Positive solutions for a class of integral boundary value condition of fractional differential equations with a parameter

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Abstract

In this work, we study a class of integral boundary value condition of fractional differential equations with a parameter. The existence and uniqueness of positive solutions to the boundary value problem is established. Further, we present some properties of positive solutions to the boundary value problem dependent on the parameter. The method employed is a fixed point theorem of concave operators in partial ordering Banach spaces. As applications, two examples are given to illustrate our main results. ©2017 All rights reserved.

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1. Introduction

In this article, we study the following integral boundary value condition of fractional differential equations with a parameter

$$\begin{cases} D_{0^+}^\alpha u(t) + \lambda f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)ds, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$, $\lambda > 0$ is a parameter. $D_{0^+}^\alpha$ is the standard Riemann-Liouville fractional derivative of order α which is defined as follows:

$$D_{0^+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)ds, \quad n = [\alpha] + 1,$$

here Γ denotes the Euler gamma function and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$, see [10].

Recently, fractional differential equations with integral boundary conditions have been extensively studied. For a small sample of such work, we refer the reader to [1–3, 5, 7, 9, 11, 17, 18] and the references

therein. As it has been stated in [3], integral boundary value problems have many various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow and population dynamics. The existence of solutions for integral boundary value problems is an important problem. In literature, most of the authors have studied the existence and multiplicity of positive solutions. Recently, in paper [11], the authors considered a class of fractional equations involving the Riemann-Liouville fractional derivative with integral boundary value conditions

$$\begin{cases} D_{0^+}^\alpha u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)ds, \end{cases} \tag{1.2}$$

where $2 < \alpha \leqslant 3$, $q, g \in L^1[0,1]$. The authors obtained some properties of Green function of this problem and by using monotone iteration method, established the existence of one positive solution for the problem (1.2). But the uniqueness of positive solutions is not treated in [11]. From literature, we know that the existence and uniqueness results of positive solutions for fractional differential equations with integral boundary conditions are still very few. Moreover, there are also few papers reported on integral boundary conditions of fractional differential equations with a parameter. So it is important to bridge this gap between known integer-order boundary value problems studies and unknown fractional boundary value problems theory.

Inspired by the work [11] and many known results in [12, 14–16], we will study the problem (1.1) with concave nonlinearity. Different from the works mentioned above, in this paper we will use a fixed point theorem of concave operators in partial ordering Banach spaces to show the existence and uniqueness of positive solutions for the problem (1.1). Moreover, we present some good properties of positive solutions to the boundary value problem dependent on the parameter. The methods used here are different from those in previous articles.

In this paper, our basic assumptions on functions $f(t, x), g(t)$ here are:

(H_1) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous;

(H_2) $f(t, x)$ is concave in x for each $t \in [0, 1]$;

(H_3) $g : [0, 1] \to [0, +\infty)$ with $g \in L^1[0, 1]$ and $\delta := \int_0^1 s^{\alpha-1}(1-s)g(s)ds > 0$, $\sigma := \int_0^1 s^{\alpha-1}g(s)ds < 1$;

(H_4) there exist constants $a, b > 0$ such that $f(t, 0) \geqslant a$, $f(t, 1) \leqslant b$, for all $t \in [0, 1]$;

(H_5) $\lim\limits_{x \to \infty} \dfrac{f(t,x)}{x} = 0$ uniformly on $[0, 1]$.

Remark 1.1. From (H_2), we know that $f(t, x)$ is increasing in $x \in [0, +\infty)$.

2. Preliminaries

In the following, we list some basic concepts in ordered Banach spaces and a fixed point theorem which will be used later. For convenience of readers, we suggest that one refer to [4, 6, 8] for details.

Let $(E, \| \cdot \|)$ be a real Banach space, θ is the zero element of E. A non-empty set $P \subset E$ is a cone. Then E is partially ordered by P, i.e., $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. Set $\mathring{P} = \{x \in P | x$ is an interior point of $P\}$, a cone P is called to be solid if $\mathring{P}$ is non-empty. The notation $x \gg y$ indicates $x - y \in \mathring{P}$. Moreover, P is called normal if there is a constant $N > 0$ such that for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; in this case N is the infimum of such constants, it is called the normality constant of P.

