Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2457–2470
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

System of N fixed point operator equations with N-pseudo-contractive mapping in reflexive Banach spaces

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Communicated by S. S. Chang

Abstract

The purpose of this paper is to study the problem of the system of N fixed point operator equations with N-variables pseudo-contractive mapping. Firstly, the concept of N-variables pseudo-contractive mapping and relatively concepts of nonlinear mappings are presented in Banach spaces. Secondly, the existence theorems of solutions for the system of N fixed point operator equations with N-variables pseudo-contractive mapping are proved in reflexive Banach spaces by using the method of product spaces. In order to get the expected results, the normalized duality mapping of product Banach spaces is defined. Meanwhile the reflexivity of the product of reflexive Banach spaces and Opial's condition of product spaces of Banach spaces are also discussed.

Keywords: N-variables pseudo-contractive mapping, N fixed point, system of operator equations, product spaces, reflexive Banach space, Opial's condition.
2010 MSC: 47H05, 47H10.

1. Introduction and preliminaries

Recently, the multivariate fixed point theorems of N-variables nonlinear mappings have been studied by some authors, for examples [9] and [17]. Many interesting results and its applications have been also given. In 2016, Su et al. [17] presented the concept of multivariate fixed point and proved a multivariate fixed point theorem for the N-variables contraction mappings which further generalizes Banach contraction mapping principle. On the other hand, the pseudo-contractive mappings and strictly pseudo-contractive mappings are important classes of nonlinear operators in the field of nonlinear functional analysis and applications. The fixed point theorems and the iterative algorithms of such mappings are also important, so which have been studied by many authors (see [1–8, 10–16, 18–28]).

Let E be a real Banach space and let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|f\|^2 = \|x\|^2\}, \quad x \in E.$$

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doi:10.22436/jnsa.010.05.16

Received 2017-03-06

Recall that, a mapping T with domain $D(T)$ and range $R(T)$ in a Banach space is said to be pseudo-contractive, if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \|x-y\|^2, \quad \forall\, x, y \in D(T).$$

Recall that, a mapping with domain $D(T)$ and range $R(T)$ in a Banach space is said to be strongly pseudo-contractive, if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant k\|x-y\|^2, \quad \forall\, x, y \in D(T),$$

where $k \in (0,1)$ is a constant.

In 1974, Deimling [4] proved the following fixed point theorem.

Theorem 1.1 ([4]). *Let E be a real Banach space, K a nonempty closed convex subset of E, and $T : K \to K$ a continuous strongly pseudo-contractive mapping. Then T has a unique fixed point in K.*

Recall that, a Banach space E is said to satisfy the Opial's condition, if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall\ y \in E,\ y \neq x.$$

Recall that, a mapping T in a Banach space is said to be demi-closed at the zero if for any sequence $\{x_n\}$ which converges weakly to x_0 and $\{Tx_n\}$ converges strongly to zero, then $Tx_0 = 0$.

In 2008, Zhou [26] proved two demi-closed principles for continuous pseudo-contractive mappings. These demi-closed principles are very useful for our main results.

Lemma 1.2 ([26]). *Let E be a real reflexive Banach space which satisfies Opial's condition. Let K be a nonempty closed convex subset of E, and $T : K \to K$ be a continuous pseudo-contractive mapping. Then $I - T$ is demi-closed at zero.*

Lemma 1.3 ([26]). *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E, and $T : K \to K$ be a continuous pseudo-contractive mapping. Then, $I - T$ is demi-closed at zero.*

Definition 1.4. Let T be a mapping with domain $D(T)$ and range $R(T)$ in a Banach space.

(1) T is said to be invariant, if $R(T) \subset D(T)$;

(2) T is said to be strong invariant, if the range $R(T)$ is bounded and there exists some constant $t_0 \in (0,1)$ such that $tR(T) \subset D(T)$ for all $t \in [t_0, 1]$.

It is obvious that, a strong invariant mapping must be invariant. However, the inverse is not true.

In 2010, Su and Li [16] proved the following two fixed point theorems for pseudo-contractive mapping in Banach spaces.

Theorem 1.5 ([16]). *Let E be a real reflexive Banach space which satisfying Opial's condition, and K be a closed convex subset of E. Let T be a continuous pseudo-contractive mapping from K into itself. If T is strong invariant on the K, then T has a fixed point in K.*

Theorem 1.6 ([16]). *Let E be a real uniformly convex Banach space, and K be a closed convex subset of E. Let T be a continuous pseudo-contractive mapping from K into itself. If T is strong invariant on the K, then T has a fixed point in K.*

Definition 1.7 ([17]). Let (X, d) be a metric space, $T : X^N \to X$ be an N-variables mapping, an element $p \in X$ is called a multivariate fixed point (or a fixed point of order N, see [9]) of T if

$$p = T(p, p, \cdots, p).$$

The purpose of this paper is to study the problem of the system of N fixed point operator equations with N-variables pseudo-contractive mapping. Firstly, the concept of N-variables pseudo-contractive mapping and relatively concepts of nonlinear mappings are presented in Banach spaces. Secondly, the existence theorems of solutions for the system of N fixed point operator equations with N-variables pseudo-contractive mapping are proved in reflexive Banach spaces by using the method of product spaces. In order to get the expected results, the normalized duality mapping of product Banach spaces is defined. Meanwhile the reflexivity of the product of reflexive Banach spaces and Opial's condition of product spaces of Banach spaces are also discussed.

