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Lyapunov-type inequalities for fractional quasilinear problems via variational methods

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Abstract

In this paper, by variational methods, some Lyapunov-type inequalities are established for fractional quasilinear problems involving left and right Riemann-Liouville fractional derivative operators. To the authors' knowledge, this is the first work, where Lyapunov-type inequalities for fractional boundary value problems are investigated by using variational methods. As an application of the obtained inequalities, we extend the notion of generalized eigenvalues to a fractional quasilinear system, and we derive some geometric properties of the fractional generalized spectrum.

Keywords: Lyapunov inequality, fractional derivative, variational method, fractional quasilinear system, fractional generalized eigenvalues.
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1. Introduction

In order to study the stability of solutions of second order differential equations, Liapounoff [14] established the following result, which provides a necessary condition for the existence of a nontrivial solution of Hill's equation under Dirichlet boundary conditions.

Theorem 1.1. *If the boundary value problem*

$$\begin{cases} u''(t) + q(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{1.1}$$

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Inequality (1.1) is known as "Lyapunov inequality". Such inequality has several applications in various problems in connection with differential equations, including oscillation theory, asymptotic theory, eigenvalue problems, disconjugacy, etc. For more details on Lyapunov-type inequalities and their applications, we refer the reader to the monographs [4, 16].

In [9], Elbert extended inequality (1.1) to the one-dimensional p-Laplacian equation. More precisely, he proved that, if u is a nontrivial solution of the problem

$$\begin{cases} (|u'|^{p-2}u')' + q(t)|u|^{p-2}u = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.2}$$

where $1 < p < \infty$ and $q \in L^1(a, b)$, then

$$\int_a^b |q(t)|\, dt > \frac{2^p}{(b-a)^{p-1}}. \tag{1.3}$$

Observe that for $p = 2$, (1.3) reduces to (1.1).

In [8], Nápoli and Pinasco considered the quasilinear system of resonant type

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = f(t)|u(t)|^{\mu-2}|v(t)|^{\nu}u(t), \\ -(|v'(t)|^{q-2}v'(t))' = g(t)|u(t)|^{\mu}|v(t)|^{\nu-2}v(t), \end{cases} \tag{1.4}$$

on the interval (a, b), with Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0. \tag{1.5}$$

Under the assumptions $p, q > 1$, $f, g \in L^1(a, b)$, $f, g \geqslant 0$, $\alpha, \beta \geqslant 0$, and

$$\frac{\mu}{p} + \frac{\nu}{q} = 1,$$

it was proved (see [8, Theorem 1.5]) that, if (1.4)-(1.5) has a nontrivial solution, then

$$2^{\mu+\nu} \leqslant (b-a)^{\frac{\mu}{p'}+\frac{\nu}{q'}} \left(\int_a^b f(s)\, ds\right)^{\frac{\mu}{p}} \left(\int_a^b g(s)\, ds\right)^{\frac{\nu}{q}}. \tag{1.6}$$

Some nice applications to generalized eigenvalues are also presented in [8]. Inequality (1.6) was extended to more general problems by different authors (see [1, 5, 6, 20, 22] and references therein).

On the other hand, due to the positive impact of fractional calculus on several applied sciences, several authors investigated Lyapunov type inequalities for various classes of fractional boundary value problems. The first work in this direction is due to Ferreira [10], where he generalized Theorem 1.1 for boundary value problems in which the classical derivative u'' is replaced by a Riemann-Liouville fractional derivative. The same author [11] obtained another generalization of Theorem 1.1 for fractional boundary value problems involving Caputo fractional derivative. The basic idea used in both cited works consists in transforming the fractional boundary value problem into an equivalent integral form and then find the maximum of the modulus of its Green's function. Following Ferreira, many other Lyapunov-type inequalities were established for different fractional boundary value problems. In this direction, we refer the reader to [2, 7, 13, 15, 18, 21] and references therein.

The main drawback of the Green's function method is that the Green's function might be very difficult to study. Moreover, in some situations, it is not easy to compute such function. In order to avoid such problems, we suggest in this paper a variational technique for fractional boundary value problems.

In this work, we deal with fractional quasilinear problems involving left and right Riemann-Liouville fractional derivative operators. More precisely, we consider the fractional boundary value problem

$$\frac{{}_tD_b^{\alpha}\left(|{}_aD_t^{\alpha}u|^{p-2}{}_aD_t^{\alpha}u\right) + {}_aD_t^{\alpha}\left(|{}_tD_b^{\alpha}u|^{p-2}{}_tD_b^{\alpha}u\right)}{2} = q(t)|u|^{p-2}u, \tag{1.7}$$

for a.e. $t \in [a, b]$, under Dirichlet boundary conditions

$$u(a) = u(b) = 0. \tag{1.8}$$

Here, ${}_aD_t^\alpha$ denotes the left Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$, and ${}_tD_b^\alpha$ denotes the right Riemann-Liouville fractional derivative of order α. Note that for $\alpha = 1$, problem (1.7)-(1.8) reduces to the p-Laplacian problem (1.2). Using variational techniques, after introducing the adequate functional spaces, Lyapunov-type inequalities are established for the fractional boundary value problem (1.7)-(1.8).

Next, we extend the above study to the case of a fractional quasilinear system. More precisely, we investigate the quasilinear system

$$\begin{cases} \dfrac{{}_tD_b^\alpha\left(|{}_aD_t^\alpha u|^{p-2}{}_aD_t^\alpha u\right) + {}_aD_t^\alpha\left(|{}_tD_b^\alpha u|^{p-2}{}_tD_b^\alpha u\right)}{2} = f(t)|u(t)|^{\mu-2}|v(t)|^{\nu}u(t), \\ \dfrac{{}_tD_b^\beta\left(|{}_aD_t^\beta v|^{q-2}{}_aD_t^\beta v\right) + {}_aD_t^\beta\left(|{}_tD_b^\beta v|^{q-2}{}_tD_b^\beta v\right)}{2} = g(t)|u(t)|^{\mu}|v(t)|^{\nu-2}v(t), \end{cases} \tag{1.9}$$

for a.e. $t \in (a, b)$, under Dirichlet boundary conditions (1.5). Via variational methods, Lyapunov-type inequalities are established for (1.9)-(1.5). The obtained inequalities are applied to fractional generalized eigenvalue problems, and some Protter's type results are obtained for the generalized fractional spectrum.

The paper is organized as follows: In Section 2, we recall some basic concepts on fractional calculus. The main references used in this section are [3, 12, 19]. In order to prove Lyapunov-type inequalities for the considered problems using a variational method, we need the introduction of an appropriate space of functions. In Section 3, such a space is introduced and some preliminaries results are proved. In Section 4, Lyapunov-type inequalities are established for the fractional quasilinear problem (1.7)-(1.8). In Section 5, we are concerned with the fractional quasilinear system (1.9)-(1.5). Via a variational technique, Lyapunov-type inequalities are established for the considered system. As an application of the obtained inequalities, we extend in Section 6 the notion of generalized eigenvalues introduced by Protter [17] to the fractional case, and we derive some geometric properties of the generalized spectrum.

