[2] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory: Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, (1990). 2.2

[3] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 591–597. 1

[4] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147–150. 1

[5] P.-L. Lions, *Approximation de points fixes de contractions*, C. R. Acad. Sci. Paris Ser. A-B, **284** (1977), 1357–1359. 3.10

[6] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510. 1

[7] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241** (2000), 46–55. 1

[8] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory Appl., **2005** (2005), 103–123. 2.3

[9] H.-K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256. 2.4

[10] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298** (2004), 279–291. 2.1

[11] H.-K. Xu, M. A. Alghamdi, N. Shahzad, *The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2015** (2015), 12 pages. 1, 1

[12] Y.-H. Yao, R.-D. Chen, J.-C. Yao, *Strong convergence and certain control conditions for modified Mann iteration*, Nonlinear Anal., **68** (2008), 1687–1693. 1

[13] Y.-H. Yao, Y.-C. Liou, T.-L. Lee, N.-C. Wong, *An iterative algorithm based on the implicit midpoint rule for nonexpansive mappings*, J. Nonlinear Convex Anal., **17** (2016), 655–668. 1

[14] Y.-H. Yao, N. Shahzad, *New methods with perturbations for nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2011** (2011), 9 pages. 1

[15] Y.-H. Yao, N. Shahzad, *Viscosity implicit midpoint methods for nonexpansive mappings*, J. Nonlinear Sci. Appl., (In press). 1

[16] Y.-H. Yao, N. Shahzad, Y.-C. Liou, *Modified semi-implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory Appl., **2015** (2015), 15 pages. 1