2. The normalized duality mapping of product Banach spaces

Lemma 2.1. *Let X be a Banach space with the norm $\|\cdot\|$. We consider on the Cartesian product space*

$$X^N = X \times X \times \cdots \times X,$$

the following functional

$$\|x\|_* = \sqrt{\sum_{i=1}^{N} \|x_i\|^2}, \quad \forall\, x = (x_1, x_2, \cdot\cdot\cdot, x_N) \in X^N.$$

Then $(X^N, \|\cdot\|_*)$ is a Banach space.

Proof. We first need to check the following conditions:

(1) $\|x\|_* \geqslant 0$ and $\|x\|_* = 0 \Leftrightarrow x = 0$;

(2) $\|\lambda x\|_* = |\lambda| \|x\|_*$;

(3) $\|x + y\|_* \leqslant \|x\|_* + \|y\|_*$.

The conditions (1), (2) are obvious. Next, we only need to check the condition (3). From the definition of $\|x\|_*$, we have, by using Minkowski inequality, that

$$\begin{aligned} \|x+y\|_* &= \sqrt{\sum_{i=1}^{N} \|x_i + y_i\|^2} \\ &\leqslant \sqrt{\sum_{i=1}^{N} (\|x_i\| + \|y_i\|)^2} \\ &\leqslant \sqrt{\sum_{i=1}^{N} \|x_i\|^2} + \sqrt{\sum_{i=1}^{N} \|y_i\|^2} \quad \text{(Minkowski inequality)} \\ &= \|x\|_* + \|y\|_*. \end{aligned}$$

We secondly need to prove that, $(X^N, \|\cdot\|_*)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in the linear normed space $(X^N, \|\cdot\|_*)$, where

$$x_n = (x_{n,1}, x_{n,2}, \cdot\cdot\cdot, x_{n,N}), \quad n = 1, 2, 3, \cdots.$$

In this case, we have that

$$\lim_{n,m\to\infty} \|x_n - x_m\|_* = \lim_{n,m\to\infty} \sqrt{\sum_{i=1}^{N} \|x_{n,i} - x_{m,i}\|^2} = 0,$$

which implies that

$$\lim_{n,m\to\infty} \|x_{n,i} - x_{m,i}\| = 0, \quad \forall\, i = 1, 2, 3, \cdots, N.$$

Since $(X, \|\cdot\|)$ is a Banach space, there exist $x_1, x_2, \cdots, x_N$ such that

$$\lim_{n,m\to\infty} \|x_{n,i} - x_i\| = 0, \quad \forall\, i = 1, 2, 3, \cdots, N.$$

Let $x = (x_1, x_2, \cdots, x_N)$, then, we have

$$\lim_{n\to\infty} \|x_n - x\|_* = \lim_{n\to\infty} \sqrt{\sum_{i=1}^{N} \|x_{n,i} - x_i\|^2} = 0,$$

which implies that, the sequence $\{x_n\}$ converges, in $\|\cdot\|_*$, to x. Hence $(X^N, \|\cdot\|_*)$ is complete. This completes the proof. □

Lemma 2.2. $(X^N, \|\cdot\|_*)^* = ((X, \|\cdot\|)^*)^N$.

Proof. For any $f \in (X^N, \|\cdot\|_*)^*$, define $f_i,\ i = 1, 2, 3, \cdots, N$ as follows:

$$\begin{aligned}
f_1(x) &= f(x, 0, 0, 0, 0, \cdots, 0),\\
f_2(x) &= f(0, x, 0, 0, 0, \cdots, 0),\\
f_3(x) &= f(0, 0, x, 0, 0, \cdots, 0),\\
f_4(x) &= f(0, 0, 0, x, 0, \cdots, 0),\\
&\ \vdots \qquad \vdots\\
f_N(x) &= f(0, 0, 0, 0, \cdots, 0, x).
\end{aligned}$$

Then, $f_i \in (X^*, \|\cdot\|),\ i = 1, 2, 3, \cdots, N$ and

$$(f_1, f_2, \cdots, f_N) \in (X^N, \|\cdot\|_*)^*.$$

Define a mapping F from $(X^N, \|\cdot\|_*)^*$ into $((X, \|\cdot\|)^*)^N$ as follows:

$$F : f \mapsto (f_1, f_2, \cdots, f_N).$$

Define a mapping G from $((X, \|\cdot\|)^*)^N$ into $(X^N, \|\cdot\|_*)^*$ as follows:

$$G : (f_1, f_2, \cdots, f_N) \mapsto f,$$

where

$$f(x_1, x_2, \cdots, x_N) = \sum_{i=1}^{N} f_i(x_i).$$

In this case, we have

$$\begin{aligned}
(GFf)(x_1, x_2, \cdots, x_N) &= G(f_1, f_2, \cdots, f_N)(x_1, x_2, \cdots, x_N)\\
&= \sum_{i=1}^{N} f_i(x_i)\\
&= f(x_1, x_2, \cdots, x_N),
\end{aligned}$$

