Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2501–2509
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Weak condition for generalized f-weakly Picard mappings on partial metric spaces

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Communicated by P. Kumam

Abstract

Recently, Minak and Altun introduced the notions of multivalued weak contractions and multivalued weakly Picard operators on partial metric spaces. They also obtained two fixed point theorems with the notions of multivalued (δ, L)– weak contractions and multivalued (α, L)-weak contractions. In this paper, we introduce the notion of generalized multivalued (f, α, β)-weak contraction on partial metric spaces. We also establish some coincidence and common fixed point theorems. Our results extend and generalize some well-known common fixed point theorems on partial metric spaces. ©2017 All rights reserved.

Keywords: Partial metric, common fixed point, hybrid maps, weakly Picard operators.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

In recent years, many works on domain theory have been made in order to equip semantics domain with a notion of distance. In particular, Matthews [18] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. They generalized the concept of a metric space in the sense that the self-distance from a point to itself need not be equal to zero. They are useful in modeling partially defined information, which often appears in computer science. In literature [18], the contraction fixed point theorem is extended to partial metric spaces. This highlights an additional feature: the fixed point has self-distance 0, which although trivial in metric spaces can be useful for reasoning about posets found in computer science. In the context of computer science where a computable function can also be proved to be a contraction, the partial metric extension of the contraction fixed point theorem can be used to prove that the unique fixed point, which is the programs output, will be totally computed [18]. Further applications of partial metrics to problems in theoretical computer science were discussed in [11, 12, 25, 26, 28, 29].

We start with recalling some basic definitions and lemmas on partial metric spaces. The definition of a partial metric space is given by Matthews [18].

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doi:10.22436/jnsa.010.05.19

Received 2016-12-21

Definition 1.1. Let X be a nonempty set, a function $p : X \times X \to R^+$ is called a partial metric if and only if for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \leqslant p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, z) \leqslant p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p), where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [31, 32].

Each partial metric p on X generates a τ_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X; \varepsilon > 0\}$, where $\{B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

From this fact it immediately follows that a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ with respect to τ_p if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. According to [18], a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ with respect to τ_{p^s} if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_m, x_n). \tag{1.1}$$

Following [18], a sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\mathcal{T}(p)$ to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is easy to see that every closed subset of a complete partial metric space is complete.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to R^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

and

$$p^w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\}, \tag{1.2}$$

are equivalent metric on X.

Lemma 1.2 ([18])**.** *Let (X, p) be a partial metric space.*

(1) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*
(2) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore $\lim_{n\to\infty} p^s(a, x_n) = 0$ if and only if $p(a, a) = \lim_{n\to\infty} p(a, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

In [18], Matthews obtained a partial metric version of the Banach fixed point theorem. Afterward, Acar et al. [1, 2], Altun et al. [4, 5, 7, 8], Karapinar and Erhan [17], Oltra and Valero [21], Romaguera [22, 23] and Valero [31], gave some generalizations of the result of Matthews. Also, Ciric et al. [13], Samet et al. [27] and Shatanawi et al. [30] proved some common fixed point results in partial metric spaces. But so far all of fixed point theorems have been given for single-valued mappings. To prove Nadler's [20] fixed point theorem for multivalued maps on partial metric spaces, Aydi et al. [9] introduced the concept of partial Hausdorff distance a parallel manner to that in the Hausdorff metric in their nice paper. Then, they give some properties of partial Hausdorff distance, some important lemmas and a fundamental fixed point theorem for multivalued mappings. We can find some nice fixed point results for single-valued and multivalued maps on partial metric space in [3, 15, 19, 24, 32].

