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Some explicit identities for the modified higher-order degenerate q-Euler polynomials and their zeroes

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Abstract

Recently, Kim et al. [D. S. Kim, T. Kim, Ars Combin., **126** (2016), 435–441], [D. S. Kim, T. Kim, J. Nonlinear Sci. Appl., **9** (2016), 443–451], [T. Kim, D. S. Kim, H.-I. Kwon, Filomat, **30** (2016), 905–912] and [T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, D. V. Dolgy, J. Nonlinear Sci. Appl., **9** (2016), 1077–1082] studied symmetric identities of higher-order degenerate q-Euler polynomials. In this paper, we define the modified higher-order degenerate q-Euler polynomials and give some identities for these polynomials. Also we give numerical investigations of the zeroes of the modified higher-order q-Euler polynomials and the zeroes of the modified higher-order degenerate q-Euler polynomials.

Furthermore, we demonstrate the shapes and zeroes of the modified higher-order q-Euler polynomials and the modified higher-order degenerate q-Euler polynomials by using a computer.

Keywords: Identities of symmetry, modified q-Euler polynomials, modified higher-order degenerate q-Euler polynomials.
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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. We normalized the p-adic norm as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$ and the q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim et al.

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(see [11–13, 18, 20]) to be

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x=\frac{[2]_q}{2}\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)q^x(-1)^x, \tag{1.1}$$

where $[x]_{-q}=\frac{1-(-q)^x}{1+q}$. Note that

$$\lim_{q\to 1}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x=\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) \tag{1.2}$$

is the ordinary fermionic p-adic integral on $\mathbb{Z}_p$ (see [2, 4, 5, 9, 14, 17, 19, 22, 25, 26]). From (1.1), we have

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0), \quad \text{where } f_1(x)=f(x+1). \tag{1.3}$$

From (1.2), we have

$$I_{-1}(f_1)+I_{-1}(f)=2f(0), \quad \text{where } f_1(x)=f(x+1). \tag{1.4}$$

Recall that the Carlitz's q-Euler numbers are defined by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^m d\mu_{-q}(x)=\mathcal{E}_{m,q} \quad (\text{see [10, 12]}).$$

From (1.3) with $f(x)=[x]_q^m$, we can derive

$$q\int_{\mathbb{Z}_p} [x+1]_q^m d\mu_{-q}(x)+\int_{\mathbb{Z}_p} [x]_q^m d\mu_{-q}(x)=\begin{cases}[2]_q, & \text{if } m=0,\\ 0, & \text{if } m>0.\end{cases}$$

We note that

$$[x+1]_q^m=\left(\frac{1-q^{x+1}}{1-q}\right)^m=(1+q[x]_q)^m=\sum_{l=0}^{m}\binom{m}{l}q^l[x]_q^l \tag{1.5}$$

and hence

$$\int_{\mathbb{Z}_p} [x+1]_q^m d\mu_{-q}(x)=\sum_{l=0}^{m}\binom{m}{l}q^l\int_{\mathbb{Z}_p}[x]_q^l d\mu_{-q}(x)=\sum_{l=0}^{m}\binom{m}{l}q^l\mathcal{E}_{l,q}=(q\mathcal{E}_q+1)^m. \tag{1.6}$$

Combining (1.6) and (1.3), the Carlitz's q-Euler numbers $\mathcal{E}_{m,q}$ satisfy as follows:

$$q(q\mathcal{E}_q+1)^m+\mathcal{E}_{m,q}=\begin{cases}[2]_q, & \text{if } m=0,\\ 0, & \text{if } m>0,\end{cases}$$

with the usual convention about replacing $\mathcal{E}_q^m$ by $\mathcal{E}_{m,q}$, (see [1, 3–5, 8]).

Then, the modified q-Euler numbers $E_{m,q}$ are defined by Kim et al. (see [8, 12, 23]) as follows:

$$\int_{\mathbb{Z}_p} [x]_q^m d\mu_{-1}(x)=E_{m,q}.$$

From (1.5), we have

$$\int_{\mathbb{Z}_p} [x+1]_q^m d\mu_{-1}(x)=\sum_{l=0}^{m}\binom{m}{l}q^l\int_{\mathbb{Z}_p}[x]_q^l d\mu_{-1}(x)=\sum_{l=0}^{m}\binom{m}{l}q^l E_{l,q}=(qE_q+1)^m. \tag{1.7}$$

Combining (1.7) and (1.4), the modified q-Euler numbers $E_{m,q}$ satisfy the followings:

$$(qE_q+1)^m+E_{m,q}=\begin{cases}2, & \text{if } m=0,\\ 0, & \text{if } m>0.\end{cases} \tag{1.8}$$

It is well-known that the Euler numbers are defined by the generating function

$$\frac{2}{e^t+1}=e^{Et}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \tag{1.9}$$

with the usual convention about replacing E^n by E_n. From (1.9), we have

$$2=e^{Et}(e^t+1)=e^{(E+1)t}+e^{Et}=\sum_{n=0}^{\infty}\left((E+1)^n+E_n\right)\frac{t^n}{n!}.$$

Thus, we have

$$(E+1)^n+E_n=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases} \tag{1.10}$$

We note that $\lim_{q\to 1}E_{n,q}=E_n$ and that if q approaches to 1, then the equation (1.8) is equal to the equation (1.10).