2. Reminder about fractional calculus

In this section, we recall some basic definitions and properties of fractional calculus which are used further in this paper. For more details on fractional calculus, we refer the reader to [3, 19, 23], and references therein.

Let $(a, b) \in \mathbb{R}^2$ with $a < b$.

2.1. *Some functional spaces*

For any $1 \leqslant p < \infty$, $L^p(a, b)$ denotes the classical Lebesgue space of p-integrable real-valued functions, endowed with its usual norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}, \quad f \in L^p(a, b).$$

By $C[a, b]$, we denote the set of real-valued and continuous functions on $[a, b]$, endowed with its usual norm

$$\|f\|_\infty = \max\{|f(t)| : a \leqslant t \leqslant b\}, \quad f \in C[a, b].$$

We denote by $AC[a, b]$ the space of real-valued and absolutely continuous functions on $[a, b]$. It is known that $f \in AC[a, b]$ if and only if there exist constant $c \in \mathbb{R}$ and a function $\varphi \in L^1(a, b)$ such that

$$f(t) = c + \int_a^t \varphi(s)\,ds, \quad t \in [a, b].$$

Consequently, if $f \in AC[a, b]$, then

$$\begin{cases} f(a) = c, \\ \dfrac{df}{dt}(t) = \varphi(t), \quad \text{a.e. } t \in [a, b]. \end{cases}$$

Let $\alpha \in (0, 1]$ be fixed. By $AC_a^\alpha[a, b]$, we denote the set of all functions $f : [a, b] \to \mathbb{R}$ having the representation

$$f(t) = c(t-a)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)}{(t-s)^{1-\alpha}}\, ds, \quad \text{a.e. } t \in [a, b],$$

with $c \in \mathbb{R}$ and $\varphi \in L^1(a, b)$. Here, Γ denotes Euler's Gamma function. It can be easily seen that $AC_a^1[a, b] = AC[a, b]$. Similarly, we define the space $AC_b^\alpha[a, b]$ as the set of functions $f : [a, b] \to \mathbb{R}$ having the representation

$$f(t) = d(b-t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_t^b \frac{\psi(s)}{(s-t)^{1-\alpha}}\, ds, \quad \text{a.e. } t \in [a, b],$$

with $d \in \mathbb{R}$ and $\psi \in L^1(a, b)$.

Lemma 2.1 ([3]). *We have*

$$AC[a, b] \subset AC_a^\alpha[a, b] \cap AC_b^\alpha[a, b].$$

2.2. Fractional operators

Definition 2.2 ([19]). The left and right fractional integral, in the sense of Riamann-Liouville, of order $\alpha > 0$ of $u \in L^1(a, b)$, are given by

$${}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(s)}{(t-s)^{1-\alpha}}\, ds, \quad \text{a.e. } t \in [a, b],$$

and

$${}_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{u(s)}{(s-t)^{1-\alpha}}\, ds, \quad \text{a.e. } t \in [a, b],$$

respectively.

For the following lemma, we refer to [19] for property (2.1), and [12] for property (2.2).

Lemma 2.3. *If $\alpha > 0$ and $1 \leqslant p < \infty$, then*

$${}_aI_t^\alpha u \in L^p(a, b) \quad \textit{for all } u \in L^p(a, b). \tag{2.1}$$

If $\alpha > 0$, $1 \leqslant p < \infty$, and $\alpha p > 1$, then

$${}_aI_t^\alpha u \in C[a, b] \quad \textit{for all } u \in L^p(a, b). \tag{2.2}$$

Remark 2.4. Analogous properties hold true for the right-sided integral.

Lemma 2.5 ([19]). *The left and right Riemann-Liouville fractional integral operators have the properties of a semigroup, i.e., for all $\alpha, \beta > 0$,*

$${}_aI_t^\alpha({}_aI_t^\beta u(t)) = {}_aI_t^{\alpha+\beta} u(t) \ \textit{ and } \ {}_tI_b^\alpha({}_tI_b^\beta u(t)) = {}_tI_b^{\alpha+\beta} u(t)$$

in any point $t \in [a, b]$ for $u \in C[a, b]$, and for almost every point $t \in [a, b]$ if $u \in L^1(a, b)$.

Definition 2.6 ([3]). We say that $u \in L^1(a, b)$ possesses a left-sided Riemann-Liouville derivative ${}_aD_t^\alpha u$ of order $\alpha \in (0, 1)$ if the function ${}_aI_t^{1-\alpha} u$ has an absolutely continuous representative. In this case, ${}_aI_t^{1-\alpha} u$ is identified to its absolutely continuous representative and ${}_aD_t^\alpha u$ is defined by

$${}_aD_t^\alpha u(t) = \frac{d}{dt}\left({}_aI_t^{1-\alpha} u(t)\right), \quad \text{a.e. } t \in [a, b].$$

Definition 2.7 ([3]). We say that $u \in L^1(a,b)$ possesses a right-sided Riemann-Liouville derivative ${}_tD_b^\alpha u$ of order $\alpha \in (0,1)$ if the function ${}_tI_b^{1-\alpha}u$ has an absolutely continuous representative. In this case, ${}_tI_b^{1-\alpha}u$ is identified to its absolutely continuous representative and ${}_tD_b^\alpha u$ is defined by

$$ {}_tD_b^\alpha u(t) = -\frac{d}{dt}\left({}_tI_b^{1-\alpha}u(t)\right), \quad \text{a.e. } t \in [a,b]. $$

From [19], if $u \in AC[a,b]$, then ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$ are defined for almost every point in $[a,b]$. However, Bourdin and Idczak [3] obtained recently a necessary and sufficient condition for the existence of Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$. More precisely, they proved the following results.