In the following, we recall the concept of partial Hausdorff distance and some properties: Let (X, p) be partial metric space and $A \subseteq X$, then A is said to be bounded if there exist $x_0 \in X$ and $M \geqslant 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(a, a) + M$. A is closed if and only if $A = \overline{A}$, where

$\overline{A}$ is the closure of A with respect to τ_p (τ_p is the topology induced by p). Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of (X, p). For $A, B \in CB^p(X)$ and $x \in X$, define

$$P(x, A) = \inf\{p(x, a) : a \in A\}, \quad \delta_p(A, B) = \sup\{P(a, B) : a \in A\},$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Lemma 1.3 ([9]). *Let (X, p) be a partial metric space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if $P(x, A) = p(x, x)$.*

Proposition 1.4 ([9]). *Let (X, p) be a partial metric space. For any $A, B, C \in CB_p(X)$, we have the following:*

(1) $\delta_p(A, A) = \sup_{a \in A} p(a, a)$;
(2) $\delta_p(A, A) \leqslant \delta_p(A, B)$;
(3) $\delta_p(A, B) = 0$ *implies* $A \subseteq B$;
(4) $\delta_p(A, B) \leqslant \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 1.5 ([9]). *Let (X, p) be a partial metric space. For any $A, B, C \in CB_p(X)$, we have the following:*

(1) $H_p(A, A) \leqslant H_p(A, B)$;
(2) $H_p(A, B) = H_p(B, A)$;
(3) $H_p(A, B) \leqslant H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

Remark 1.6. An example is given by Minak and Altun in [19] that $H_p(A, A) = H_p(A, B) = H_p(B, A)$, but $A \neq B$. That is, H_p is not a partial metric on $CB_p(X)$. Nevertheless, as shown in [9]. we have the following property:

$$H_p(A, B) = 0 \ \text{ implies } \ A = B.$$

Also, it is easy to see that for all $A \in CB_P(X)$

$$P(a, B) \leqslant \delta_p(A, B) \leqslant H_p(A, B).$$

The following lemma is very important to give fixed point results for multivalued maps on partial metric space.

Lemma 1.7 ([9]). *Let (X, p) be a partial metric space, $A, B \in CB_p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a, b) \leqslant hH_p(A, B)$.*

Lemma 1.7 can be expressed with the following version.

Lemma 1.8 ([10]). *Let (X, p) be a partial metric space, $A, B \in CB_p(X)$ and $\varepsilon > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a, b) \leqslant H_p(A, B) + \varepsilon$.*

Using the partial Hausdorff distance H_p, Aydi et al. [9] proved the following fixed point theorem for multivalued mappings.

Theorem 1.9. *Let (X, p) be a complete partial metric space. If $T : X \to CB_p(X)$ is a mapping such that*

$$H_p(Tx, Ty) \leqslant kp(x, y),$$

for all $x, y \in X$, where $k \in (0, 1)$, then T has a fixed point.

The following theorem is a generalized version of Theorem 1.9, which is given by Altun and Minak in literature [6].

Theorem 1.10. *Let (X, p) be a complete partial metric space and let $T : X \to CB_p(X)$ be a multivalued map. Assume*

$$H_p(Tx, Ty) \leqslant \alpha(p(x, y))p(x, y),$$

for all $x, y \in X$, where α is an $\mathcal{MT}$-function (that is, it satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$). Then T

has a fixed point.

In literature [19], Minak and Altun generalized above theorems as follows:

Theorem 1.11. *Let* (X, p) *be a complete partial metric space and* $T : X \to CB_p(X)$ *be a multivalued map such that*

$$H_p(Tx, Ty) \leqslant kp(x, y) + LP^w(y, Tx),$$

for all $x, y \in X$*, where* $k \in (0, 1)$, $L \geqslant 0$ *and* $P^w(y, Tx) = \inf\{p^w(y, z) : z \in Tx\}$*. Then* T *has a fixed point.*

Theorem 1.12. *Let* (X, p) *be a complete partial metric space and let* $T : X \to CB_p(X)$ *be a multivalued map such that there exist an* $\mathcal{MT}$*-function* α *and a constant* $L \geqslant 0$ *satisfying*

$$H_p(Tx, Ty) \leqslant \alpha(p(x, y))p(x, y) + LP^w(y, Tx),$$

for all $x, y \in X$*. Then* T *has a fixed point.*

Recently, Huang et al. [16] gave two more general results on a partial metric space.