The purpose of this paper is to define the modified higher-order degenerate q-Euler polynomials which are defined from fermionic p-adic integral on $\mathbb{Z}_p$ and to give some explicit identities for those polynomials. Furthermore, we demonstrate the shapes of the modified higher-order q-Euler polynomials $E^{(r)}_{n,q}(x)$ and the modified higher-order degenerate q-Euler polynomials $E^{(r)}_{n,\lambda,q}(x)$ (see Figure 1) and investigated the zeroes of $E^{(r)}_{n,q}(x)$ and $E^{(r)}_{n,\lambda,q}(x)$ by using a computer.

2. The modified higher-order degenerate q-Euler polynomials

Let $r\in\mathbb{N}$ and $\lambda,t\in\mathbb{C}$ be such that $|\lambda t|_p<p^{-\frac{1}{p-1}}$. We note that if we take $f(x)=e^{xt}$, then, by (1.4), we have

$$\int_{\mathbb{Z}_p}e^{xt}d\mu_{-1}(x)=\frac{2}{e^t+1},\quad \text{(see [14, 16, 17, 19])}. \tag{2.1}$$

By (2.1), we have

$$\begin{aligned}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}e^{(x_1+x_2+\cdots+x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)&=\int_{\mathbb{Z}_p}e^{x_1t}d\mu_{-1}(x_1)\cdots\int_{\mathbb{Z}_p}e^{x_rt}d\mu_{-1}(x_r)\\ &=\left(\frac{2}{e^t+1}\right)^r=\sum_{n=0}^{\infty}E^{(r)}_n\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

where $E^{(r)}_n$ are called the higher-order Euler numbers (see [15, 19, 26]). We also note that

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}e^{(x_1+x_2+\cdots+x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\sum_{n=0}^{\infty}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}(x_1+\cdots+x_r)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *Let* $n \in \mathbb{N} \cup \{0\}$. *Then we have*

$$E_n^{(r)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

In [11], the modified higher-order q-Euler numbers are defined by Kim to be

$$E_{n,q}^{(r)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} [x_1 + x_2 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

The next diagram illustrates the variations of several types of degenerate q-Euler polynomials and numbers. Those polynomials in the first row and the third row of the diagram are introduced by Carliz et al. [1, 3–5, 8] and Kim et al. [12, 18, 20], respectively. A research of these has yielded fruitful results in number theory and combinatorics (see [6, 7, 21, 24]). The motivation of this paper is to investigate some explicit identities for those polynomials in the second row of the diagram.

$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}(x) \frac{t^n}{n!}$ (degenerate q-Euler polynomials) (see [12, 18, 20])	$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!}$ (higher-order degenerate q-Euler polynomials) (see [12, 18, 20])
$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,\lambda,q}(x) \frac{t^n}{n!}$ (modified degenerate q-Euler polynomials)	$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!}$ (modified higher-order degenerate q-Euler polynomials)
$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}$ (modified q-Euler polynomials) (see [8, 12, 23])	$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}$ (modified higher-order q-Euler polynomials) (see [8, 12, 23])

Recently, Kim defined the higher-order degenerate q-Euler polynomials given by the generating function (see [18, 20]) as follows:

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (1+\lambda t)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!}.$$

Accordingly, we define the modified higher-order degenerate q-Euler polynomials given by the generating function as follows:

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.4}$$

Note that $\lim_{\lambda \to 0} E_{n,\lambda,q}^{(r)}(x) = E_{n,q}^{(r)}(x)$, where $E_{n,q}^{(r)}(x)$ are the higher-order q-Euler polynomials.