Lemma 2.8. *Let $\alpha \in (0,1)$ and $u \in L^1(a,b)$. Then ${}_aD_t^\alpha u(t)$ exists almost everywhere on $[a,b]$ if and only if $u \in AC_a^\alpha[a,b]$, that is, u has the following representation:*

$$ u(t) = c(t-a)^{\alpha-1} + {}_aI_t^\alpha \varphi(t), \quad \textit{a.e. } t \in [a,b], $$

where $c \in \mathbb{R}$ and $\varphi \in L^1(a,b)$. In this case, we have

$$ {}_aD_t^\alpha u(t) = \varphi(t), \quad \textit{a.e. } t \in [a,b]. $$

Lemma 2.9. *Let $\alpha \in (0,1)$ and $u \in L^1(a,b)$. Then ${}_tD_b^\alpha u(t)$ exists almost everywhere on $[a,b]$ if and only if $u \in AC_b^\alpha[a,b]$, that is, u has the following representation:*

$$ u(t) = d(b-t)^{\alpha-1} + {}_tI_b^\alpha \psi(t), \quad \textit{a.e. } t \in [a,b], $$

where $d \in \mathbb{R}$ and $\psi \in L^1(a,b)$. In this case, we have

$$ {}_tD_b^\alpha u(t) = \psi(t), \quad \textit{a.e. } t \in [a,b]. $$

Lemma 2.10 ([19]). *Let $\alpha \in (0,1)$. Then the following equality holds:*

$$ {}_aD_t^\alpha({}_aI_t^\alpha u) = u, \quad u \in L^1(a,b). $$

Now, we recall the formula of integration by parts involving Riemann-Liouville fractional derivatives. At first, we need to introduce some functional spaces.

Definition 2.11. For every $\alpha \in (0,1)$ and every $1 \leqslant p < \infty$, we denote by $AC_a^{\alpha,p}[a,b]$ the functional space defined by

$$ AC_a^{\alpha,p}[a,b] = \{u \in L^1(a,b) : {}_aD_t^\alpha u \in L^p(a,b)\}. $$

Definition 2.12. For every $\alpha \in (0,1)$ and every $1 \leqslant p < \infty$, we denote by $AC_b^{\alpha,p}[a,b]$ the functional space defined by

$$ AC_b^{\alpha,p}[a,b] = \{u \in L^1(a,b) : {}_tD_b^\alpha u \in L^p(a,b)\}. $$

From [3], we have the following formula of integration by parts.

Lemma 2.13. *If $0 < \frac{1}{p} < \alpha < 1$ and $0 < \frac{1}{r} < \alpha < 1$, then*

$$ \int_a^b ({}_aD_t^\alpha u)(t)\cdot v(t)\,dt = \int_a^b u(t)\cdot({}_tD_b^\alpha v)(t)\,dt + u(b)({}_tI_b^{1-\alpha}v)(b) - v(a)({}_aI_t^{1-\alpha}u)(b), $$

for all $u \in AC_a^{\alpha,p}[a,b]$ and $v \in AC_b^{\alpha,r}[a,b]$.

3. Preliminaries results

In this section, we prove some preliminaries results which are used further in this paper.

Lemma 3.1. *Suppose that $0 < \frac{1}{p} < \alpha < 1$. Then*

$$ {}_aI_t^\alpha({}_aD_t^\alpha u(t)) = u(t), \; t \in [a,b], \; u(a) = 0, $$

for any $u \in AC_a^{\alpha,p}[a,b] \cap C[a,b]$.

Proof. Let $u \in AC_a^{\alpha,p}[a,b] \cap C[a,b]$. From Lemma 2.8, we have

$$ {}_aI_t^{\alpha}({}_aD_t^{\alpha}u(t)) - u(t) = c(t-a)^{\alpha-1}, \quad \text{a.e. } t \in [a,b], $$

where $c \in \mathbb{R}$. On the other hand, by Lemma 2.3, since ${}_aD_t^{\alpha}u \in L^p(a,b)$ and $u \in C[a,b]$, then ${}_aI_t^{\alpha}({}_aD_t^{\alpha}u) - u \in C[a,b]$. Therefore, we have $c = 0$, and

$$ {}_aI_t^{\alpha}({}_aD_t^{\alpha}u(t)) = u(t), \ t \in [a,b]. $$

Since $u \in C[a,b]$, we have

$$ u(a) = \lim_{t \to a^+} u(t) = \lim_{t \to a^+} \frac{1}{\Gamma(\alpha)} \int_a^t \frac{{}_aD_s^{\alpha}u(s)}{(t-s)^{1-\alpha}}\, ds. $$

Using Hölder's inequality, we have

$$ \left| \int_a^t \frac{{}_aD_s^{\alpha}u(s)}{(t-s)^{1-\alpha}}\, ds \right| \leqslant \left(\int_a^b |{}_aD_s^{\alpha}u(s)|^p\, ds \right)^{\frac{1}{p}} \left(\int_a^t (t-s)^{\frac{p(\alpha-1)}{p-1}}\, ds \right)^{\frac{p-1}{p}} = \|{}_aD_t^{\alpha}u\|_p \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} (t-a)^{\alpha - \frac{1}{p}}. $$

Passing to the limit as $t \to a^+$, we obtain

$$ \lim_{t \to a^+} \int_a^t \frac{{}_aD_s^{\alpha}u(s)}{(t-s)^{1-\alpha}}\, ds = 0, $$

which yields $u(a) = 0$. □

Similarly, using Lemma 2.9 and Remark 2.4, we obtain the following result.

Lemma 3.2. *Suppose that* $0 < \frac{1}{p} < \alpha < 1$. *Then*

$$ {}_tI_b^{\alpha}({}_tD_b^{\alpha}u(t)) = u(t), \ t \in [a,b], \ u(b) = 0, $$

for any $u \in AC_b^{\alpha,p}[a,b] \cap C[a,b]$.

Suppose that $0 < \frac{1}{p} < \alpha < 1$, and consider the functional space $\chi^{\alpha,p}[a,b]$ defined by

$$ \chi^{\alpha,p}[a,b] = AC_a^{\alpha,p}[a,b] \cap AC_b^{\alpha,p}[a,b] \cap C[a,b]. $$

For $u \in \chi^{\alpha,p}[a,b]$, we denote

$$ \|u\|_{\alpha,p}^p = \|{}_aD_t^{\alpha}u\|_p^p + \|{}_tD_b^{\alpha}u\|_p^p. $$

Lemma 3.3. *Suppose that* $0 < \frac{1}{p} < \alpha < 1$. *Then,* $(\chi^{\alpha,p}[a,b], \|\cdot\|_{\alpha,p})$ *is a Banach space.*

Proof. At first, observe that $\|\cdot\|_{\alpha,p}$ is a norm in $\chi^{\alpha,p}[a,b]$. Indeed, if $\|u\|_{\alpha,p} = 0$ for some $u \in \chi^{\alpha,p}[a,b]$, then $\|{}_aD_t^{\alpha}u\|_p = 0$. Therefore, by Lemma 3.1, we obtain $\|u\|_p = 0$. The other properties of a norm can be easily checked. Now, let $\{u_n\}$ be a Cauchy sequence in $\chi^{\alpha,p}[a,b]$. Then $\{{}_aD_t^{\alpha}u_n\}$ and $\{{}_tD_b^{\alpha}u_n\}$ are Cauchy sequences in $L^p(a,b)$. Therefore, there exists $(f,g) \in L^p(a,b) \times L^p(a,b)$ such that

$$ \lim_{n\to\infty} \|{}_aD_t^{\alpha}u_n - f\|_p = \lim_{n\to\infty} \|{}_tD_b^{\alpha}u_n - g\|_p = 0. $$