so that $GFf = f$ and hence $G = F^{-1}$. On the other hand, we have

$$\begin{aligned}
(FG(f_1, f_2, \cdots, f_N))(x_1, x_2, \cdots, x_N) &= (Ff)(x_1, x_2, \cdots, x_N)\\
&= (f_1, f_2, \cdots, f_N)(x_1, x_2, \cdots, x_N),
\end{aligned}$$

so that $FG(f_1, f_2, \cdots, f_N) = (f_1, f_2, \cdots, f_N)$ and hence $F = G^{-1}$. From above two hands, we know F is one to one mapping from $(X^N, \|\cdot\|_*)^*$ onto $((X, \|\cdot\|)^*)^N$.

Next, we prove F is also a linear isomorphism. For any $f, g \in (X^N, \|\cdot\|_*)^*$ and $\alpha, \beta \in R = (-\infty, +\infty)$, we have,

$$\begin{aligned} F(\alpha f + \beta g)(x_1, x_2, \cdots, x_N) &= ((\alpha f + \beta g)_1, (\alpha f + \beta g)_2, \cdots, (\alpha f + \beta g)_N)(x_1, x_2, \cdots, x_N) \\ &= (\alpha(f_1, f_2, \cdots, f_N) + \beta(g_1, g_2, \cdots, g_N))(x_1, x_2, \cdots, x_N), \\ &= (\alpha Ff + \beta Fg)(x_1, x_2, \cdots, x_N), \end{aligned}$$

so that

$$F(\alpha f + \beta g) = \alpha Ff + \beta Fg.$$

In what follows, we prove $\|Ff\| = \|f\|$ for all $f \in (X^N, \|\cdot\|_*)^*$. From the definition of F, we know,

$$Ff(x_1, x_2, \cdots, x_N) = \sum_{i=1}^{N} f_i(x_i) = f(x_1, x_2, \cdots, x_N),$$

for all $(x_1, x_2, \cdots, x_N) \in X^N = \Pi_{i=1}^N X$. We also know,

$$\|(x_1, x_2, \cdots, x_N)\|_* = \sqrt{\sum_{i=1}^{N} \|x_i\|^2} = \|(x_1, x_2, \cdots, x_N)\|, \quad \text{(in product of norm)}.$$

From above two hands, we know $\|Ff\| = \|f\|$ for all $f \in (X^N, \|\cdot\|_*)^*$. This completes the proof. □

Question 2.3 (Open question). *For any $f \in (X^N, \|\cdot\|_*)^*$, the $(f_1, f_2, \cdots, f_N)$ is defined by above. Is $\|f\| = \sqrt{\sum_{i=1}^N \|f_i\|^2}$ right?*

Lemma 2.4. *Let X be a Banach space with the norm $\|\cdot\|$. Let $J : X \to 2^{X^*}$ be the duality mapping defined by*

$$Jx = \{f \in X^* : f(x) = \|x\|^2 = \|f\|^2\}, \quad \forall\, x \in X.$$

For any $y = (y_1, y_2, \cdots, y_N) \in X^N$, we consider the following functional

$$\langle x, J_* y\rangle_* = \sum_{i=1}^{N} \langle x_i, Jy_i\rangle, \quad \forall\, x = (x_1, x_2, \cdots, x_N) \in X^N.$$

Then the following conclusions hold:

(1) *for any given $y \in X^N$, $J_* y$ is a linear continuous functional defined on X^N and its functional value at x is $\langle x, J_* y\rangle_*$;*

(2) *for all $x \in X^N$, $\langle x, J_* x\rangle = \|x\|_*^2 = \|J_* x\|^2$.*

Proof.

(1) For any $x = (x_1, x_2, \cdots, x_N)$, $z = (z_1, z_2, \cdots, z_N) \in X^N$ and $\alpha, \beta \in R = (-\infty, +\infty)$, we have that

$$\begin{aligned} \langle \alpha x + \beta z, Jy\rangle_* &= \sum_{i=1}^{N} \langle \alpha x_i + \beta z_i, Jy_i\rangle, \\ &= \sum_{i=1}^{N} (\alpha\langle x_i, Jy_i\rangle + \beta\langle z_i, Jy_i\rangle) \end{aligned}$$

$$\begin{aligned}&= \alpha \sum_{i=1}^{N} \langle x_i, Jy_i\rangle + \beta \sum_{i=1}^{N} \langle x_i, Jy_i\rangle \\ &= \alpha\langle x, J_* y\rangle + \beta\langle z, J_* y\rangle.\end{aligned}$$

That is, the functional $J_* y$ is linear. Next, we prove $J_* y$ is continuous. It is obvious that

$$\lim_{x\to 0}\langle x, J_* y\rangle_* = \lim_{x\to 0}\sum_{i=1}^{N}\langle x_i, Jy_i\rangle = 0.$$

Hence $J_* y$ is continuous.