We observe that

$$\begin{aligned}(1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q} &= \sum_{n=0}^{\infty}\binom{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q}{n}\lambda^n t^n\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q\right)_n \frac{\lambda^n}{n!}t^n\\ &= \sum_{n=0}^{\infty}\left(\tfrac{1}{\lambda}[x_1+\cdots+x_r+x]_q\right)\left(\tfrac{1}{\lambda}[x_1+\cdots+x_r+x]_q-1\right)\\ &\quad\cdots\left(\tfrac{1}{\lambda}[x_1+\cdots+x_r+x]_q-n+1\right)\lambda^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left([x_1+x_2+\cdots+x_r+x]_q\right)\left([x_1+x_2+\cdots+x_r+x]_q-\lambda\right)\\ &\quad\cdots\left([x_1+x_2+\cdots+x_r+x]_q-(n-1)\lambda\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left([x_1+x_2+\cdots+x_r+x]_q\right)_{n,\lambda}\frac{t^n}{n!},\end{aligned} \tag{2.5}$$

where $([x]_q)_{n,\lambda} = [x]_q([x]_q-\lambda)([x]_q-2\lambda)\cdots([x]_q-(n-1)\lambda)$. By (2.4), we have

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}(1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\sum_{n=0}^{\infty}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}\left([x_1+x_2+\cdots+x_r+x]_q\right)_{n,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Using (2.4) and (2.6), we obtain the following Witt's formula.

Theorem 2.2 (Witt's formula)**.** *For* $n \in \mathbb{N}\cup\{0\}$*, we have*

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}\left([x_1+x_2+\cdots+x_r+x]_q\right)_{n,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = E^{(r)}_{n,\lambda,q}(x). \tag{2.7}$$

We observe that

$$\left([x_1+x_2+\cdots+x_r+x]_q\right)_{n,\lambda} = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}[x_1+x_2+\cdots+x_r+x]_q^l, \tag{2.8}$$

where $S_1(n,l)$ is the Stirling numbers of the first kind. By (2.7) and (2.8), we have

$$\begin{aligned}E^{(r)}_{n,\lambda,q}(x) &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}\left([x_1+x_2+\cdots+x_r+x]_q\right)_{n,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-1}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}[x_1+x_2+\cdots+x_r+x]_q^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}E_{l,q}(x).\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 2.3. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$E_{n,\lambda,q}^{(r)}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}E_{l,q}^{(r)}(x).$$

Remark that $\lim_{\lambda\to 0} E_{n,\lambda,q}^{(r)}(x) = E_{n,q}^{(r)}(x)$ are the modified higher-order q-Euler polynomials and that $\lim_{q\to 1} E_{n,q}^{(r)}(x) = E_{n}^{(r)}(x)$ are the higher-order Euler polynomials. We note that

$$\begin{aligned}
E_{n,q} &= \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x) \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} q^{lx} d\mu_{-1}(x) \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x q^{lx} \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N\to\infty} \frac{1+q^{lp^N}}{1+q^l} \\
&= \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{ml} \\
&= 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{ml} \\
&= 2\sum_{m=0}^{\infty} (-1)^m \Big([m]_q\Big)^n.
\end{aligned}$$

Summarizing this, we have the following equation.

Theorem 2.4. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$E_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x) = 2\sum_{m=0}^{\infty} (-1)^m [m]_q^n.$$

For $r \in \mathbb{N}$, we derive

$$\begin{aligned}
E_{n,q}^{(r)}(x) &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} [x_1+x_2+\cdots+x_r+x]_q^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&= \left(\frac{1}{1-q}\right)^n \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} \left(1-q^{x_1+\cdots+x_r+x}\right)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&= \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} q^{(x_1+\cdots+x_r+x)l} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left(\lim_{N\to\infty} \sum_{x_1,\cdots,x_r=0}^{p^N-1} (-1)^{x_1+\cdots+x_r} q^{lx_1+\cdots+lx_r} \right) q^{lx} \qquad (2.9)
\end{aligned}$$

$$= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(\frac{2}{1+q^l}\right)\left(\frac{2}{1+q^l}\right)\cdots\left(\frac{2}{1+q^l}\right)q^{lx}$$
$$= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{lm_1+\cdots+lm_r}q^{lx}$$
$$= \frac{2^r}{(1-q)^n}\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(m_1+m_2+\cdots+m_r+x)}$$
$$= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n.$$

By (2.9), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$E_{n,q}^{(r)}(x) = 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n, \quad \text{(see [8, 12, 23])}.$$

Theorem 2.6. *For $w_1, w_2, \cdots, w_n \in \mathbb{N}$ with $w_i \equiv 1$ (mod 2), $(i = 1, 2, \cdots, n)$, and $m \geqslant 0$, the following expressions*

$$\sum_{p=0}^{m}\sum_{i=0}^{p}\binom{p}{i}\lambda^{m-p}S_1(m,p)\left(\frac{[w_{\sigma(n)}]_q}{\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q}\right)^{p-i}$$
$$\times\, \mathcal{E}_{i,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(w_{\sigma(n)}x)T^{(p)}_{n,q^{w_{\sigma(n)}}}(w_{\sigma(1)},\cdots,w_{\sigma(n-1)}|i+1)$$

are the same for any permutation σ in the symmetry group of degree n.