Theorem 1.13. *Let* (X, p) *be a partial metric space,* $f : X \to X$ *and* $T : X \to CB(X)$ *be a multivalued map such that there exist two constants* $\theta \in (0, 1)$ *and* $L \geqslant 0$ *satisfying*

$$H_p(Tx, Ty) \leqslant \theta p(fx, fy) + LP^w(fy, Tx),$$

for all $x, y \in X$ *where* $P^w(fy, Tx) = \inf\{p^w(fy, z) : z \in Tx\}$ *and* p^w *as in* (1.2)*. Suppose* $TX \subset fX$ *and* fX *is a complete subspace of* X*. Then* f *and* T *have a coincidence point* $u \in X$*. Further if* $ffu = fu$*, then* f *and* T *have a common fixed point.*

Theorem 1.14. *Let* (X, p) *be a partial metric space,* $f : X \to X$ *and* $T : X \to CB(X)$ *such that there exist an* $\mathcal{MT}$*-function* α *and a constant* $L \geqslant 0$ *satisfying*

$$H_p(Tx, Ty) \leqslant \alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx),$$

for all $x, y \in X$ *where* $P^w(fy, Tx) = \inf\{p^w(fy, z) : z \in Tx\}$ *and* p^w *as in* (1.2)*. Suppose* $TX \subset fX$ *and* fX *is a complete subspace of* X*. Then* f *and* T *have a coincidence point* $u \in X$*. Further if* $ffu = fu$*, then* f *and* T *have a common fixed point.*

The aim of this paper is to introduce the notion of generalized multivalued (f, α, β)-weak contraction on partial metric space as the parallel manner on metric space. We also establish some coincidence and common fixed point theorems with the notion of generalized multivalued (f, α, β)-weak contraction on partial metric space.

2. Main results

As a departure, let us recall the notion of a hybrid generalized multivalued contraction mapping on partial metric spaces.

Definition 2.1 ([16])**.** Let (X, p) be a partial metric space, $f : X \to X$ and $T : X \to CB_p(X)$ be a multivalued operator. T is said to be multivalued f weakly Picard operator if and only if for each $x \in X$ and $fy \in Tx(y \in X)$, there exists a sequence $\{x_n\}$ in X such that

(1) $x_0 = x, x_1 = y$;
(2) $fx_{n+1} \in Tx_n$ for all $n = 0, 1, 2, \cdots$;
(3) the sequence $\{fx_n\}$ converges to fu where u is the coincidence point of f and T.

Definition 2.2 ([16]). Let $\{x_n\}$ be a sequence in X satisfying condition (1) and (2) in Definition 2.1, then the sequence $O_f(x_0) = \{fx_n : n = 1, 2, \cdots\}$ is said to be an f-orbit of T at x_0.

Definition 2.3. Let (X, p) be a partial metric space, $f : X \to X$ and $T : X \to CB_p(X)$ be a multivalued operator. T is said to be a generalized multivalued f-weakly contraction or a generalized multivalued (f, α, β)-weak contraction if and only if there exist a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$ for every $t \in [0, \infty)$ and a function $\beta : [0, \infty) \to [0, +\infty)$ such that

$$H_p(Tx, Ty) \leqslant \alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx), \tag{2.1}$$

for all $x, y \in X$ where $P^w(fy, Tx) = \inf\{p^w(fy, z) : z \in Tx\}$ and p^w as in (1.2). Especially, if $\beta(x) = L(const.) \geqslant 0$ for every $x \in [0, \infty)$, then T is said to be a generalized multivalued (f, α, L)-weak contraction.