Definition 2.1. Assume D is a convex subset in E. An operator $A : D \to E$ is called a concave operator if $A(tx + (1-t)y) \geqslant tAx + (1-t)Ay$, for $x, y \in D$ and $t \in [0, 1]$.

Definition 2.2. Let P be a cone in E and $e \in P \backslash \theta$, set $E_e = \{x \in E: $ there exists $\lambda > 0$ such that $-\lambda e \leqslant x \leqslant \lambda e\}$, and $\|x\|_e = \inf\{\lambda > 0 : -\lambda e \leqslant x \leqslant \lambda e\}$, for all $x \in E_e$. Then E_e becomes a normed linear space under the norm $\| \cdot \|_e$, and $\|x\|_e$ is called the *e*-norm of the element $x \in E_e$.

Lemma 2.3 ([6, Theorem 1.5.1]). *Assume cone P is normal. Then the following conclusions hold:*

(i) *E_e is a Banach space with e-norm, and there exists a constant $l > 0$ such that $\|x\| \leqslant l\|x\|_e$, for all $x \in E_e$;*

(ii) *$P_e = E_e \cap P$ is a normal solid cone in E_e, and $\mathring{P}_e = \{x \in E_e: $ there exists $\tau = \tau(x) > 0$ such that $x \geqslant \tau e\}$.*

Lemma 2.4 ([18]). *Assume (H_3) holds. Let $2 < \alpha \leqslant 3$ and $y \in C[0,1]$, then the unique solution of the following boundary value problem*

$$\begin{cases} D_{0^+}^\alpha u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)\,ds, \end{cases}$$

is given by

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \quad t \in [0,1],$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \quad (t,s) \in [0,1] \times [0,1],$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$G_2(t,s) = \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 G_1(\tau,s)g(\tau)\,d\tau.$$

Lemma 2.5 ([13]). *The function $G_1(t,s)$ given in Lemma 2.4 has the following properties:*

$$\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leqslant G_1(t,s) \leqslant \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \quad \forall\, t,s \in [0,1].$$

Lemma 2.6. *The Green function $G(t,s)$ given in Lemma 2.4 satisfies the inequalities*

$$\frac{\delta s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} \leqslant G(t,s) \leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)}, \quad t,s \in [0,1].$$

Proof. From Lemma 2.4 in [11], the right inequality holds. So we only need to prove the left inequality. From Lemma 2.5, we obtain

$$G(t,s) = G_1(t,s) + G_2(t,s) \geqslant G_2(t,s) = \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 G_1(\tau,s)g(\tau)\,d\tau$$

$$\geqslant \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 \frac{\tau^{\alpha-1}(1-\tau)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(\tau)\,d\tau$$

$$= \frac{t^{\alpha-1}s(1-s)^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} \int_0^1 \tau^{\alpha-1}(1-\tau)g(\tau)\,d\tau$$

$$= \frac{\delta s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)}.$$

$\square$

3. Existence, uniqueness and properties of positive solutions for the problem (1.1)

In this section, we apply the following theorem to study the problem (1.1).

Lemma 3.1 ([4]). *Let P be a normal solid cone and $A : P \to P$ be a concave operator. Assume that $A\theta \gg \theta$. Then the following conclusions hold:*

(i) *there exists $0 < \lambda^* \leqslant \infty$ such that the operator equation*

$$u = \lambda A u, \tag{3.1}$$

has a unique solution $u(\lambda)$ in P for $0 \leqslant \lambda < \lambda^*$; when $\lambda \geqslant \lambda^*$, (3.1) has no solution in P;

(ii) *for any $u_0 \in P$, set $u_0(\lambda) = u_0$, $u_n(\lambda) = \lambda A u_{n-1}(\lambda)$, $n = 1, 2, 3, \cdots$. Then, for $0 < \lambda < \lambda^*$ we have $u_n(\lambda) \to u(\lambda)$ as $n \to \infty$;*