(2) From the definition of $J_* y$, we have for all $y = (y_1, y_2, \cdots, y_N) \in X^N$ that

$$\langle y, J_* y\rangle_* = \sum_{i=1}^{N}\langle x_i, Jx_i\rangle = \sum_{i=1}^{N}\|y_i\|^2 = \|y\|_*^2,$$

and hence

$$\|J_* y\|\|y\|_* \geqslant \langle y, J_* y\rangle_* = \|y\|_*^2.$$

That is, $\|J_* y\| \geqslant \|y\|_*$. On the other hand, we have by using Cauchy-Schwartz inequality that

$$\begin{aligned}\frac{|\langle x, J_* y\rangle|}{\|x\|_*} &= \frac{\sum_{i=1}^{N}\langle x_i, Jy_i\rangle}{\sqrt{\sum_{i=1}^{N}\|x_i\|^2}} \\ &\leqslant \frac{\sum_{i=1}^{N}\|x_i\|\|y_i\|}{\sqrt{\sum_{i=1}^{N}\|x_i\|^2}} \\ &\leqslant \sqrt{\sum_{i=1}^{N}\|y_i\|^2} \\ &= \|y\|_*,\end{aligned}$$

for all $x = (x_1, x_2, \cdots, x_N) \neq 0$. Hence $\|J_* y\| \leqslant \|y\|_*$. From above results, we know that,

$$\langle y, J_* y\rangle_* = \|J_* y\| = \|y\|_*,$$

for all $y \in X^N$. This completes the proof. □

Remark 2.5. From above Lemma 2.4, we know that,

$$J_* : X^N \to (X^N)^*,$$

is namely a branch of the normalized duality mapping $J_N : X^N \to 2^{(X^N)^*}$ defined by

$$J_N(x) = \{f \in (X^N)^* : \langle f, x\rangle = \|f\|^2 = \|x\|^2\},$$

for all $x \in X^N$. That is, $J_*(x) \in J_N(x)$, for all $x \in X^N$.

Question 2.6 (Open question)**.** *Let X be a Banach space with the single-valued normalized duality mapping J. Is J_N single-valued?*

3. Cartesian product of reflexive Banach space with Opial's condition

Theorem 3.1. *Let X be a reflexive Banach space with the norm $\|\cdot\|$, $X^N = X \times X \times \cdots \times X$ be Cartesian product space of X. Let*

$$\|x\|_* = \sqrt{\sum_{i=1}^{N}\|x_i\|^2}, \quad \forall\, x = (x_1, x_2, \cdots, x_N) \in X^N.$$

Then $(X^N, \|\cdot\|_*)$ is a reflexive Banach space.

Proof. We only need to prove the reflexivity of $(X^N, \|\cdot\|_*)$. Let J be the natural embedded mapping from X into its secondary conjugate space X^{**}. For any $x = (x_1, x_2, \cdots, x_N) \in X^N$, we define a functional J_x^* on Banach space $(X^N, \|\cdot\|_*)^* (= ((X, \|\cdot\|)^*)^N)$ as follows

$$\begin{aligned} J_x^*(f) &= J_x^*(f_1, f_2, \cdots, f_N) \\ &= \sum_{i=1}^{N} f_i(x_i), \end{aligned}$$

for all $f = (f_1, f_2, \cdots, f_N) \in (X^N, \|\cdot\|_*)^*$, where $f_i \in (X, \|\cdot\|)^*,\ i = 1, 2, \cdots, N$. Observe that

$$\begin{aligned} J_x^*(f+g) &= J_x^*(f_1+g_1, f_2+g_2, \cdots, f_N+g_N) \\ &= \sum_{i=1}^{N} (f_i+g_i)(x_i) \\ &= \sum_{i=1}^{N} (f_i)(x_i) + \sum_{i=1}^{N} (g_i)(x_i) \\ &= J_x^*(f) + J_x^*(g), \end{aligned}$$

and

$$\begin{aligned} J_x^*(\alpha f) &= J_x^*(\alpha f_1 + \alpha f_2 + \cdots + \alpha f_N) \\ &= \sum_{i=1}^{N} (\alpha f_i)(x_i) \\ &= \alpha \sum_{i=1}^{N} (f_i)(x_i) \\ &= \alpha J_x^*(f), \end{aligned}$$

for all

$$f = (f_1, f_2, \cdots, f_N),\ g = (g_1, g_2, \cdots, g_N) \in (X^N, \|\cdot\|_*)^* = ((X, \|\cdot\|)^*)^N,$$

and $\alpha \in R$, where $f_i, g_i \in (X, \|\cdot\|)^*,\ i = 1, 2, \cdots, N$. Then J_x^* is a linear functional for all $x \in (X^N, \|\cdot\|_*)$. On the other hand, by Hölder inequality, we have

$$\begin{aligned} |J_x^*(f)| &= |J_x^*(f_1 + f_2 + \cdots + f_N)| \\ &= |\sum_{i=1}^{N} (f_i)(x_i)| \\ &\leqslant \sum_{i=1}^{N} |(f_i)(x_i)| \\ &\leqslant \sum_{i=1}^{N} |(f_i)||(x_i)| \\ &\leqslant \sqrt{\sum_{i=1}^{N} |f_i|^2} \sqrt{\sum_{i=1}^{N} |x_i|^2}, \end{aligned} \tag{3.1}$$

so that, the functional J_x^* is continuous for all $x \in (X^N, \|\cdot\|_*)$.