Remark 2.4. Due to the symmetry of p and H_p, in order to check that T is a multivalued (f, α, β)-weak contraction on (X, p), we also have to check the dual of (2.1), that is to check that T verifies

$$H_p(Tx, Ty) \leqslant \alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fx, Ty))P^w(fx, Ty).$$

Now, we give a more general result on a partial metric space. For this we need the following lemma.

Lemma 2.5 ([14]). *Let $\alpha : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}$-function, then the function $\gamma : [0, \infty) \to [0, 1)$ defined as $\gamma(t) = \frac{1+\alpha(t)}{2}$ is also an $\mathcal{MT}$-function.*

Theorem 2.6. *Let (X, p) be a partial metric space, $f : X \to X$ and $T : X \to CB(X)$ be a multivalued (f, α, β)-weak contraction such that $TX \subset fX$. Suppose fX is complete. Then f and T have a coincidence point $u \in X$. Further, if $ffu = fu$, then f and T have a common fixed point.*

Proof. Define a function $\gamma : [0, \infty) \to [0, 1)$ as $\gamma(t) = \frac{1+\alpha(t)}{2}$, then from Lemma 2.5 $\gamma(t)$ is also an $\mathcal{MT}$-function. Let $x, y \in X$ be two arbitrary points with $fx \neq fy$, $u \in Tx$ and $\varepsilon = \frac{1-\alpha(p(fx,fy))}{2} p(fx, fy) > 0$ (note that since $fx \neq fy$ then $p(fx, fy) > 0$), then from Lemma 1.8 we can find $v \in Ty$ such that $p(u, v) \leqslant H_p(Tx, Ty) + \varepsilon$. Therefore, from (2.1) we have

$$\begin{aligned} p(u, v) &\leqslant H_p(Tx, Ty) + \frac{1 - \alpha(p(fx, fy))}{2} p(fx, fy) \\ &\leqslant \alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) + \frac{1 - \alpha(p(fx, fy))}{2} p(fx, fy) \\ &= \frac{1 + \alpha(p(fx, fy))}{2} p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) \\ &= \gamma(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx). \end{aligned} \tag{2.2}$$

Now, let $x_0 \in X$ and $y_0 = fx_0$. Since $Tx_0 \subset fX$, there exists a point $x_1 \in X$ such that $y_1 = f(x_1) \in Tx_0$. If $y_0 = y_1$, i.e., $fx_0 = fx_1$, then $fx_0 \in Tx_0$, that is x_0 is a coincidence point of f and T and so the proof is complete. Let $fx_0 \neq fx_1$, then from (2.2) there exists $y_2 = f(x_2) \in Tx_1$ such that

$$\begin{aligned} p(y_1, y_2) = p(fx_1, fx_2) &\leqslant \gamma(p(fx_0, fx_1))p(fx_0, fx_1) + \beta(P^w(fx_1, Tx_0))P^w(fx_1, Tx_0) \\ &= \gamma(p(fx_0, fx_1))p(fx_0, fx_1). \end{aligned}$$

If $y_1 = y_2$, i.e., $fx_1 = fx_2$, then $fx_1 \in Tx_1$, that is x_1 is a coincidence point of f and T and so the proof is complete. Let $fx_1 \neq fx_2$, then from (2.2) there exists $y_3 = f(x_3) \in Tx_2$ such that

$$\begin{aligned} p(y_2, y_3) = p(fx_2, fx_3) &\leqslant \gamma(p(fx_1, fx_2))p(fx_1, fx_2) + \beta(P^w(fx_2, Tx_1))P^w(fx_2, Tx_1) \\ &= \gamma(p(fx_1, fx_2))p(fx_1, fx_2). \end{aligned}$$