Since ${}_aI_t^{\alpha}, {}_tI_b^{\alpha} : L^p(a,b) \to L^p(a,b)$ are bounded operators (see [19]), it follows from Lemmas 3.1 and 3.2 that

$$ \lim_{n\to\infty} \|u_n - {}_aI_t^{\alpha}f\|_p = \lim_{n\to\infty} \|u_n - {}_tI_b^{\alpha}g\|_p = 0. $$

Therefore, by the uniqueness of the limit, we can put

$$u = {}_aI_t^\alpha f = {}_tI_b^\alpha g \in L^p(a, b).$$

Moreover, by property (2.2), since $f \in L^p(a, b)$, we have $u \in C[a, b]$. Using Lemma 2.10, we have ${}_aD_t^\alpha u = f \in L^p(a, b)$ and ${}_tD_b^\alpha u = g \in L^p(a, b)$. Hence $u \in \chi^{\alpha,p}[a, b]$ and

$$\lim_{n\to\infty} \|u_n - u\|_{\alpha,p} = 0.$$

Consequently, $(\chi^{\alpha,p}[a, b], \|\cdot\|_{\alpha,p})$ is a Banach space. □

Remark 3.4. Note that from Lemmas 3.1 and 3.2, we have

$$u(a) = u(b) = 0, \quad u \in \chi^{\alpha,p}[a, b].$$

4. Lyapunov type inequalities for problem (1.7)-(1.8)

In this section, Lyapunov-type inequalities are established for the fractional boundary value problem (1.7)-(1.8) under the assumptions

$$0 < \frac{1}{p} < \alpha < 1 \quad \text{and} \quad q \in L^1(a, b). \tag{4.1}$$

We define two kinds of solutions for problem (1.7)-(1.8).

Definition 4.1 (Strong solution)**.** A function $u : [a, b] \to \mathbb{R}$ is said to be a strong solution of (1.7)-(1.8) if

(i) $u \in \chi^{\alpha,p}[a, b]$;

(ii) ${}_tD_b^\alpha \left(|{}_aD_t^\alpha u|^{p-2}\,{}_aD_t^\alpha u\right) \in L^p(a, b)$;

(iii) ${}_aD_t^\alpha \left(|{}_tD_b^\alpha u|^{p-2}\,{}_tD_b^\alpha u\right) \in L^p(a, b)$;

(iv) u satisfies (1.7) for a.e. $t \in [a, b]$.

Definition 4.2 (Weak solution)**.** A function $u \in \chi^{\alpha,p}[a, b]$ is said to be a weak solution of (1.7)-(1.8) if

$$\begin{aligned}\int_a^b |{}_aD_t^\alpha u(t)|^{p-2}\,{}_aD_t^\alpha u(t)\,{}_aD_t^\alpha v(t)\,dt + \int_a^b |{}_tD_b^\alpha u(t)|^{p-2}\,{}_tD_b^\alpha u(t)\,{}_tD_b^\alpha v(t)\,dt \\ = 2\int_a^b q(t)|u(t)|^{p-2}u(t)v(t)\,dt,\end{aligned} \tag{4.2}$$

for every $v \in \chi^{\alpha,p}[a, b]$.

Proposition 4.3. *If $u : [a, b] \to \mathbb{R}$ is a strong solution of* (1.7)-(1.8)*, then u is a weak solution of* (1.7)-(1.8).

Proof. Let $u : [a, b] \to \mathbb{R}$ be a strong solution of (1.7)-(1.8). Multiplying (1.7) by $v \in \chi^{\alpha,p}[a, b]$, we obtain

$$\begin{aligned}\int_a^b {}_tD_b^\alpha \left(|{}_aD_t^\alpha u|^{p-2}\,{}_aD_t^\alpha u\right) v(t)\,dt + \int_a^b {}_aD_t^\alpha \left(|{}_tD_b^\alpha u|^{p-2}\,{}_tD_b^\alpha u\right) v(t)\,dt \\ = 2\int_a^b q(t)|u(t)|^{p-2}u(t)v(t)\,dt.\end{aligned}$$

On the other hand, from (ii), we have

$$\left(|{}_aD_t^\alpha u|^{p-2}\,{}_aD_t^\alpha u\right) \in AC_b^{\alpha,p}[a, b].$$

Similarly, from (iii), we have

$$\left(|{}_tD_b^\alpha u|^{p-2}\,{}_tD_b^\alpha u\right) \in AC_a^{\alpha,p}[a, b].$$

Therefore, by Remark 3.4, and using the formula of integration by parts given by Lemma 2.13, we obtain (4.2), which means that $u \in \chi^{\alpha,p}[a, b]$ is a weak solution of (1.7)-(1.8). □

The main result in this section is given by the following theorem.

Theorem 4.4. *Under assumptions* (4.1), *if* (1.7)-(1.8) *admits a nontrivial weak solution* $u \in \chi^{\alpha,p}[a,b]$ *such that* $|u(c)| = \|u\|_\infty$, $c \in (a,b)$, *then*

$$\int_a^b q^+(s)\, ds \geqslant \left(\frac{2(\alpha p-1)}{p-1}\right)^{p-1} \frac{[\Gamma(\alpha)]^p}{\left((c-a)^{\frac{\alpha p-1}{p-1}} + (b-c)^{\frac{\alpha p-1}{p-1}}\right)^{p-1}}, \tag{4.3}$$

where $q^+(t) = \max\{q(t), 0\}$, *for* $t \in [a,b]$.

Proof. Let $u \in \chi^{\alpha,p}[a,b]$ be a nontrivial weak solution of (1.7)-(1.8). By Lemma 3.1, we have

$$u(c) = {}_aI_c^\alpha({}_aD_c^\alpha u(c)),$$

that is,

$$u(c) = \frac{1}{\Gamma(\alpha)} \int_a^c \frac{{}_aD_s^\alpha u(s)}{(c-s)^{1-\alpha}}\, ds.$$

Then

$$|u(c)| \leqslant \frac{1}{\Gamma(\alpha)} \int_a^c \frac{|{}_aD_s^\alpha u(s)|}{(c-s)^{1-\alpha}}\, ds. \tag{4.4}$$

Similarly, by Lemma 3.2, we have

$$u(c) = {}_cI_b^\alpha({}_cD_b^\alpha u(c)),$$

that is,

$$u(c) = \frac{1}{\Gamma(\alpha)} \int_c^b \frac{{}_sD_b^\alpha u(s)}{(s-c)^{1-\alpha}}\, ds.$$