From the above conclusion, we can define a mapping J^* from Banach space $(X^N, \|\cdot\|_*)$ into its secondary conjugate space $(X^N, \|\cdot\|_*)^{**}$ by the relation $J^* : x \mapsto J_x^*$ for all $x \in (X^N, \|\cdot\|_*)$. Next, we are going to prove that, the mapping J^* is the natural embedded mapping from Banach space $(X^N, \|\cdot\|_*)$ into its secondary conjugate space $(X^N, \|\cdot\|_*)^{**}$.

Firstly, by using Hahn-Banach theorem, we know, $x, y \in (X^N, \|\cdot\|_*)$, $x \neq y$ implies that $J^*_x \neq J^*_y$. Secondly, for any $x, y \in (X^N, \|\cdot\|_*)$, $\alpha, \beta \in R$, we have that

$$\begin{aligned} J^*_{\alpha x+\beta y}(f) &= f(\alpha x+\beta y) \\ &= \alpha f(x)+\beta f(y) \\ &= \alpha J^*_x(f)+\beta J^*_y(f), \end{aligned}$$

for all $f \in (X^N, \|\cdot\|_*)^*$. Hence

$$J^*_{\alpha x+\beta y} = \alpha J^*_x+\beta J^*_y.$$

Thirdly, from the above inequality (3.1), we have that, $|J^*_x| \leqslant |x|$ for all $x \in (X^N, \|\cdot\|_*)$. On the other hand, by using Hahn-Banach theorem, for any $x \in (X^N, \|\cdot\|_*)$, there exists a $f_0 \in (X^N, \|\cdot\|_*)^*$ such that $|f_0| = 1$, $f_0(x) = |x|$, hence $|J^*_x| \geqslant |J^*_x(f_0)| = |f_0(x)| = |x|$. Therefore, $|J^*_x| = |x|$ for all $x \in (X^N, \|\cdot\|_*)$. That is, the mapping J^* is the natural embedded mapping from Banach space $(X^N, \|\cdot\|_*)$ into its secondary conjugate space $(X^N, \|\cdot\|_*)^{**}$.

Finally, it is easy to see that the mapping J^* can be also represented as

$$J^* = (J, J, \cdots, J),$$

and

$$J^*_x = (J_{x_1}, J_{x_2}, \cdots, J_{x_N}),$$

for all $x = (x_1, x_2, \cdots, x_N) \in X^N$. From the reflexivity of Banach space X, we know $J(X) = X^{**}$, so that $J^*((X^N, \|\cdot\|_*)) = (X^N, \|\cdot\|_*)^{**}$. This completes the proof of the reflexivity of Banach space $(X^N, \|\cdot\|_*)$. □

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space which satisfies Opial's condition. Let $X^N = X \times X \times \cdots \times X$ be Cartesian product space of X. Let*

$$\|x\|_* = \sqrt{\sum_{i=1}^N \|x_i\|^2}, \quad \forall\, x = (x_1, x_2, \cdots, x_N) \in X^N.$$

Then $(X^N, \|\cdot\|_*)$ satisfies Opial's condition.

Proof. Let

$$x_n = (x_{1,n}, x_{2,n}, \cdots, x_{N,n}), \quad n = 1, 2, 3, \cdots$$

be a sequence which converges weakly to a point

$$x = (x_1, x_2, \cdots, x_N),$$

in Banach space $(X^N, \|\cdot\|_*)$. From the process of proof of Theorem 3.1, we know that $\{x_{i,n}\}$ converges weakly to x_i for all $i = 1, 2, \cdots, N$. Since $(X, \|\cdot\|)$ satisfies Opial's condition, we have that

$$\limsup_{n\to\infty} \|x_{i,n} - x_i\| < \limsup_{n\to\infty} \|x_{i,n} - y_i\|,$$

for any

$$y = (y_1, y_2, \cdots, y_N) \in (X^N, \|\cdot\|_*),$$

and $x \neq y$ which implies that

$$\limsup_{n\to\infty} \|x_n - x\|_* = \limsup_{n\to\infty} \sqrt{\sum_{i=1}^N \|x_{i,n} - x_i\|}$$

$$< \limsup_{n\to\infty} \sqrt{\sum_{i=1}^{N} \|x_{i,n} - y_i\|}$$
$$= \limsup_{n\to\infty} \|x_n - y\|_*.$$

Then $(X^N, \|\cdot\|_*)$ satisfies Opial's condition. This completes the proof. □

4. N-variables nonlinear mappings in normed spaces

Definition 4.1. Let X be a smooth Banach space with the norm $\|\cdot\|$. Let $T : X^N \to X$ be an N-variables mapping.