3. The modified higher-order degenerate q-Euler polynomials and the higher-order q-zeta functions

In [11, 12], Kim introduced the generating function of the higher-order q-Euler polynomials. From the generating function of the higher-order q-Euler polynomials, we have

$$\begin{aligned} F_q^{(r)}(x,t) &= \sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!} \\ &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\sum_{n=0}^{\infty}[m_1+\cdots+m_r+x]_q^n\frac{t^n}{n!} \\ &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}e^{[m_1+\cdots+m_r+x]_q t}, \quad \text{(see [12, 18, 20])}. \end{aligned} \tag{3.1}$$

From (3.1), Kim [11] defined the higher-order q-zeta functions as follows:

$$\zeta_{E,q}^{(r)}(s,x) = \frac{1}{\Gamma(s)}\int_0^{\infty}F_q^{(r)}(x,-t)t^{s-1}dt, \tag{3.2}$$

where $\Gamma(s) = \int_0^{\infty}y^{s-1}e^{-y}dy$. By (3.1) and (3.2) we derive

$$\begin{aligned} \zeta_{E,q}^{(r)}(s,x) &= \frac{1}{\Gamma(s)}2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\int_0^{\infty}e^{-[m_1+\cdots+m_r+x]_q t}t^{s-1}dt \\ &= \frac{2^r}{\Gamma(s)}\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\frac{1}{[m_1+\cdots+m_r+x]_q^s}\int_0^{\infty}y^{s-1}e^{-y}dy \\ &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\frac{1}{[m_1+\cdots+m_r+x]_q^s}. \end{aligned} \tag{3.3}$$

By (3.3), we obtain the following theorem.

Theorem 3.1. *For* $r \in \mathbb{N}, s \in \mathbb{C}$ *with Re(s)* > 0, *we have*

$$\zeta_{E,q}^{(r)}(s,x) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]_q^s}, \quad \text{(see [11, 12])}.$$

For $s, x \in \mathbb{C}$ with Re $(x) > 0$, $a_1, \cdots, a_r \in \mathbb{C}$, the Barnes-type multiple q-zeta functions are defined by Kim [12] as follows:

$$\zeta_{E,q}^{(r)}(s,x|w_1,\cdots,w_r;a_1,\cdots,a_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1a_1+\cdots+m_ra_r}}{[x+w_1m_1+\cdots+w_rm_r]_q^s},$$

where the parameters $w_1, \cdots, w_r$ are positive. Note that $\zeta_{E,q}^{(r)}(s,x|1,\cdots,1;0,\cdots,0) = \zeta_{E,q}^{(r)}(s,x)$.

By (3.1), we have

$$\zeta_{E,q}^{(r)}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty F_q^{(r)}(x,-t)t^{s-1}dt = \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x)\frac{(-1)^m}{m!} \int_0^\infty t^{s-1+m}dt. \tag{3.4}$$

Let $s = -n$ $(n \in \mathbb{N})$. Then, by (3.4), we have

$$\begin{aligned} \zeta_{E,q}^{(r)}(-n,t) &= \lim_{s\to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x)\frac{(-1)^m}{m!} \int_0^\infty t^{-n-1+m}dt \\ &= \left(\lim_{s\to -n}\frac{1}{\Gamma(s)}\right)\left(E_{n,q}^{(r)}(x)\frac{(-1)^n}{n!}\right)2\pi i \\ &= \frac{n!}{2\pi i}(-1)^n E_{n,q}^{(r)}(x)\frac{(-1)^n}{n!}2\pi i = E_{n,q}^{(r)}(x). \end{aligned} \tag{3.5}$$

where

$$\Gamma(-n) = \int_0^\infty e^{-t}t^{-n-1}dt = \lim_{t\to 0} 2\pi i \frac{1}{n!}\left(\frac{d}{dt}\right)^n \left(t^{n+1}e^{-t}t^{-n-1}\right) = 2\pi i\frac{1}{n!}(-1)^n \lim_{t\to 0} e^{-t} = 2\pi i\frac{1}{n!}(-1)^n.$$

By (3.5), we obtain the following theorem.