Then

$$|u(c)| \leqslant \frac{1}{\Gamma(\alpha)} \int_c^b \frac{|{}_sD_b^\alpha u(s)|}{(s-c)^{1-\alpha}}\, ds. \tag{4.5}$$

Next, adding (4.4) to (4.5), we obtain

$$2|u(c)| \leqslant \frac{1}{\Gamma(\alpha)} \left(\int_a^c \frac{|{}_aD_s^\alpha u(s)|}{(c-s)^{1-\alpha}}\, ds + \int_c^b \frac{|{}_sD_b^\alpha u(s)|}{(s-c)^{1-\alpha}}\, ds \right) \leqslant \frac{1}{\Gamma(\alpha)} \int_a^b \frac{\max\left\{|{}_aD_s^\alpha u(s)|, |{}_sD_b^\alpha u(s)|\right\}}{|s-c|^{1-\alpha}}\, ds.$$

Using Hölder's inequality, we obtain

$$2|u(c)| \leqslant \frac{1}{\Gamma(\alpha)} \left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\, ds \right)^{\frac{1}{p'}} \left(\int_a^b \max\{|{}_aD_s^\alpha u(s)|^p, |{}_sD_b^\alpha u(s)|^p\}\, ds \right)^{\frac{1}{p}}, \tag{4.6}$$

where $p' = \frac{p}{p-1}$. On the other hand, taking $v = u$ in (4.2), we obtain

$$\int_a^b |{}_aD_s^\alpha u(s)|^p\, ds + \int_a^b |{}_sD_b^\alpha u(s)|^p\, ds = 2\int_a^b q(s)|u(s)|^p\, ds,$$

which yields

$$\begin{aligned} \int_a^b \max\{|{}_aD_s^\alpha u(s)|^p, |{}_sD_b^\alpha u(s)|^p\}\, ds &\leqslant \int_a^b \left(|{}_aD_s^\alpha u(s)|^p + |{}_sD_b^\alpha u(s)|^p\right) ds \\ &\leqslant 2\int_a^b q^+(s)|u(s)|^p\, ds. \end{aligned} \tag{4.7}$$

Combining (4.6) with (4.7), we obtain

$$2|u(c)| \leqslant \frac{1}{\Gamma(\alpha)}\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\frac{1}{p'}}\left(2\int_a^b q^+(s)|u(s)|^p\,ds\right)^{\frac{1}{p}},$$

which implies that

$$2|u(c)| \leqslant \frac{2^{\frac{1}{p}}}{\Gamma(\alpha)}\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\frac{1}{p'}}\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{p}}|u(c)|.$$

Next, we divide the above inequality by $|u(c)|$ (since u is nontrivial) to get

$$2 \leqslant \frac{2^{\frac{1}{p}}}{\Gamma(\alpha)}\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\frac{1}{p'}}\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{p}},$$

which yields

$$\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{p}} \geqslant \frac{2^{1-\frac{1}{p}}\Gamma(\alpha)}{\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\frac{1}{p'}}}.$$

Note that

$$\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds = \int_a^c \frac{1}{(c-s)^{p'(1-\alpha)}}\,ds + \int_c^b \frac{1}{(s-c)^{p'(1-\alpha)}}\,ds = \frac{p-1}{\alpha p-1}\left((c-a)^{\frac{\alpha p-1}{p-1}} + (b-c)^{\frac{\alpha p-1}{p-1}}\right).$$

Therefore, we have

$$\int_a^b q^+(s)\,ds \geqslant \left(\frac{2(\alpha p-1)}{p-1}\right)^{p-1}\frac{[\Gamma(\alpha)]^p}{\left((c-a)^{\frac{\alpha p-1}{p-1}} + (b-c)^{\frac{\alpha p-1}{p-1}}\right)^{p-1}},$$

which yields the desired result. □

Observe that the function

$$\varphi(s) = (s-a)^{\frac{\alpha p-1}{p-1}} + (b-s)^{\frac{\alpha p-1}{p-1}}, \quad s \in [a,b]$$

has a maximum at the point $s^* = \frac{a+b}{2}$. Therefore,

$$\varphi(c) \leqslant \varphi(s^*) = 2^{\frac{p(1-\alpha)}{p-1}}(b-a)^{\frac{\alpha p-1}{p-1}},$$

which yields

$$\frac{1}{\left((c-a)^{\frac{\alpha p-1}{p-1}} + (b-c)^{\frac{\alpha p-1}{p-1}}\right)^{p-1}} \geqslant \frac{1}{2^{p(1-\alpha)}(b-a)^{\alpha p-1}}. \tag{4.8}$$

Therefore, from (4.3) we deduce the following result.

Corollary 4.5. *Under assumptions (4.1), if (1.7)-(1.8) admits a nontrivial weak solution $u \in \chi^{\alpha,p}[a,b]$, then*

$$\int_a^b q^+(s)\,ds \geqslant \left(\frac{2}{b-a}\right)^{\alpha p-1}\left(\frac{\alpha p-1}{p-1}\right)^{p-1}[\Gamma(\alpha)]^p. \tag{4.9}$$

Remark 4.6. Observe that for $\alpha = 1$, problem (1.7)-(1.8) reduces to the p-Laplacian problem (1.2). Let us suppose that q is a non-negative function. In such case, if (1.2) admits a nontrivial solution, from (1.3) we have

$$\int_a^b q(s)\,ds \geqslant 2\left(\frac{2}{b-a}\right)^{p-1}. \tag{4.10}$$

However, if we take $\alpha = 1$ in (4.9), we obtain

$$\int_a^b q(s)\,ds \geqslant \left(\frac{2}{b-a}\right)^{p-1}. \tag{4.11}$$

It is clear that (4.10) is "better" than (4.11). So, we address the following question to the readers: is it possible to improve inequality (4.9) for $0 < \frac{1}{p} < \alpha < 1$?

5. Lyapunov type inequalities for problem (1.9)-(1.5)

In this section, we are concerned with the fractional quasilinear system (1.9) under Dirichlet boundary conditions (1.5). System (1.9) is investigated under the assumptions:

$$\mu \geqslant 0,\ \nu \geqslant 0,\ 0 < \frac{1}{p} < \alpha < 1,\ 0 < \frac{1}{q} < \beta < 1 \tag{5.1}$$

and

$$\frac{\mu}{p} + \frac{\nu}{q} = 1. \tag{5.2}$$

We suppose also that

$$(f, g) \in L^1(a, b) \times L^1(a, b). \tag{5.3}$$

As in the case of a single equation, we define two kinds of solutions for (1.9)-(1.5).