(1) T is said to be nonexpansive, if
$$\|T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\| \leqslant \|x_i - y_i\|, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(2) T is said to be pseudo-contractive, if
$$\langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle \leqslant \|x_i - y_i\|^2, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(3) T is said to be strongly pseudo-contractive, if there exists a constant $k \in (0, 1)$ such that
$$\langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle \leqslant k\|x_i - y_i\|^2, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(4) T is said to be monotone, if
$$\langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle \geqslant 0, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(5) T is said to be β-strongly monotone, if
$$\langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle \geqslant \beta\|x_i - y_i\|^2, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(6) T is said to be α-inverse-strongly strong monotone, if
$$\langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle \geqslant \alpha\|Tx_i - Ty_i\|^2, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N),\ (y_1, y_2, \cdots, y_N) \in X^N$;

(7) T is said to be γ-strongly positive, if
$$\langle T(x_1, x_2, \cdots, x_N), J(x_1, x_2, \cdots, x_N)\rangle \geqslant \gamma\|x_i\|^2, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N) \in X^N$;

(8) T is said to be k-strictly pseudo-contractive, if for all $i = 1, 2, 3, \cdots, N$,
$$\|T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\|^2 \leqslant \|x_i - y_i\|^2 + k\|(I - T)(x) - (I - T)(y)\|^2,$$
for any $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots, y_N) \in X^N$, where $k \in [0, 1)$ is a constant.

Remark 4.2. T is pseudo-contractive, if and only if for all $i = 1, 2, 3, \cdots, N$,

$$\|T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\|^2 \leqslant \|x_i - y_i\|^2 + \|(I-T)(x) - (I-T)(y)\|^2,$$

for any $(x_1, x_2, \cdots, x_N), (y_1, y_2, \cdots, y_N) \in X^N$.

Conclusion 4.3. *Let T be an N-variables pseudo-contractive mapping from a smooth Banach space X into itself. Let $T^* : X^N \to X^N$ be a mapping defined by*

$$T^*(x_1, x_2, \cdots, x_N) = (T(x_1, x_2, \cdots, x_N), T(x_1, x_2, \cdots, x_N), \cdots, T(x_1, x_2, \cdots, x_N)),$$

for any $(x_1, x_2, \cdots, x_N) \in X^N$. Then T^* is a pseudo-contractive mapping from Banach space X^N into itself.

Proof. From Lemma 2.4, we have for all $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$ that

$$\begin{aligned}
&\langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*((x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N))\rangle_* \\
&\qquad = \langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*(x_1 - y_1, x_2 - y_2, \cdots, x_N - y_N)\rangle_* \\
&\qquad = \sum_{i=1}^{N} \langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle\rangle \\
&\qquad \leqslant \sum_{i=1}^{N} \|x_i - y_i\|^2 \\
&\qquad = \|(x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N)\|_*^2.
\end{aligned}$$

From above inequality, we know that T^* is a pseudo-contractive mapping from Banach space $(X^N, \|\cdot\|_*)$ into itself. This completes the proof. □

Conclusion 4.4. *Let T be an N-variables monotone mapping from a smooth Banach space X into itself. Let $T^* : X^N \to X^N$ be a mapping defined by*

$$T^*(x_1, x_2, \cdots, x_N) = (T(x_1, x_2, \cdots, x_N), T(x_1, x_2, \cdots, x_N), \cdots, T(x_1, x_2, \cdots, x_N)),$$

for any $(x_1, x_2, \cdots, x_N) \in X^N$. Then T^* is a monotone mapping from Banach space X^N into itself.

Proof. From Lemma 2.4, we have for all $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$ that

$$\begin{aligned}
&\langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*((x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N))\rangle_* \\
&\qquad = \langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*(x_1 - y_1, x_2 - y_2, \cdots, x_N - y_N)\rangle_* \\
&\qquad = \sum_{i=1}^{N} \langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N),\ J(x_i - y_i)\rangle\rangle \geqslant 0.
\end{aligned}$$

From above inequality, we know that T^* is a monotone mapping from Banach space $(X^N, \|\cdot\|_*)$ into itself. This completes the proof. □

5. N fixed point operator equations with N-pseudo-contractive mapping

Theorem 5.1. *Let T be an N-variables continuous strongly pseudo-contractive mapping from a smooth Banach space X into itself. Then the system of N fixed point operator equation:*

$$\begin{cases}
T(x_{1,1}, x_{1,2}, \cdots, x_{1,N}) = x_1, \\
T(x_{2,1}, x_{2,2}, \cdots, x_{2,N}) = x_2, \\
T(x_{3,1}, x_{3,2}, \cdots, x_{3,N}) = x_3, \\
\qquad \vdots \\
T(x_{i,1}, x_{i,2}, \cdots, x_{i,N}) = x_i, \\
\qquad \vdots \\
T(x_{N,1}, x_{N,2}, \cdots, x_{N,N}) = x_N,
\end{cases} \tag{5.1}$$

has a unique solution $x^* = (x_1^*, x_2^*, \cdots, x_N^*)$, *where*

$$x_{i,1}, x_{i,2}, \cdots, x_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$

and

$$x_{1,j}, x_{2,j}, \cdots, x_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$

are the permutations of elements $x_1, x_2, x_3, \cdots, x_n$.