Theorem 3.2. *For* $n \in \mathbb{N}$, *we have*

$$\zeta_{E,q}^{(r)}(-n,x) = E_{n,q}^{(r)}(x), \quad \text{(see [12])}.$$

From Theorem 2.3 and Theorem 2.5, and (2.8), we have

$$\begin{aligned} E_{n,\lambda,q}^{(r)}(x) &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}E_{l,q}^{(r)}(x) \\ &= 2^r \sum_{l=0}^{n} \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^l S_1(n,l)\lambda^{n-l} \\ &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \sum_{l=0}^{n}[m_1+\cdots+m_r+x]_q^l S_1(n,l)\lambda^{n-l} \\ &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}\left([m_1+\cdots+m_r+x]_q\right)_{n,\lambda}. \end{aligned} \tag{3.6}$$

By (3.6), we obtain the following theorem.

Theorem 3.3. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$E_{n,\lambda,q}^{(r)}(x) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \left([m_1+\cdots+m_r+x]_q\right)_{n,\lambda}. \tag{3.7}$$

Applying (3.7) and using (2.5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!} &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \sum_{n=0}^{\infty} \frac{\left([m_1+\cdots+m_r+x]_q\right)_{n,\lambda} t^n}{n!} \\ &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} (1+\lambda t)^{\frac{[m_1+\cdots+m_r+x]_q}{\lambda}}. \end{aligned} \tag{3.8}$$

By (3.8), we obtain the following theorem.

Theorem 3.4. *For* $r \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!} = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} (1+\lambda t)^{\frac{[m_1+\cdots+m_r+x]_q}{\lambda}}. \tag{3.9}$$

Replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (3.9), and by using (3.1), we have

$$\sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x)\frac{\lambda^{-m}(e^{\lambda t}-1)^m}{m!} = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!},$$

and

$$\begin{aligned} \sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x)\lambda^{-m}\frac{1}{m!}(e^{\lambda t}-1)^m &= \sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x)\lambda^{-m}\frac{1}{m!}m! \sum_{n=m}^{\infty} S_2(n,m)\lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \lambda^{n-m} E_{m,\lambda,q}^{(r)}(x) S_2(n,m)\frac{t^n}{n!}. \end{aligned} \tag{3.10}$$

By (3.10), we obtain the following theorem.

Theorem 3.5. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$E_{n,q}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} E_{m,\lambda,q}^{(r)}(x) S_2(n,m).$$

By replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ by in (3.9), and using (3.8), we have

$$\begin{aligned} \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x)\lambda^{-m}\frac{(\log(1+\lambda t))^m}{m!} &= \sum_{m=0}^{\infty} 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]\frac{1}{\lambda}\log(1+\lambda t)} \\ &= \sum_{m=0}^{\infty} 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{\log(1+\lambda t)^{\frac{1}{\lambda}[m_1+\cdots+m_r+x]_q}} \\ &= \sum_{m=0}^{\infty} 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} (1+\lambda t)^{\frac{1}{\lambda}[m_1+\cdots+m_r+x]_q} \\ &= \sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty} E_{m,q}^{(r)}(x)\lambda^{-m}\frac{(\log(1+\lambda t))^m}{m!} &= \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \lambda^{n-m}E_{m,q}^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.12}$$

By comparing the coefficients of (3.11) and (3.12), we obtain the following theorem.

Theorem 3.6. *For* $n \in \mathbb{N}\cup\{0\}$, *we have*

$$E_{n,\lambda,q}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m}E_{m,q}^{(r)}(x)S_1(n,m).$$

4. Zeroes of the modified higher-order q-Euler polynomials and the modified higher-order degenerate q-Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeroes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$. We display the shapes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$. Next we investigate the zeroes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$. Let $q \in \mathbb{C}, |q| < 1$. For $n = 1, \cdots, 10$, we can draw a plot of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $E_{n,\lambda,q}^{(r)}(x)$ and $E_{n,q}^{(r)}(x)$, $-5 \leqslant x \leqslant 5$ (Figure 1).

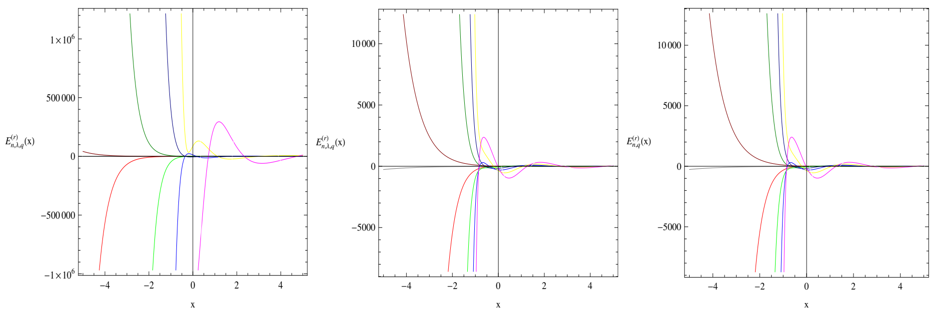

Figure 1: Curve of the $E_{n,\lambda,q}^{(r)}(x)$ and $E_{n,q}^{(r)}(x)$.