(iii) $u(\cdot) : [0, \lambda^*) \to P$ *is continuous, and strongly increasing (i.e., $0 \leqslant \lambda_1 < \lambda_2 < \lambda^* \implies u(\lambda_1) \ll u(\lambda_2)$). Moreover, $u(t\lambda) \leqslant t u(\lambda)$ for $0 \leqslant \lambda < \lambda^*$, $0 \leqslant t \leqslant 1$;*

(iv) $\lambda \to \lambda^* - 0$ *implies $\|u(\lambda)\| \to \infty$;*

(v) *if there exist $v_0 \in P$ and $\lambda_0 > 0$ such that $\lambda_0 A v_0 \leqslant v_0$, then $\lambda^* > \lambda_0$.*

Remark 3.2. If $A : P \to P$ is concave, then A is increasing (see [4]).

Throughout our following considerations, we will work in $E = C[0,1]$, a Banach space with endowed with the norm $\|u\|_E = \max_{t \in [0,1]} |u(t)|$. Let $P = \{u \in C[0,1], u(t) \geqslant 0, \forall t \in [0,1]\}$, then P is a normal solid cone of E. Denote $e(t) = t^{\alpha - 1}$, $t \in [0,1]$. Then $e \in C[0,1]$, $0 \leqslant e(t) \leqslant 1$, $e(t) \not\equiv 0$. We get $e \in P \backslash \{\theta\}$.

Let $X = E_e = \{u \in E : \exists \tau > 0, \text{s.t.} - \tau e(t) \leqslant u(t) \leqslant \tau e(t), \forall t \in [0,1]\}$, endowed with the norm $\|u\|_X = \inf\{\tau > 0 : -\tau e(t) \leqslant u(t) \leqslant \tau e(t), \forall t \in [0,1]\}$. Set $\widetilde{P} = X \bigcap P$. From Lemma 2.3, we know that X is a Banach space, $\widetilde{P}$ is a normal solid cone of X and $\overset{\circ}{\widetilde{P}} = \{u \in X:$ there exists $\zeta > 0$ such that $u(t) \geqslant \zeta e(t), \forall t \in [0,1]\}$. In addition, there exists a constant $l > 0$ such that $\|u\|_E \leqslant l\|u\|_X$, for all $u \in X$.

Theorem 3.3. *Assume (H_1)-(H_4) hold. Then the following conclusions hold:*

(i) *there exists $0 < \lambda^* \leqslant \infty$ such that, for any given $\lambda \in (0, \lambda^*)$, the problem (1.1) has a unique positive solution u_λ in $\widetilde{P}$; when $\lambda \geqslant \lambda^*$, the problem (1.1) has no positive solution in $\widetilde{P}$;*

(ii) *for any $u_0 \in \widetilde{P}$ and for any given $\lambda \in (0, \lambda^*)$, constructing successively the sequence*

$$u_m(t) = \lambda \int_0^1 G(t, s) f(s, u_{m-1}(s)) ds, \quad m = 1, 2, 3, \cdots,$$

then we have

$$\max_{t \in [0,1]} |u_m(t) - u_\lambda(t)| \to 0 \text{ as } m \to \infty;$$

(iii) $\max_{t \in [0,1]} |u_\lambda(t) - u_{\lambda_0}(t)| \to 0$ *as $\lambda \to \lambda_0$ where $0 < \lambda_0 < \lambda^*$, and if $0 < \lambda_1 < \lambda_2 < \lambda^*$ then*

$$u_{\lambda_1}(t) \leqslant u_{\lambda_2}(t), \quad \forall t \in [0,1],$$

and $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$;

(iv) $u_{r\lambda}(t) \leqslant r u_\lambda(t)$, $r, t \in [0,1]$ *where $\lambda \in (0, \lambda^*)$.*