Definition 5.1 (Strong solution)**.** A pair of functions $(u, v) : [a, b] \to \mathbb{R}^2$ is said to be a strong solution of (1.9)-(1.5) if

(i) $u \in \chi^{\alpha,p}[a, b]$;

(ii) ${}_tD_b^\alpha\left(|{}_aD_t^\alpha u|^{p-2}\,{}_aD_t^\alpha u\right) \in L^p(a, b)$;

(iii) ${}_aD_t^\alpha\left(|{}_tD_b^\alpha u|^{p-2}\,{}_tD_b^\alpha u\right) \in L^p(a, b)$,

and

(i') $v \in \chi^{\beta,q}[a, b]$;

(ii') ${}_tD_b^\beta\left(|{}_aD_t^\beta v|^{q-2}\,{}_aD_t^\beta v\right) \in L^q(a, b)$;

(iii') ${}_aD_t^\beta\left(|{}_tD_b^\beta v|^{q-2}\,{}_tD_b^\beta v\right) \in L^q(a, b)$;

(iv) (u, v) satisfies (1.9) for a.e. $t \in [a, b]$.

Definition 5.2 (Weak solution)**.** A pair of functions $(u, v) \in \chi^{\alpha,p}[a, b] \times \chi^{\beta,q}[a, b]$ is said to be a weak solution of (1.9)-(1.5) if

$$\begin{aligned}\int_a^b |{}_aD_t^\alpha u(t)|^{p-2}\,{}_aD_t^\alpha u(t)\,{}_aD_t^\alpha w_1(t)\,dt + \int_a^b |{}_tD_b^\alpha u(t)|^{p-2}\,{}_tD_b^\alpha u(t)\,{}_tD_b^\alpha w_1(t)\,dt \\ = 2\int_a^b f(t)|u(t)|^{\mu-2}|v(t)|^{\nu} u(t) w_1(t)\,dt,\end{aligned} \tag{5.4}$$

and

$$\int_a^b |{}_aD_t^\beta v(t)|^{q-2}{}_aD_t^\beta v(t)\,{}_aD_t^\beta w_2(t)\,dt + \int_a^b |{}_tD_b^\beta v(t)|^{q-2}{}_tD_b^\beta v(t)\,{}_tD_b^\beta w_2(t)\,dt = 2\int_a^b g(t)|u(t)|^\mu |v(t)|^{\nu-2}v(t)w_2(t)\,dt, \tag{5.5}$$

for any $(w_1, w_2) \in \chi^{\alpha,p}[a,b] \times \chi^{\beta,q}[a,b]$.

As in the case of a single equation, we have the following proposition.

Proposition 5.3. *If the pair of functions $(u,v) : [a,b] \to \mathbb{R}^2$ is a strong solution of (1.9)-(1.5), then (u,v) is a weak solution of (1.9)-(1.5).*

The following Lyapunov-type inequality for (1.9)-(1.5) holds.

Theorem 5.4. *Suppose that assumptions (5.1), (5.2), and (5.3) are satisfied. If (1.9)-(1.5) admits a nontrivial weak solution $(u,v) \in \chi^{\alpha,p}[a,b] \times \chi^{\beta,q}[a,b]$ such that $(u(c), v(d)) = (\|u\|_\infty, \|v\|_\infty)$, $(c,d) \in (a,b) \times (a,b)$, then*

$$\left(\int_a^b f^+(s)\,ds\right)^{\frac{\mu}{p}} \left(\int_a^b g^+(s)\,ds\right)^{\frac{\nu}{q}} \geqslant \frac{2^{\mu+\nu-1}[\Gamma(\alpha)]^\mu[\Gamma(\beta)]^\nu \left(\frac{\alpha p-1}{p-1}\right)^{\left(\frac{p-1}{p}\right)\mu} \left(\frac{\beta q-1}{q-1}\right)^{\left(\frac{q-1}{q}\right)\nu}}{\left((c-a)^{\frac{\alpha p-1}{p-1}} + (b-c)^{\frac{\alpha p-1}{p-1}}\right)^{\left(\frac{p-1}{p}\right)\mu} \left((d-a)^{\frac{\beta q-1}{q-1}} + (b-d)^{\frac{\beta q-1}{q-1}}\right)^{\left(\frac{q-1}{q}\right)\nu}}.$$

Proof. From (4.6), we have

$$2|u(c)| \leqslant \frac{1}{\Gamma(\alpha)} \left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\frac{1}{p'}} \left(\int_a^b \max\{|{}_aD_s^\alpha u(s)|^p, |{}_sD_b^\alpha u(s)|^p\}\,ds\right)^{\frac{1}{p}}, \tag{5.6}$$

where $p' = \frac{p}{p-1}$. Similarly, we have

$$2|v(d)| \leqslant \frac{1}{\Gamma(\beta)} \left(\int_a^b \frac{1}{|s-d|^{q'(1-\beta)}}\,ds\right)^{\frac{1}{q'}} \left(\int_a^b \max\left\{|{}_aD_s^\beta v(s)|^q, |{}_sD_b^\beta v(s)|^q\right\}\,ds\right)^{\frac{1}{q}}, \tag{5.7}$$

where $q' = \frac{q}{q-1}$. Taking $w_1 = u$ in (5.4), we obtain

$$\int_a^b |{}_aD_s^\alpha u(s)|^p\,ds + \int_a^b |{}_sD_b^\alpha u(s)|^p\,ds = 2\int_a^b f(s)|u(s)|^\mu |v(s)|^\nu\,ds,$$

which yields

$$\int_a^b \left(|{}_aD_s^\alpha u(s)|^p + |{}_sD_b^\alpha u(s)|^p\right)\,ds \leqslant 2\int_a^b f^+(s)|u(s)|^\mu |v(s)|^\nu\,ds,$$

and

$$\int_a^b \max\{|{}_aD_s^\alpha u(s)|^p, |{}_sD_b^\alpha u(s)|^p\}\,ds \leqslant 2|u(c)|^\mu |v(d)|^\nu \int_a^b f^+(s)\,ds. \tag{5.8}$$

Similarly, taking $w_2 = v$ in (5.5), we obtain

$$\int_a^b \max\left\{|{}_aD_s^\beta v(s)|^q, |{}_sD_b^\beta v(s)|^q\right\}\,ds \leqslant 2|u(c)|^\mu |v(d)|^\nu \int_a^b g^+(s)\,ds. \tag{5.9}$$

Combining (5.6) with (5.8), we get

$$2^{p-1}|u(c)|^p \leqslant \frac{1}{[\Gamma(\alpha)]^p} \left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{p-1} \left(\int_a^b f^+(s)\,ds\right) |u(c)|^\mu |v(d)|^\nu. \tag{5.10}$$

Similarly combining (5.7) with (5.9), we get

$$2^{q-1}|v(d)|^q \leqslant \frac{1}{[\Gamma(\beta)]^q}\left(\int_a^b \frac{1}{|s-d|^{q'(1-\beta)}}\,ds\right)^{q-1}\left(\int_a^b g^+(s)\,ds\right)|u(c)|^\mu|v(d)|^\nu. \tag{5.11}$$