In Figure 1 (left), we choose $r = 5$, $\lambda = 1/2$ and $q = 1/2$. In Figure 1 (middle), we choose $r = 5$, $\lambda = 1/10000$ and $q = 1/2$. In Figure 1 (right), we choose $r = 5$ and $q = 1/2$. It is obvious that, by letting λ tend to 1 from the curve of $E_{n,\lambda,q}^{(r)}(x)$ of left side, we lead to the curve of the $E_{n,q}^{(r)}(x)$. By using computer, the modified higher-order q-Euler numbers $E_{n,q}^{(r)}$ and the modified higher-order degenerate q-Euler numbers $E_{n,\lambda,q}^{(r)}$ are listed in Table 1.

We investigate the beautiful zeroes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ by using a computer. We plot the zeroes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ for $n = 50, q = 1/2$ and $x \in \mathbb{C}$ (Figure 2).

Table 1: The first few $E_{n,q}^{(r)}$ and $E_{n,\lambda,q}^{(r)}$.

degree n	$E_{n,q}^{(r)}$	$E_{n,\lambda,q}^{(r)}$
	$q = 1/2, r = 5$	$q = 1/2, r = 5, \lambda = 1/10$
0	1	1
1	−1562/243	−1562/243
2	9287996/759375	10264246/759375
3	3037448168/184528125	4674974089/922640625
4	−1425517528162096/262003549978125	−240516181113919276/6550088749453125

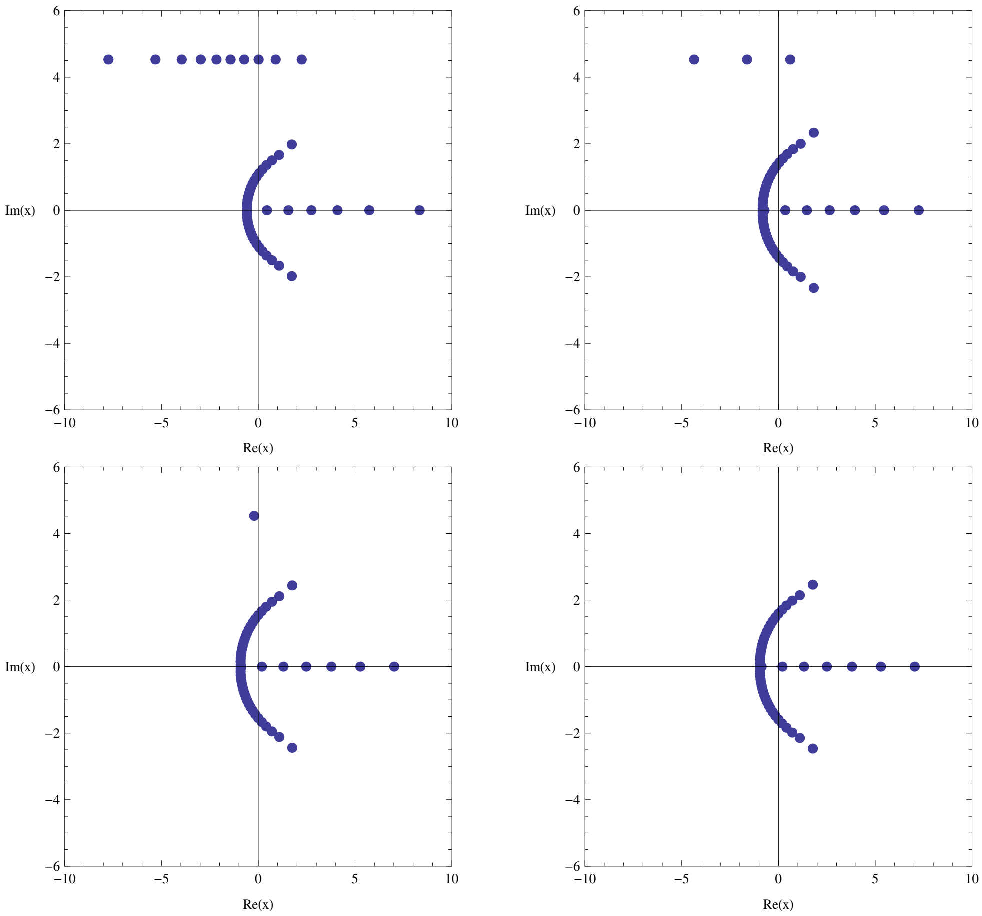

Figure 2: Zeroes of $E_{n,\lambda,q}^{(r)}(x)$ and $E_{n,q}^{(r)}(x)$.