By continuing this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = fx_n \in Tx_{n-1}$ and

$$p(y_n, y_{n+1}) = p(fx_n, fx_{n+1}) \leqslant \gamma(p(fx_{n-1}, fx_n))p(fx_{n-1}, fx_n),$$

for all $n \in N$. Since $\gamma(t) < 1$ for all $t \in [0, \infty)$, then $p(y_n, y_{n+1})$ is a nonincreasing sequence of non-negative real numbers. Hence $p(y_n, y_{n+1})$ converges to some $\lambda \geqslant 0$. Since $\gamma(t)$ is an $\mathcal{MT}$-function, then $\limsup_{s\to t^+} \gamma(s) < 1$ and $\gamma(\lambda) < 1$. Therefore, there exist $r \in [0, 1)$ and $\varepsilon > 0$ such that $\gamma(s) \leqslant r$ for all $s \in [\lambda, \lambda + \varepsilon)$. Since $p(y_n, y_{n+1}) \downarrow \lambda$ we can take $k_0 \in N$ such that $\lambda \leqslant p(y_n, y_{n+1}) \leqslant \lambda + \varepsilon$ for all $n \in N$ with $n \geqslant k_0$.

$$p(y_{n+1}, y_{n+2}) = p(fx_{n+1}, fx_{n+2}) \leqslant \gamma(p(fx_n, fx_{n+1}))p(fx_n, fx_{n+1}) \leqslant rp(fx_n, fx_{n+1}) = rp(y_n, y_{n+1}),$$

for all $n \in N$ with $n \geqslant k_0$, then we have

$$\begin{aligned}\sum_{n=1}^{\infty} p(y_n, y_{n+1}) &\leqslant \sum_{n=1}^{k_0} p(y_n, y_{n+1}) + \sum_{n=k_0+1}^{\infty} p(y_n, y_{n+1}) \\ &= \sum_{n=1}^{k_0} p(y_n, y_{n+1}) + \sum_{n=k_0}^{\infty} p(y_{n+1}, y_{n+2}) \\ &\leqslant \sum_{n=1}^{k_0} p(y_n, y_{n+1}) + \sum_{n=k_0}^{\infty} rp(y_n, y_{n+1}) \\ &\leqslant \sum_{n=1}^{k_0} p(y_n, y_{n+1}) + \sum_{n=1}^{\infty} r^n p(y_{k_0}, y_{k_0+1}) < \infty.\end{aligned}$$

Then for $m, n \in N$ with $m > n$, by omitting the negative term in modified triangular inequality we obtain

$$\begin{aligned}p(y_n, y_m) &\leqslant p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{i=n}^{m-1} p(y_i, y_{i+1}) \\ &\leqslant \sum_{i=n}^{\infty} p(y_i, y_{i+1}) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty} p(y_n, y_m) \to 0$, that is $\{y_n = fx_n\}$ is a Cauchy sequence in (fX, p). Since (fX, p) is complete, (fX, p^s) is also complete by Lemma 1.2 (2). So, there exists a point $u \in X$ such that $fx_n \to fu$ with respect to the metric p^s, that is $\lim_{n\to\infty} p^s(fx_n, fu) = 0$.

And, by (1.1), we have

$$p(fu, fu) = \lim_{n\to\infty} p(fx_n, fu) = \lim_{n,m\to\infty} p(fx_m, fx_n) = 0. \tag{2.3}$$

Now,

$$\begin{aligned}P(fu, Tu) &\leqslant p(fu, fx_{n+1}) + P(fx_{n+1}, Tu) \\ &\leqslant p(fu, fx_{n+1}) + H_p(Tx_n, Tu) \\ &\leqslant p(fu, fx_{n+1}) + \alpha(p(fx_n, fu))p(fx_n, fu) + \beta(P^w(fu, Tx_n))P^w(fu, Tx_n) \\ &\leqslant p(fu, fx_{n+1}) + \alpha(p(fx_n, fu))p(fx_n, fu) + \beta(P^w(fu, Tx_n))p^w(fu, fx_{n+1}) \\ &\leqslant p(fu, fx_{n+1}) + p(fx_n, fu) + \beta(P^w(fu, Tx_n))p^w(fu, fx_{n+1}).\end{aligned}$$

Letting $n \to \infty$ in the above inequality we get (note that p^s and p^w are equivalent metrics) $P(fu, Tu) = 0$. Therefore, from (2.3), we obtain $P(fu, Tu) = p(fu, fu)$. Thus, from Lemma 1.3, we have $fu \in Tu$, since Tu is closed.