Raising inequality (5.10) to a power $e_1 > 0$, inequality (5.11) to a power $e_2 > 0$, and multiplying the resulting inequalities, we obtain

$$\begin{aligned}2^{(p-1)e_1+(q-1)e_2} \leqslant{}& \frac{1}{[\Gamma(\alpha)]^{pe_1}[\Gamma(\beta)]^{qe_2}}\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{(p-1)e_1}\left(\int_a^b \frac{1}{|s-d|^{q'(1-\beta)}}\,ds\right)^{(q-1)e_2}\\ &\times\left(\int_a^b f^+(s)\,ds\right)^{e_1}\left(\int_a^b g^+(s)\,ds\right)^{e_2}|u(c)|^{(\mu-p)e_1+\mu e_2}|v(d)|^{\nu e_1+(\nu-q)e_2}.\end{aligned} \tag{5.12}$$

Next, we take (e_1, e_2) any solution of the homogeneous linear system

$$\begin{cases}(\mu-p)e_1+\mu e_2 = 0,\\ \nu e_1+(\nu-q)e_2 = 0.\end{cases}$$

From (5.2), the above system is equivalent to

$$p\nu e_1 = q\mu e_2.$$

Therefore, we may take $(e_1, e_2) = (q\mu, p\nu)$. Hence, from (5.12), we obtain

$$\begin{aligned}2^{\mu+\nu-1} \leqslant{}& \frac{1}{[\Gamma(\alpha)]^{\mu}[\Gamma(\beta)]^{\nu}}\left(\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds\right)^{\left(\frac{p-1}{p}\right)\mu}\left(\int_a^b \frac{1}{|s-d|^{q'(1-\beta)}}\,ds\right)^{\left(\frac{q-1}{q}\right)\nu}\\ &\times\left(\int_a^b f^+(s)\,ds\right)^{\frac{\mu}{p}}\left(\int_a^b g^+(s)\,ds\right)^{\frac{\nu}{q}}.\end{aligned}$$

Now, using the equalities

$$\int_a^b \frac{1}{|s-c|^{p'(1-\alpha)}}\,ds = \frac{p-1}{\alpha p-1}\left((c-a)^{\frac{\alpha p-1}{p-1}}+(b-c)^{\frac{\alpha p-1}{p-1}}\right)$$

and

$$\int_a^b \frac{1}{|s-d|^{q'(1-\beta)}}\,ds = \frac{q-1}{\beta q-1}\left((d-a)^{\frac{\beta q-1}{q-1}}+(b-d)^{\frac{\beta q-1}{q-1}}\right),$$

we get

$$\left(\int_a^b f^+(s)\,ds\right)^{\frac{\mu}{p}}\left(\int_a^b g^+(s)\,ds\right)^{\frac{\nu}{q}} \geqslant \frac{2^{\mu+\nu-1}[\Gamma(\alpha)]^{\mu}[\Gamma(\beta)]^{\nu}\left(\frac{\alpha p-1}{p-1}\right)^{\left(\frac{p-1}{p}\right)\mu}\left(\frac{\beta q-1}{q-1}\right)^{\left(\frac{q-1}{q}\right)\nu}}{\left((c-a)^{\frac{\alpha p-1}{p-1}}+(b-c)^{\frac{\alpha p-1}{p-1}}\right)^{\left(\frac{p-1}{p}\right)\mu}\left((d-a)^{\frac{\beta q-1}{q-1}}+(b-d)^{\frac{\beta q-1}{q-1}}\right)^{\left(\frac{q-1}{q}\right)\nu}},$$

which yields the desired result. □

From (4.8), we have

$$\frac{1}{\left((c-a)^{\frac{\alpha p-1}{p-1}}+(b-c)^{\frac{\alpha p-1}{p-1}}\right)^{p-1}} \geqslant \frac{1}{2^{p(1-\alpha)}(b-a)^{\alpha p-1}}$$

and

$$\frac{1}{\left((d-a)^{\frac{\beta q-1}{q-1}}+(b-d)^{\frac{\beta q-1}{q-1}}\right)^{q-1}} \geqslant \frac{1}{2^{q(1-\beta)}(b-a)^{\beta q-1}}.$$

Therefore, from Theorem 5.4, we deduce the following result.

Corollary 5.5. *Suppose that assumptions* (5.1), (5.2), *and* (5.3) *are satisfied. If* (1.9)-(1.5) *admits a nontrivial weak solution* $(u,v)\in \chi^{\alpha,p}[a,b]\times\chi^{\beta,q}[a,b]$, *then*

$$\begin{aligned}&\left(\int_a^b f^+(s)\,ds\right)^{\frac{\mu}{p}}\left(\int_a^b g^+(s)\,ds\right)^{\frac{\nu}{q}}\\ &\geqslant \left(\frac{2}{b-a}\right)^{\alpha\mu+\beta\nu-1}\left(\frac{\alpha p-1}{p-1}\right)^{\left(\frac{p-1}{p}\right)\mu}\left(\frac{\beta q-1}{q-1}\right)^{\left(\frac{q-1}{q}\right)\nu}[\Gamma(\alpha)]^{\mu}[\Gamma(\beta)]^{\nu}.\end{aligned}\tag{5.13}$$

Remark 5.6. Taking $(\mu,\nu)=(p,0)$ and $u=v$ in (1.9)-(1.5), inequality (5.13) reduces to inequality (4.9).

Remark 5.7. Observe that for $(\alpha,\beta)=(1,1)$, (1.9)-(1.5) reduces to the quasilinear system (1.4)-(1.5). Let us suppose that f and g are non-negative functions. From (1.6), if (1.4)-(1.5) has a nontrivial solution, then

$$\left(\int_a^b f(s)\,ds\right)^{\frac{\mu}{p}}\left(\int_a^b g(s)\,ds\right)^{\frac{\nu}{q}} \geqslant 2\left(\frac{2}{b-a}\right)^{\mu+\nu-1}.\tag{5.14}$$

On the other hand, taking $(\alpha,\beta)=(1,1)$ in (5.13), we obtain

$$\left(\int_a^b f(s)\,ds\right)^{\frac{\mu}{p}}\left(\int_a^b g(s)\,ds\right)^{\frac{\nu}{q}} \geqslant \left(\frac{2}{b-a}\right)^{\mu+\nu-1}.\tag{5.15}$$

It is clear that inequality (5.14) is "better" than inequality (5.15). So, we address the following question to readers: is it possible to improve inequality (5.13) for $0<\frac{1}{p}<\alpha<1$ and $0<\frac{1}{q}<\beta<1$?