Proof. From Lemma 2.4, we know that $u(t)$ is the solution of the problem (1.1), if and only if

$$u(t) = \lambda \int_0^1 G(t, s) f(s, u(s)) ds,$$

where $G(t, s)$ is given as in Lemma 2.4. Define an operator A by

$$(Au)(t) = \int_0^1 G(t, s) f(s, u(s)) ds.$$

It is easy to prove that u is the solution of the problem (1.1), if and only if $u = \lambda Au$. For $u \in P$, in view of the continuity of functions $G(t,s)$ and $f(t,x)$, we have $Au \in C[0,1]$. And from $G(t,s) \geqslant 0$ and $f(t,x) > 0$, for all $t \in [0,1]$, we can get $(Au)(t) \geqslant 0$, $t \in [0,1]$. So $Au \in P$. Since $f(t,\cdot)$ is concave, for $x > 1$, we have

$$f(t,1) = f(t, \frac{1}{x} \cdot x + (1 - \frac{1}{x}) \cdot 0) \geqslant \frac{1}{x} f(t,x) + (1 - \frac{1}{x}) f(t,0),$$

and thus from (H_4), we get

$$f(t,x) \leqslant xf(t,1) - (x-1)f(t,0) \leqslant xf(t,1) \leqslant bx.$$

Therefore, for $u \in P$, from Lemma 2.6 we have

$$0 \leqslant \int_0^1 G(t,s)f(s,u(s))ds \leqslant \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} f(s, \|u\|_E) ds$$

$$\leqslant \frac{t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, \|u\|_E + 1) ds$$

$$\leqslant \frac{t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(1 + \|u\|_E) ds$$

$$= \frac{b(1 + \|u\|_E)}{(1-\sigma)\alpha\Gamma(\alpha)} t^{\alpha-1} = \frac{b(1 + \|u\|_E)}{(1-\sigma)\alpha\Gamma(\alpha)} e(t)$$

$$= M_1 e(t), \quad t \in [0,1],$$

where $M_1 = \frac{b(1+\|u\|_E)}{(1-\sigma)\alpha\Gamma(\alpha)} > 0$. Thus, $0 \leqslant (Au)(t) \leqslant M_1 e(t)$, so $Au \in X$. Then $Au \in P \cap X$, i.e., $Au \in \widetilde{P}$.

Hence, we obtain $A : P \to \widetilde{P}$, and thus $A : \widetilde{P} \to \widetilde{P}$. Further, from $f(t,\cdot)$ is concave, we can see that operator A is concave. By Lemma 2.6 and (H_4),

$$A\theta(t) = \int_0^1 G(t,s)f(s,0)ds \geqslant \int_0^1 \frac{\delta s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} a\,ds$$

$$= \frac{a\delta t^{\alpha-1}}{(1-\sigma)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} ds = \frac{a\delta}{(1-\sigma)\alpha(\alpha+1)\Gamma(\alpha)} e(t), \quad t \in [0,1].$$

Since $\frac{a\delta}{(1-\sigma)\alpha(\alpha+1)\Gamma(\alpha)} > 0$, we have $A\theta \in \overset{\circ}{P}$, i.e., $A\theta \gg \theta$. From Lemma 3.1, there exists $0 < \lambda^* \leqslant \infty$ such that the operator equation $u = \lambda Au$ has a unique solution u_λ in $\widetilde{P}$ for $0 < \lambda < \lambda^*$. When $\lambda \geqslant \lambda^*$, $u = \lambda Au$ has no solution in $\widetilde{P}$. For any $u_0 \in \widetilde{P}$, set $u_m = \lambda Au_{m-1}$, $m = 1, 2, \cdots$, then for $0 < \lambda < \lambda^*$, we have $u_m \to u_\lambda$ as $m \to \infty$. $u_\lambda : (0, \lambda^*) \to \widetilde{P}$ is continuous, and strongly increasing (i.e., $0 < \lambda_1 < \lambda_2 < \lambda^* \implies u_{\lambda_1} \ll u_{\lambda_2}$). Moreover, $u_{(t\lambda)} \leqslant tu_\lambda$ for $0 \leqslant \lambda < \lambda^*$, $0 \leqslant t \leqslant 1$, $\lambda \to \lambda^* - 0$ implies $\|u_\lambda\|_X \to \infty$. That is,