In Figure 2 (top-left), we choose $n = 50, q = 1/2$, and $\lambda = 1/100$. In Figure 2 (top-right), we choose $n = 50, q = 1/2$ and $\lambda = 1/1000$. In Figure 2 (bottom-left), we choose $n = 50, q = 1/2$ and $\lambda = 1/10000$. In Figure 2 (bottom-right), we choose $n = 50, q = 1/2$ and $\lambda \to 0$.

Stacks of zeroes of the modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ for $1 \leqslant n \leqslant 40$ from a 3-D structure are presented in Figure 3.

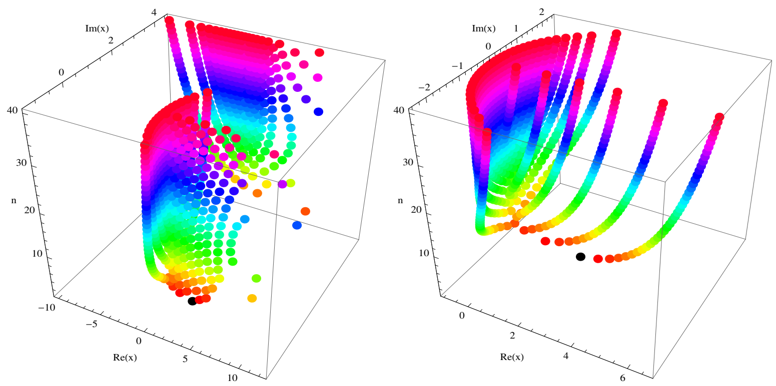

Figure 3: Stacks of zeroes of $E^{(r)}_{n,\lambda,q}(x)$ and $E^{(r)}_{n,q}(x)$ for $1 \leqslant n \leqslant 40$.

In Figure 3 (left), we choose $1 \leqslant n \leqslant 40, q = 1/2$ and $\lambda = 1/10$. In Figure 3 (right), we choose $1 \leqslant n \leqslant 40, q = 1/2$, and $\lambda \to 0$.

It was known that $E^{(r)}_{n,q}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry analytic complex functions, (see [12]). However, we observe that $E^{(r)}_{n,\lambda,q}(x), x \in \mathbb{C}$, has not $\text{Im}(x) = 0$ reflection symmetry analytic complex functions (Figures 2 and 3).

Our numerical results for approximate solutions of real zeroes of the modified higher-order q-Euler polynomials $E^{(r)}_{n,q}(x)$ and the modified higher-order degenerate q-Euler polynomials $E^{(r)}_{n,\lambda,q}(x)$ are displayed in Tables 2, 3, and 4. We observe a remarkably regular structure of the complex roots of the modified higher-order q-Euler polynomials $E^{(r)}_{n,q}(x)$ and the modified higher-order degenerate q-Euler polynomials $E^{(r)}_{n,\lambda,q}(x)$ are displayed in Table 2. We hope to verify a remarkably regular structure of the complex roots of the modified higher-order q-Euler polynomials $E^{(r)}_{n,q}(x)$ and the modified higher-order degenerate q-Euler polynomials $E^{(r)}_{n,\lambda,q}(x)$ (Table 2).

Table 2: Numbers of real and complex zeroes of $E^{(r)}_{n,\lambda,q}(x)$ and $E^{(r)}_{n,q}(x)$.

degree n	$E^{(5)}_{n,1/10,1/2}(x)$ real zeroes	$E^{(5)}_{n,1/10,1/2}(x)$ complex zeroes	$E^{(5)}_{n,1/2}(x)$ real zeroes	$E^{(5)}_{n,1/2}(x)$ complex zeroes
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	3	1	4	0
5	4	1	3	2
6	3	3	4	2
7	4	3	5	2
8	4	4	4	4
9	5	4	5	4
10	3	7	4	6

Plot of real zeroes of $E^{(r)}_{n,\lambda,q}(x)$ and $E^{(r)}_{n,q}(x)$ for $1 \leqslant n \leqslant 40$ structure are presented in Figure 4.

In Figure 4 (left), we choose $r = 5$, $\lambda = 1/10$ and $q = 1/2$. In Figure 4 (middle), we choose $r = 5$, $\lambda = 1/1000$ and $q = 1/2$. In Figure 4 (right), we choose $r = 5$ and $q = 1/2$. It is obvious that, by letting λ tend to 1 from the real zeroes of $E^{(r)}_{n,\lambda,q}(x)$ of left side, we lead to the real zeroes of the $E^{(r)}_{n,q}(x)$.