Proof. Let $T^* : X^N \to X^N$ be a mapping defined by

$$T^*(x_1, x_2, \cdots, x_N) = (T(x_{1,1}, x_{1,2}, \cdots, x_{1,N}), T(x_{2,1}, x_{2,2}, \cdots, x_{2,N}), \cdots, T(x_{N,1}, x_{N,2}, \cdots, x_{N,N})),$$

for any $(x_1, x_2, \cdots, x_N) \in X^N$. From Lemma 2.4, we have that

$$\begin{aligned}\langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ & J_*((x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N))\rangle_* \\ &= \langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*(x_1 - y_1, x_2 - y_2, \cdots, x_N - y_N)\rangle_* \\ &= \sum_{i=1}^{N} \langle T(x_{i,1}, x_{i,2}, \cdots, x_{i,N}) - T(y_{i,1}, y_{i,2}, \cdots, y_{i,N}),\ J(x_i - y_i)\rangle\rangle \\ &\leqslant \sum_{i=1}^{N} k\|x_i - y_i\|^2 \\ &= k\|(x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N)\|_*^2,\end{aligned}$$

for all $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$. From above inequality we know that T^* is a k-strongly pseudo-contractive mapping from Banach space $(X^N, \|\cdot\|_*)$ into itself, where $k \in (0, 1)$ is a constant. By using Deimling's theorem, the k-strongly pseudo-contractive mapping T^* has a unique fixed point $x^* = (x_1^*, x_2^*, \cdots, x_N^*) \in X^N$ such that

$$T^*((x_1^*, x_2^*, \cdots, x_N^*)) = (x_1^*, x_2^*, \cdots, x_N^*).$$

That is,

$$\begin{cases} T(x_{1,1}^*, x_{1,2}^*, \cdots, x_{1,N}^*) = x_1^*, \\ T(x_{2,1}^*, x_{2,2}^*, \cdots, x_{2,N}^*) = x_2^*, \\ T(x_{3,1}^*, x_{3,2}^*, \cdots, x_{3,N}^*) = x_3^*, \\ \quad\vdots \\ T(x_{i,1}^*, x_{i,2}^*, \cdots, x_{i,N}^*) = x_i^*, \\ \quad\vdots \\ T(x_{N,1}^*, x_{N,2}^*, \cdots, x_{N,N}^*) = x_N^*, \end{cases} \tag{5.2}$$

where

$$x_{i,1}^*, x_{i,2}^*, \cdots, x_{i,N}^*, \quad i = 1, 2, 3, \cdots, N,$$

and

$$x_{1,j}^*, x_{2,j}^*, \cdots, x_{N,j}^*, \quad j = 1, 2, 3, \cdots, N,$$

are the permutations of elements $x_1^*, x_2^*, x_3^*, \cdots, x_n^*$, which is consistent to the permutations:

$$x_{i,1}, x_{i,2}, \cdots, x_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$

and

$$x_{1,j}, x_{2,j}, \cdots, x_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$

in the system of operator equations (5.1). From (5.2), we know that $x^* = (x_1^*, x_2^*, x_3^*, \cdots, x_n^*)$ is a unique solution of (5.1). This completes the proof.

□

Theorem 5.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space which satisfying Opial's condition. Let T be an N-variables continuous pseudo-contractive mapping from X into itself. Assume the range $R(T)$ is bounded. Then the system of N fixed point operator equation:*

$$\begin{cases} T(x_{1,1}, x_{1,2}, \cdots, x_{1,N}) = x_1, \\ T(x_{2,1}, x_{2,2}, \cdots, x_{2,N}) = x_2, \\ T(x_{3,1}, x_{3,2}, \cdots, x_{3,N}) = x_3, \\ \quad\vdots \\ T(x_{i,1}, x_{i,2}, \cdots, x_{i,N}) = x_i, \\ \quad\vdots \\ T(x_{N,1}, x_{N,2}, \cdots, x_{N,N}) = x_N, \end{cases} \tag{5.3}$$

has at least one solution $x^* = (x_1^*, x_2^*, \cdots, x_N^*)$, where

$$x_{i,1}, x_{i,2}, \cdots, x_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$

and

$$x_{1,j}, x_{2,j}, \cdots, x_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$

are the permutations of elements $x_1, x_2, x_3, \cdots, x_n$.