  (i) for any given $\lambda \in (0, \lambda^*)$, the problem (1.1) has a unique positive solution u_λ in $\widetilde{P}$; when $\lambda \geqslant \lambda^*$, the problem (1.1) has no positive solution in $\widetilde{P}$;

 (ii) for any $u_0 \in \widetilde{P}$ and for any given $\lambda \in (0, \lambda^*)$, constructing successively the sequence

$$u_m(t) = \lambda \int_0^1 G(t,s)f(s, u_{m-1}(s))ds, \quad m = 1, 2, \cdots,$$

then

$$\max_{t \in [0,1]} |u_m(t) - u_\lambda(t)| \to 0 \text{ as } m \to \infty;$$

(iii) $\max_{t \in [0,1]} |u_\lambda(t) - u_{\lambda_0}(t)| \to 0$ as $\lambda \to \lambda_0$, where $0 < \lambda_0 < \lambda^*$, and if $0 < \lambda_1 < \lambda_2 < \lambda^*$, then there exists $\tau > 0$ such that $u_{\lambda_2}(t) - u_{\lambda_1}(t) \geqslant \tau e(t)$, $t \in [0,1]$ and thus $u_{\lambda_1}(t) \leqslant u_{\lambda_2}(t)$, for all $t \in [0,1]$ and $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$;

(iv) $u_{r\lambda}(t) \leqslant r u_\lambda(t)$, $r, t \in [0, 1]$, where $\lambda \in (0, \lambda^*)$.

So the conclusions (i), (ii), (iii), and (iv) hold. □

Theorem 3.4. *Assume* (H_1)-(H_5) *hold. Then the following conclusions hold:*

(i) *for any given* $\lambda > 0$, *the problem* (1.1) *has a unique positive solution* u_λ *in* $\widetilde{P}$;

(ii) *for any* $u_0 \in \widetilde{P}$ *and for any given* $\lambda > 0$, *constructing successively the sequence*

$$u_m(t) = \lambda \int_0^1 G(t, s) f(s, u_{m-1}(s)) ds, \quad m = 1, 2, 3, \cdots,$$

then we have

$$\max_{t \in [0,1]} |u_m(t) - u_\lambda(t)| \to 0 \ as \ m \to \infty;$$

(iii) $\max_{t \in [0,1]} |u_\lambda(t) - u_{\lambda_0}(t)| \to 0$ *as* $\lambda \to \lambda_0$ *where* $0 < \lambda_0$, *and if* $0 < \lambda_1 < \lambda_2$, *then* $u_{\lambda_1}(t) \leqslant u_{\lambda_2}(t)$, *for all* $t \in [0, 1]$ *and* $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$;

(iv) $u_{r\lambda}(t) \leqslant r u_\lambda(t)$, $r, t \in [0, 1]$ *where* $\lambda > 0$.

Proof. From the proof of Theorem 3.3, we only need to prove $\lambda^* = \infty$ for any $\lambda > 0$ is given. From (H_5), we can take a $T > 0$ large enough such that $f(t, T) \leqslant \lambda^{-1}(1 - \sigma)\alpha\Gamma(\alpha)T$, for all $t \in [0, 1]$. Set $v_0(t) = Te(t)$, so

$$\lambda(Av_0)(t) = \lambda \int_0^1 G(t, s) f(s, Te(s)) ds \leqslant \lambda \int_0^1 G(t, s) f(s, T) ds$$

$$\leqslant \lambda \int_0^1 \frac{(1 - s)^{\alpha-1}}{(1 - \sigma)\Gamma(\alpha)} f(s, T) ds \cdot t^{\alpha-1}$$

$$\leqslant \lambda \frac{t^{\alpha-1}}{(1 - \sigma)\Gamma(\alpha)} \lambda^{-1}(1 - \sigma)\alpha\Gamma(\alpha)T \int_0^1 (1 - s)^{\alpha-1} ds$$

$$= Te(t) = v_0(t),$$

$$v_0(t) - \lambda(Av_0)(t) = Te(t) - \lambda \int_0^1 G(t, s) f(s, Te(t)) ds$$

$$\leqslant Te(t) + \lambda \int_0^1 G(t, s) f(s, Te(t)) ds$$

$$\leqslant Te(t) + \lambda \int_0^1 \frac{(1 - s)^{\alpha-1}}{(1 - \sigma)\Gamma(\alpha)} f(s, T) ds \cdot e(t)$$