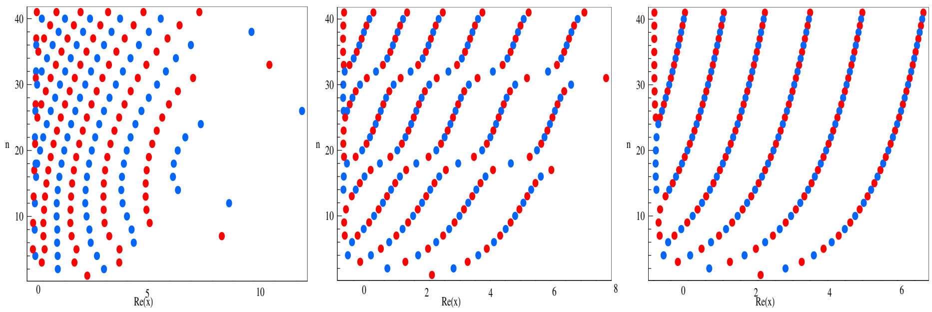

Figure 4: Real zeroes of the $E^{(r)}_{n,\lambda,q}(x)$ and $E^{(r)}_{n,q}(x)$.

Next, we calculated an approximate solution satisfying $E^{(r)}_{n,\lambda,q}(x) = 0$, $E^{(r)}_{n,q}(x) = 0$, and $x \in \mathbb{R}$. The results are given in Tables 3 and 4.

Table 3: Approximate solutions of $E^{(5)}_{n,\lambda,q}(x) = 0, q = 1/2, \lambda = 1/10, x \in \mathbb{R}$.

degree n	x
1	2.07519
2	0.674416, 2.86795
3	-0.0853565, 1.46616, 3.59324
4	-0.507236, 0.0495496, 1.56211, 3.60538
5	0.064954, 1.2468, 2.5000, 3.7532, 4.9350
6	0.642491, 2.23658, 4.2821
7	-0.00924544, 1.3286, 3.12019, 8.46041
8	-0.422343, 0.65015, 2.17641, 4.17786

Table 4: Approximate solutions of $E^{(5)}_{n,\lambda,q}(x) = 0, q = 1/2, x \in \mathbb{R}$.

degree$n$	x
1	2.07519
2	0.601538, 2.78882
3	-0.29846, 1.19492, 3.25392
4	-0.707974, 0.0540258, 1.61452, 3.60212
5	0.380137, 1.93959, 3.88122
6	-0.577628, 0.651902, 2.20503, 4.1144
7	-0.820786, -0.396232, 0.88272, 2.42936, 4.31478
8	-0.201325, 1.08272, 2.6236, 4.49051

Finally, we shall consider the more general problems. How many zeroes does $E^{(r)}_{n,q}(x)$ have? Prove or disprove: $E^{(r)}_{n,q}(x) = 0$ has n distinct solutions. Find the numbers of complex zeroes $C_{E^{(r)}_{n,q}(x)}$ of $E^{(r)}_{n,q}(x), \mathrm{Im}(x) \neq 0$. Since n is the degree of the polynomial $E^{(r)}_{n,q}(x)$, the number of real zeroes $R_{E^{(r)}_{n,q}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{E^{(r)}_{n,q}(x)} = n - C_{E^{(r)}_{n,q}(x)}$, where $C_{E^{(r)}_{n,q}(x)}$ denotes complex zeroes. See Table 2 for tabulated values of $R_{E^{(r)}_{n,q}(x)}$ and $C_{E^{(r)}_{n,q}(x)}$.

5. Conclusions

Kim et al., [17–20] studied some identities of symmetry on the higher-order degenerate q-Euler polynomials. The motivation of this paper is to investigate some explicit identities for the modified higher-order degenerate q-Euler polynomials in the second row of the diagram at page 4. So we defined the modified higher degenerate q-Euler polynomials in the equation (2.4) and obtained the formulas (see Theorems 2.2-2.5). We also obtained the explicit identities related with the modified higher-order degenerate q-Euler polynomials and the higher-order q-zeta functions (see Theorems 3.1-3.6).

Finally, we demonstrated the comparing three facts between modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ as follows:

(1) We displayed the shape of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ (see Figure 1) and investigated the zeroes of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ by using a computer (see Figure 2 and Table 1).

(2) We presented stacks of zeroes of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ for $1 \leqslant n \leqslant 40$ from a 3-D structure (see Figure 3) and verified a regular structure of the complex roots of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ (see Figure 4 and Table 2).

(3) We calculated an approximate solution satisfying $E_{n,q}^{(r)}(x) = 0$, $E_{n,\lambda,q}^{(r)}(x) = 0$, and $x \in \mathbb{R}$ (see Tables 3-4).

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This paper is dedicated to Professor Yeol Je Cho on the occasion of his 65th birthday.

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