6. Fractional generalized eigenvalues

The concept of generalized eigenvalues was introduced by Protter [17] for a system of linear elliptic operators. The first work dealing with generalized eigenvalues for p-Laplacian systems is due to Nápoli and Pinasco [8]. Inspired by that work, we present in this section some applications to fractional generalized eigenvalues related to problem (1.9)-(1.5).

Let us consider the fractional generalized eigenvalue problem

$$\begin{cases}\dfrac{{}_tD_b^{\alpha}\left(|{}_aD_t^{\alpha}u|^{p-2}{}_aD_t^{\alpha}u\right)+{}_aD_t^{\alpha}\left(|{}_tD_b^{\alpha}u|^{p-2}{}_tD_b^{\alpha}u\right)}{2}=\lambda\mu w(t)|u(t)|^{\mu-2}|v(t)|^{\nu}u(t),\\[2ex] \dfrac{{}_tD_b^{\beta}\left(|{}_aD_t^{\beta}v|^{q-2}{}_aD_t^{\beta}v\right)+{}_aD_t^{\beta}\left(|{}_tD_b^{\beta}v|^{q-2}{}_tD_b^{\beta}v\right)}{2}=\gamma\nu w(t)|u(t)|^{\mu}|v(t)|^{\nu-2}v(t)\end{cases}\tag{6.1}$$

for a.e. $t\in(a,b)$, under Dirichlet boundary conditions (1.5). If (6.1)-(1.5) admits a nontrivial weak solution (u,v), we say that (λ,γ) is a generalized eigenvalue of (6.1)-(1.5). The set of generalized eigenvalues is called generalized spectrum, and it is denoted by σ. We assume that

$$0<\frac{1}{p}<\alpha<1,\ 0<\frac{1}{q}<\beta<1, w\geqslant 0,\ w\in L^1(a,b),\ w\not\equiv 0,$$

and the non-negative parameters μ, ν satisfy (5.2).

In this section, some Protter's type results for the generalized spectrum σ are obtained.

The following result provides lower bounds of the generalized eigenvalues of (6.1)-(1.5).

Theorem 6.1. *Let* (λ, γ) *be a generalized eigenvalue of* (6.1)-(1.5). *Then*

$$\gamma \geqslant h(\lambda), \tag{6.2}$$

where $h : (0, \infty) \to (0, \infty)$ *is the function defined by*

$$h(r) = \frac{1}{\nu}\left(\frac{C}{(\mu r)^{\frac{\mu}{p}} \int_a^b w(s)\, ds}\right)^{\frac{q}{\nu}}, \quad r > 0 \tag{6.3}$$

with

$$C = \left(\frac{2}{b-a}\right)^{\alpha\mu+\beta\nu-1} \left(\frac{\alpha p - 1}{p-1}\right)^{\left(\frac{p-1}{p}\right)\mu} \left(\frac{\beta q - 1}{q-1}\right)^{\left(\frac{q-1}{q}\right)\nu} [\Gamma(\alpha)]^{\mu}[\Gamma(\beta)]^{\nu}. \tag{6.4}$$

Proof. Let $(\lambda, \gamma) \in \sigma$. Then (6.1)-(1.5) admits a nontrivial weak solution $(u, v) \in \chi^{\alpha,p}[a, b] \times \chi^{\beta,q}[a, b]$. Applying Corollary 5.5 with $f = \lambda\mu w$ and $g = \gamma\nu w$, and using condition (5.2), we obtain

$$(\lambda\mu)^{\frac{\mu}{p}}(\gamma\nu)^{\frac{\nu}{q}} \int_a^b w(s)\, ds \geqslant C,$$

where C is given by (6.4). Therefore, we have

$$\gamma^{\frac{\nu}{q}} \geqslant \frac{C}{\nu^{\frac{\nu}{q}}(\lambda\mu)^{\frac{\mu}{p}} \int_a^b w(s)\, ds},$$

which yields

$$\gamma \geqslant \frac{1}{\nu}\left(\frac{C}{(\lambda\mu)^{\frac{\mu}{p}} \int_a^b w(s)\, ds}\right)^{\frac{q}{\nu}} = h(\lambda),$$

where h is the function defined by (6.3). □

Now, we deduce from Theorem 6.1 the following Protter's type results for the generalized spectrum.

Corollary 6.2. *There exists a constant* $c_{a,b} > 0$ *that depends on* a *and* b *such that no point of the generalized spectrum* σ *is contained in the ball* $B(0, c_{a,b})$, *where*

$$B(0, c_{a,b}) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_\infty < c_{a,b}\right\},$$

and $\|\cdot\|_\infty$ *is the Chebyshev norm in* $\mathbb{R}^2$.

Proof. Let $(\lambda, \mu) \in \sigma$. From (6.2), we have

$$\lambda^{\frac{\mu}{p}}\gamma^{\frac{\nu}{q}} \geqslant \frac{C}{\mu^{\frac{\mu}{p}}\nu^{\frac{\nu}{q}} \int_a^b w(s)\, ds}. \tag{6.5}$$

On the other hand, using condition (5.2), we have

$$\lambda^{\frac{\mu}{p}}\gamma^{\frac{\nu}{q}} \leqslant \|(\lambda, \mu)\|_\infty^{\frac{\mu}{p}+\frac{\nu}{q}} = \|(\lambda, \mu)\|_\infty.$$

Therefore, we obtain

$$\|(\lambda, \mu)\|_\infty \geqslant c_{a,b},$$

where

$$c_{a,b} = \frac{C}{\mu^{\frac{\mu}{p}}\nu^{\frac{\nu}{q}} \int_a^b w(s)\, ds}.$$

The proof is finished. □

Corollary 6.3. *Let* (λ, γ) *be fixed. There exists an interval* J *of sufficiently small measure, such that, if* $I = [a, b] \subset J$*, then there are no nontrivial solutions of* (6.1)-(1.5).

Proof. Suppose that (6.1)-(1.5) admits a nontrivial solution. Since

$$\frac{C}{\int_a^b w(s)\,ds} \to +\infty \text{ as } b - a \to 0^+,$$

where C is the constant defined by (6.4), there exists $\delta > 0$ such that

$$b - a < \delta \implies \frac{C}{\int_a^b w(x)\,dx} > (\mu\lambda)^{\frac{\mu}{p}} (\nu\gamma)^{\frac{\nu}{q}}.$$

Let $J = [a, a + \delta]$. Hence, if $I \subset J$, we have

$$\frac{C}{\mu^{\frac{\mu}{p}} \nu^{\frac{\nu}{q}} \int_a^b w(s)\,ds} > \lambda^{\frac{\mu}{p}} \gamma^{\frac{\nu}{q}},$$

which is a contradiction with (6.5). Therefore, if $I \subset J$, there are no nontrivial solutions of (6.1)-(1.5). □

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