$$\leqslant \left[T + \lambda \int_0^1 \frac{b(1 + T)}{(1 - \sigma)\Gamma(\alpha)} (1 - s)^{\alpha-1} ds \right] e(t)$$

$$= \left[T + \frac{b\lambda(1 + T)}{(1 - \sigma)\alpha\Gamma(\alpha)} \right] e(t), \quad t \in [0, 1].$$

Therefore, $v_0 - \lambda Av_0 \in \widetilde{P}$. That is, $\lambda Av_0(t) \leqslant v_0$. From Lemma 3.1 (v), we have $\lambda^* > \lambda$. Because λ is arbitrary, we get $\lambda^* = \infty$. □

To illustrate how our main results can be used in practice we present two examples.

Example 3.5. Consider the following integral boundary value condition of fractional differential equations with a parameter

$$
\begin{cases}
D_{0^+}^{\frac{7}{3}} u(t) + \lambda(1 + \sin t)[\arctan u(t) + \frac{\pi}{4}] = 0, & t \in (0,1), \\
u(0) = u'(0) = 0, \quad u(1) = \int_0^1 s^{\frac{2}{3}} u(s) ds.
\end{cases}
\tag{3.2}
$$

In this example, $\alpha = \frac{7}{3}$, $\lambda > 0$ is a parameter and

$$
g(t) = t^{\frac{2}{3}}, \ f(t,x) = (1 + \sin t)(\arctan x + \frac{\pi}{4}), \quad \forall\, t \in [0,1].
$$

Then we have $\delta = \int_0^1 s^{\frac{4}{3}}(1-s)s^{\frac{2}{3}} ds = \frac{1}{12} > 0$, $\sigma = \int_0^1 s^{\frac{4}{3}} s^{\frac{2}{3}} ds = \frac{1}{3} < 1$. Evidently, $f(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $f(t,\cdot)$ is increasing and concave. Take $a = \frac{\pi}{4}$, $b = \pi$, then

$$
f(t,0) = (1 + \sin t)(0 + \frac{\pi}{4}) \geqslant \frac{\pi}{4} = a, \quad f(t,1) = (1 + \sin t)(\frac{\pi}{4} + \frac{\pi}{4}) \leqslant \pi = b.
$$

Hence, all the conditions of Theorem 3.3 are satisfied. So we have the following conclusions:

(i) there exists $0 < \lambda^* \leqslant \infty$ such that, for any given $\lambda \in (0, \lambda^*)$, the problem (3.2) has a unique positive solution u_λ in $\widetilde{P}$; when $\lambda \geqslant \lambda^*$, the problem (3.2) has no positive solution in $\widetilde{P}$, here

$$
\widetilde{P} = \{u \in C[0,1] : \exists \tau > 0, \ \text{s.t. } 0 \leqslant u(t) \leqslant \tau t^{\frac{4}{3}}, \ \forall\, t \in [0,1]\};
$$

(ii) for any $u_0 \in \widetilde{P}$ and for any given $\lambda \in (0, \lambda^*)$, constructing successively the sequence

$$
u_m(t) = \lambda \int_0^1 G(t,s)(1 + \sin s)[\arctan u_{m-1}(s) + \frac{\pi}{4}] ds, \quad m = 1,2,3,\cdots,
$$

then we have

$$
\max_{t \in [0,1]} |u_m(t) - u_\lambda(t)| \to 0 \ \text{ as } \ m \to \infty;
$$

(iii) $\max\limits_{t \in [0,1]} |u_\lambda(t) - u_{\lambda_0}(t)| \to 0$ as $\lambda \to \lambda_0$, where $0 < \lambda_0 < \lambda^*$, and if $0 < \lambda_1 < \lambda_2 < \lambda^*$, then

$$
u_{\lambda_1}(t) \leqslant u_{\lambda_2}(t), \quad \forall t \in [0,1],
$$

and $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$;

(iv) $u_{r\lambda}(t) \leqslant r u_\lambda(t)$, $r, t \in [0,1]$, where $\lambda \in (0, \lambda^*)$.