Let $z = fu \in Tu$, then $fz = ffu = fu = z$. Using the notion of generalized multivalued (f, α, β)-weak contraction, we get

$$\begin{aligned} H_p(Tu, Tz) &\leqslant \alpha(p(fu, fz))p(fu, fz) + \beta(P^w(fz, Tu))P^w(fz, Tu) \\ &= \alpha(p(fu, fu))p(fu, fu) + \beta(P^w(fu, Tu))P^w(fu, Tu) = 0. \end{aligned}$$

From $P(fz, Tz) = P(fu, Tz) \leqslant H_p(Tu, Tz)$, then $P(fz, Tz) = 0$. Therefore, from (2.3) we obtain $P(fz, Tz) = p(fu, fu) = p(fz, fz)$. Thus, from Lemma 1.3 we have $z = fz \in Tz$, since Tz is closed. Thus f and T have a common fixed point. This completes the proof. □

Remark 2.7. Let $\beta(x) = L(const.) \geqslant 0$, for every $x \in [0, \infty)$ in Theorem 2.6. We get Theorem 1.14.

Remark 2.8. Substituting $f = I$, the identity map on X and $\beta(x) = L(const.) \geqslant 0$, for every $x \in [0, \infty)$ in Theorem 2.6, we get at once Theorem 1.12.

Finally, we introduce an example satisfying the hypotheses of Theorem 2.6 to support the usability of our results. In doing so, we are essentially inspired by Aydi et al. [10].

Example 2.9. Let $X = \{0, 1, 2, 3\}$ be endowed with the partial metric $p : X \times X \to R^+$ defined by

$$p(0,0) = p(1,1) = p(2,2) = 0, \quad p(3,3) = \frac{1}{5}, \quad p(0,1) = p(1,0) = \frac{2}{5}, \quad p(0,2) = p(2,0) = \frac{1}{3},$$
$$p(1,2) = p(2,1) = \frac{2}{3}, \quad p(0,3) = p(3,0) = \frac{1}{2}, \quad p(1,3) = p(3,1) = \frac{3}{5}, \quad p(2,3) = p(3,2) = \frac{7}{10}.$$

Also define the mappings $f : X \to X$ and $T : X \to CB_p(X)$ by

$$fx = \begin{cases} 0 & \text{if } x \in \{0,1\}, \\ 1 & \text{if } x = 2, \\ 2 & \text{if } x = 3, \end{cases} \quad , \quad Tx = \begin{cases} \{0\} & \text{if } x \in \{0,1,2\}, \\ \{1,2\} & \text{if } x = 3, \end{cases}$$

and the $\mathcal{MT}$-function $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \frac{6t}{5+2t^2}$ for any $t \geqslant 0$ and the function $\beta : [0, \infty) \to [0, +\infty)$ by $\beta(t) = 4t$ for any $t \geqslant 0$. Note that Tx is closed and bounded for all $x \in X$ under the given partial metric p. We shall show that (2.1) holds for all $x, y \in X$. We distinguish the following cases:

(1) If $x, y \in \{0, 1, 2\}$, then $H_p(Tx, Ty) = H_p(\{0\}, \{0\}) = 0$ and (2.1) is satisfied obviously.