Proof. Let $T^* : X^N \to X^N$ be a mapping defined by

$$T^*(x_1, x_2, \cdots, x_N) = (T(x_{1,1}, x_{1,2}, \cdots, x_{1,N}), T(x_{2,1}, x_{2,2}, \cdots, x_{2,N}), \cdots, T(x_{N,1}, x_{N,2}, \cdots, x_{N,N})),$$

for any $(x_1, x_2, \cdots, x_N) \in X^N$. From Lemma 2.4, we have that

$$\begin{aligned} &\langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*((x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N))\rangle_* \\ &\qquad = \langle T^*(x_1, x_2, \cdots, x_N) - T^*(y_1, y_2, \cdots, y_N),\ J_*(x_1 - y_1, x_2 - y_2, \cdots, x_N - y_N)\rangle_* \\ &\qquad = \sum_{i=1}^{N} \langle T(x_{i,1}, x_{i,2}, \cdots, x_{i,N}) - T(y_{i,1}, y_{i,2}, \cdots, y_{i,N}),\ J(x_i - y_i)\rangle\rangle \\ &\qquad \leqslant \sum_{i=1}^{N} \|x_i - y_i\|^2 \\ &\qquad = \|(x_1, x_2, \cdots, x_N) - (y_1, y_2, \cdots, y_N)\|_*^2, \end{aligned}$$

for all $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$. From above inequality, we know that T^* is a pseudo-contractive mapping from Banach space $(X^N, \|\cdot\|_*)$ into itself. Since T is continuous, so that T^* is also continuous. Since the range $R(T)$ is bounded in X, so that the range $R(T^*)$ is also bounded in X^N. Therefore T^* is strong invariant. From Theorem 3.1 and Theorem 3.2, we know that $(X^N, \|\cdot\|_*)$ is a reflexive Banach space which satisfying Opial's condition. By using Theorem 1.5, the continuous pseudo-contractive mapping T^* has at least one fixed point $x^* = (x_1^*, x_2^*, \cdots, x_N^*) \in X^N$ such that

$$T^*((x_1^*, x_2^*, \cdots, x_N^*)) = (x_1^*, x_2^*, \cdots, x_N^*).$$

That is,

$$\begin{cases} T(x_{1,1}^*, x_{1,2}^*, \cdots, x_{1,N}^*) = x_1^*, \\ T(x_{2,1}^*, x_{2,2}^*, \cdots, x_{2,N}^*) = x_2^*, \\ T(x_{3,1}^*, x_{3,2}^*, \cdots, x_{3,N}^*) = x_3^*, \\ \quad\vdots \\ T(x_{i,1}^*, x_{i,2}^*, \cdots, x_{i,N}^*) = x_i^*, \\ \quad\vdots \\ T(x_{N,1}^*, x_{N,2}^*, \cdots, x_{N,N}^*) = x_N^*, \end{cases} \tag{5.4}$$

where
$$x^*_{i,1}, x^*_{i,2}, \cdots, x^*_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$
and
$$x^*_{1,j}, x^*_{2,j}, \cdots, x^*_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$
are the permutations of elements $x^*_1, x^*_2, x^*_3, \cdots, x^*_n$, which is consistent to the permutations:
$$x_{i,1}, x_{i,2}, \cdots, x_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$
and
$$x_{1,j}, x_{2,j}, \cdots, x_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$
in the system of operator equations (5.3). From (5.4), we know that, $x^* = (x^*_1, x^*_2, x^*_3, \cdots, x^*_n)$ is a solution of (5.3). This completes the proof. □

Theorem 5.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space which satisfying Opial's condition. Let T be an N-variables nonexpansive mapping from X into itself. Assume the range $R(T)$ is bounded. Then the system of N fixed point operator equation:*
$$\begin{cases} T(x_{1,1}, x_{1,2}, \cdots, x_{1,N}) = x_1, \\ T(x_{2,1}, x_{2,2}, \cdots, x_{2,N}) = x_2, \\ T(x_{3,1}, x_{3,2}, \cdots, x_{3,N}) = x_3, \\ \quad\vdots \\ T(x_{i,1}, x_{i,2}, \cdots, x_{i,N}) = x_i, \\ \quad\vdots \\ T(x_{N,1}, x_{N,2}, \cdots, x_{N,N}) = x_N, \end{cases}$$
has at least one solution $x^* = (x^*_1, x^*_2, \cdots, x^*_N)$, where
$$x_{i,1}, x_{i,2}, \cdots, x_{i,N}, \quad i = 1, 2, 3, \cdots, N,$$
and
$$x_{1,j}, x_{2,j}, \cdots, x_{N,j}, \quad j = 1, 2, 3, \cdots, N,$$
are the permutations of elements $x_1, x_2, x_3, \cdots, x_n$.

Proof. We only need to check that, every N-variables nonexpansive mapping must be N-variables continuous pseudo-contractive mapping. In fact that if T is an N-variables nonexpansive mapping, then
$$\|T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\| \leqslant \|x_i - y_i\|, \quad i = 1, 2, 3, \cdots, N,$$
for any $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$. In this case, we have that
$$\begin{aligned} \langle T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\ J(x_i - y_i)\rangle &\leqslant \|T(x_1, x_2, \cdots, x_N) - T(y_1, y_2, \cdots, y_N)\|\|x_i - y_i\| \\ &\leqslant \|x_i - y_i\|^2, \quad i = 1, 2, 3, \cdots, N, \end{aligned}$$
for any $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in X^N$. So that, T is an N-variables pseudo-contractive mapping. It is obvious that T is also continuous. This completes the proof. □

Acknowledgment

This project is supported by the major project of Hebei North University under grant No. ZD201304.

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