Example 3.6. Consider the following integral boundary value condition of fractional differential equations with a parameter

$$
\begin{cases}
D_{0^+}^{2.5} u(t) + \lambda\{\sum\limits_{i=1}^{n_0} a_i(t)[u(t)]^{\frac{1}{i+1}} + h(t)\} = 0, & t \in (0,1), \\
u(0) = u'(0) = 0, \quad u(1) = \int_0^1 s u(s) ds.
\end{cases}
\tag{3.3}
$$

In this problem, $\alpha = 2.5$, $\lambda > 0$, $n_0 \geqslant 1$ is an integer and

$$
g(t) = t, \ f(t,x) = \sum_{i=1}^{n_0} a_i(t) x^{\frac{1}{i+1}} + h(t), \quad \forall\, t \in [0,1],
$$

where $a_i(t)$ is continuous and nonnegative ($i = 1,2,\cdots,n_0$) and $h : [0,1] \to (0,+\infty)$ is continuous and nondecreasing. Then we have $\delta = \int_0^1 s^\alpha(1-s) ds = \frac{4}{63} > 0$, $\sigma = \int_0^1 s^\alpha ds = \frac{2}{7} < 1$. Evidently,

$f(t,x): [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $f(t,\cdot)$ is increasing and concave. Take $a \in (0, h(0))$, $b \geqslant \sum_{i=1}^{n_0} \|a_i\| + h(1)$, then

$$f(t,0) = h(t) \geqslant h(0) \geqslant a, \quad f(t,1) = \sum_{i=1}^{n_0} a_i(t) + h(t) \leqslant \sum_{i=1}^{n_0} \|a_i\| + h(1) \leqslant b.$$

Further, it is easy to prove that $\lim_{x\to\infty} \frac{f(t,x)}{x} = 0$ uniformly on [0,1], therefore from Theorem 3.4, $\lambda^* = \infty$. Hence, all the conditions of Theorem 3.4 are satisfied. So we have the following conclusions:

(i) for any given $\lambda \in (0, +\infty)$, the problem (3.3) has a unique positive solution u_λ in $\widetilde{P}$, here

$$\widetilde{P} = \{u \in C[0,1] : \exists \tau > 0, \text{ s.t. } 0 \leqslant u(t) \leqslant \tau t^{\frac{3}{2}}, \ \forall \ t \in [0,1]\};$$

(ii) for any $u_0 \in \widetilde{P}$ and for any given $\lambda \in (0, +\infty)$, constructing successively the sequence

$$u_m(t) = \lambda \int_0^1 G(t,s) \left\{ \sum_{i=1}^{n_0} a_i(s)[u_{m-1}(s)]^{\frac{1}{i+1}} + h(s) \right\} ds, \quad m = 1, 2, 3, \cdots,$$

then we have

$$\max_{t\in[0,1]} |u_m(t) - u_\lambda(t)| \to 0 \text{ as } m \to \infty;$$

(iii) $\max_{t\in[0,1]} |u_\lambda(t) - u_{\lambda_0}(t)| \to 0$ as $\lambda \to \lambda_0$, where $0 < \lambda_0 < +\infty$, and if $0 < \lambda_1 < \lambda_2 < +\infty$, then

$$u_{\lambda_1}(t) \leqslant u_{\lambda_2}(t), \quad \forall t \in [0,1],$$

and $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$;

(iv) $u_{r\lambda}(t) \leqslant r u_\lambda(t)$, $r, t \in [0,1]$, where $\lambda \in (0, +\infty)$.

4. Conclusions

In this work, we study a class of integral boundary value problem with a parameter. We obtain some new results on the existence and uniqueness of positive solutions. Moreover, we establish some good properties of positive solutions to the integral boundary value problem dependent on the parameter. The method used here is different from the literature and so is the existence and uniqueness results to integral boundary value conditions of fractional differential equations.

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