(2) If $x = 0,\ y = 3$, then

$$\begin{aligned} \alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f0, f3))p(f0, f3) + \beta(P^w(f3, T0))P^w(f3, T0) \\ &= \alpha(p(0,2))p(0,2) + \beta(P^w(2, \{0\}))P^w(2, \{0\}) \\ &= \alpha(\frac{1}{3})\frac{1}{3} + \beta(\frac{1}{3})\frac{1}{3} = \frac{18}{47} + \frac{4}{9} \\ &= \frac{330}{423} \geqslant \frac{2}{5} = H_p(\{0\}, \{1,2\}) = H_p(T0, T3). \end{aligned}$$

(3) If $x = 1,\ y = 3$, then

$$\begin{aligned} \alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f1, f3))p(f1, f3) + \beta(P^w(f3, T1))P^w(f3, T1) \\ &= \alpha(p(0,2))p(0,2) + \beta(P^w(2, \{0\}))P^w(2, \{0\}) \\ &= \alpha(\frac{1}{3})\frac{1}{3} + \beta(\frac{1}{3})\frac{1}{3} = \frac{18}{47} + \frac{4}{9} \\ &= \frac{330}{423} \geqslant \frac{2}{5} = H_p(\{0\}, \{1,2\}) = H_p(T1, T3). \end{aligned}$$

(4) If $x = 2,\ y = 3$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f2, f3))p(f2, f3) + \beta(P^w(f3, T2))P^w(f3, T2)\\ &= \alpha(p(1, 2))p(1, 2) + \beta(P^w(2, \{0\}))P^w(2, \{0\})\\ &= \alpha(\frac{2}{3})\frac{2}{3} + \beta(\frac{1}{3})\frac{1}{3} = \frac{24}{53} + \frac{4}{9}\\ &= \frac{428}{477} \geqslant \frac{2}{5} = H_p(\{0\}, \{1, 2\}) = H_p(T2, T3).\end{aligned}$$

(5) If $x = 3,\ y = 0$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f3, f0))p(f3, f0) + \beta(P^w(f0, T3))P^w(f0, T3)\\ &= \alpha(p(2, 0))p(2, 0) + \beta(P^w(0, \{1, 2\}))P^w(0, \{1, 2\})\\ &= \alpha(\frac{1}{3})\frac{1}{3} + \beta(\frac{1}{3})\frac{1}{3} = \frac{18}{47} + \frac{4}{9}\\ &= \frac{330}{423} \geqslant \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T0).\end{aligned}$$

(6) If $x = 3,\ y = 1$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f3, f1))p(f3, f1) + \beta(P^w(f1, T3))P^w(f1, T3)\\ &= \alpha(p(2, 0))p(2, 0) + \beta(P^w(0, \{1, 2\}))P^w(0, \{1, 2\})\\ &= \alpha(\frac{1}{3})\frac{1}{3} + \beta(\frac{1}{3})\frac{1}{3} = \frac{18}{47} + \frac{4}{9}\\ &= \frac{330}{423} \geqslant \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T1).\end{aligned}$$

(7) If $x = 3,\ y = 2$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + \beta(P^w(fy, Tx))P^w(fy, Tx) &= \alpha(p(f3, f2))p(f3, f2) + \beta(P^w(f2, T3))P^w(f2, T3)\\ &= \alpha(p(2, 1))p(2, 1) + \beta(P^w(1, \{1, 2\}))P^w(1, \{1, 2\})\\ &= \alpha(\frac{2}{3})\frac{2}{3} + 0 = \frac{24}{53}\\ &\geqslant \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T2).\end{aligned}$$

(8) If $x = y = 3$, then $H_p(Tx, Ty) = H_p(\{1, 2\}, \{1, 2\}) = 0$ and (2.1) is satisfied obviously. Thus, all the conditions of Theorem 2.6 are satisfied and $x = 0$ is a common fixed point of f and T in X.

Acknowledgment

The authors thank the editor and the referees for their valuable comments and suggestions. This research has been supported by the National Natural Science Foundation of China (11461043, 11661053) and supported partly by the Provincial Natural Science Foundation of Jiangxi, China (20161BAB201009) and the Science and Technology Project of Educational Commission of Jiangxi Province, China (150013).

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