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Hybrid projection algorithms for finite total asymptotically strict quasi-ϕ-pseudo-contractions

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Abstract

The purpose of this article is to prove strong convergence theorems for finding a common fixed point of finite total asymptotically strict quasi-ϕ-pseudo-contractions by using a hybrid projection algorithm in Banach spaces. As applications, we apply our main results to find a common solution of a system of generalized mixed equilibrium problems. Finally, some results of numerical simulations are given for supporting our results.

Keywords: Total asymptotically strict quasi-ϕ-pseudo-contraction, generalized mixed equilibrium problem, fixed point, Banach space, hybrid method.
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1. Introduction

Fixed point theory as an important branch of nonlinear analysis theory has been applied in the study of nonlinear phenomena. In fact, lots of real world problems arising in economics, medicine, image reconstruction, engineering, and physics can be studied via fixed point techniques. Mann-like valued iterative methods are efficient and popular tool to study solutions of nonlinear equator equations, monotone variational equilibrium problems and inclusion problems, see [4, 5, 7, 12, 18] and the references therein. However, Mann-like valued iterative methods are only weak convergent without any compact assumptions imposed on the framework of the space or the operators [8]. In image recovery and control theory, problems arise in infinite dimension spaces. In such problems, norm convergence is often much more desirable than weak convergence since it translates the physically tangible property. In 1991, Güler [9] showed that the rate of convergence of the value sequence $\{f(x_n)\}$ is better when $\{x_n\}$ converges strongly than it converges weakly. Such properties have a direct impact when the process is executed directly in the underlying infinite dimensional space. Hybrid projection technique, which was first introduced by Haugazeau [10], has extensively been investigated for fixed point problems, variational inequality problems, equilibrium problems and inclusion problems since they can generate a strong convergent iterative

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sequence without any compact assumption, see [6, 13, 14, 24] and the references therein. Nonexpansive mappings fixed point theory can be applied to solve the solutions of diverse problems such as equilibrium problems, variational inequality problems, and convex feasibility problems, however, strict pseudo-contractions have more powerful applications than nonexpansive mappings in solving these problems, in particular, inverse problems [20].

In recent years, construction of an iterative algorithm for seeking fixed points of nonexpansive mappings, strict pseudo-contractions and more general mappings has extensively been investigated. In 2010, Zhou and Gao [27] studied a new projection algorithm for strict quasi-ϕ-pseudocontractions and obtained a strong convergence theorem. Qin et al. [17] proved a strong convergence theorem for fixed points of an asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense in some Banach space. In 2015, Wang and Yang [23] introduced a new nonlinear mapping, which was called total asymptotically strict quasi-ϕ-pseudo-contraction, and prove a strong convergence theorem for finding fixed point of this kind of mappings.

Motivated and inspired by the works going in this directions, we propose a general hybrid projection iterative algorithm for a finite family of total asymptotically strict quasi-ϕ-pseudo-contractions and prove strong convergence results in the framework of Banach spaces. The results presented in this paper improve or enrich the known corresponding results announced in the literature sources listed in this work.

2. Preliminaries

In this section, we collect some preliminaries including definitions and lemmas which will be used to prove our main results. Throughout this paper, we assume that E is a real Banach space with the dual E^*, C is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^*. We note that in a Hilbert space H, J is the identity operator.

A Banach space E is said to be strictly convex, if $\|\frac{x+y}{2}\| < 1$ for all $x,\ y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex, if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $x,\ y \in U_E$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for all $x,\ y \in U_E$. The following facts are well-known:

(1) if E^* is strictly convex then J is single-valued;

(2) if E^* is uniformly smooth then J is uniformly continuous on bounded subsets of E;

(3) if E^* is a reflexive and smooth Banach space, then J is single-valued and demicontinuous;

(4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(5) E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall\ x,\ y \in E. \tag{2.2}$$

It is obvious from the definition of the function ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall\, x,\ y,\ z \in E. \tag{2.3}$$

Observe that in a Hilbert space H, (2.2) is reduced to $\phi(x, y) = \|x - y\|^2$, for all $x,\ y \in H$. If E is a reflexive, strictly convex, and smooth Banach space, then for all $x,\ y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E. The generalized projection [2] $\Pi_C : E \to C$ is a mapping defined by

$$\Pi_C x = \min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

In Hilbert spaces, $\Pi_C = P_C$, where $P_C : H \to C$ is the metric projection from a Hilbert space H onto a nonempty, closed, and convex subset C of H.

Let $T : C \to C$ be a mapping, the set of fixed points of T is denoted by $F(T)$, that is,

$$F(T) := \{x \in C : Tx = x\}.$$

A point p is said to be an asymptotic fixed point of T [19], if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to \infty$, then $Tx = y$. A mapping T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x \in K}\{\|T^{n+1}x - T^n x\|\} = 0.$$

Next, we recall the following definitions of nonlinear operators.

Definition 2.1. Let $T : C \to C$ be a mapping, $F(T)$ and $\widehat{F}(T)$ denote the set of fixed points and the set of asymptotic fixed points, respectively.

(1) T is called relatively nonexpansive [3], if $\widehat{F}(T) = F(T) \neq \emptyset$, and

$$\phi(p, Tx) \leqslant \phi(p, x), \quad \forall\, x \in C,\ \forall\, p \in F(T).$$

(2) T is said to be relatively asymptotically nonexpansive [1], if $\widehat{F}(T) = F(T) \neq \emptyset$, and

$$\phi(p, T^n x) \leqslant (1 + k_n)\phi(p, x), \quad \forall\, x \in C,\ \forall\, p \in F(T),\ \forall\, n \geqslant 1,$$

where $\{k_n\} \subset [0, \infty)$ is a sequence such that $k_n \to 0$ as $n \to \infty$.

(3) T is said to be hemi-relatively nonexpansive [21, 22], if $F(T) \neq \emptyset$, and

$$\phi(p, Tx) \leqslant \phi(p, x), \quad \forall\, x \in C,\ \forall\, p \in F(T).$$

(4) T is said to be asymptotically quasi-ϕ-nonexpansive [16], if $F(T) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leqslant (1 + k_n)\phi(p, x), \quad \forall\, x \in C,\ \forall\, p \in F(T),\ \forall\, n \geqslant 1.$$

(5) T is said to be generalized asymptotically quasi-ϕ-nonexpansive [17], if $F(T) \neq \emptyset$, and there exist two sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu \to 0$, and $\{\nu_n\}$ with $\nu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leqslant (1 + \mu_n)\phi(p, x) + \nu_n, \quad \forall\, x \in C,\ \forall\, p \in F(T),\ \forall\, n \geqslant 1.$$

(6) T is said to be a strict quasi-ϕ-pseudo-contraction [27], if $F(T) \neq \emptyset$, and there exists a constant $k \in [0, 1)$ such that

$$\phi(p, Tx) \leqslant \phi(p, x) + k\phi(x, Tx), \quad \forall x \in C, \ \forall\, p \in F(T).$$

(7) T is said to be an asymptotically strict quasi-ϕ-pseudo-contraction [17], if $F(T) \neq \emptyset$, and there exist a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu \to 0$ as $n \to \infty$ and a constant $k \in [0, 1)$ such that

$$\phi(p, T^n x) \leqslant (1 + \mu_n)\phi(p, x) + k\phi(x, T^n x), \quad \forall\, x \in C, \ \forall\, p \in F(T), \ \forall\, n \geqslant 1.$$

(8) T is said to be an asymptotically strict quasi-ϕ-pseudo-contraction in the intermediate sense [17], if $F(T) \neq \emptyset$, and there exist a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ and a constant $k \in [0, 1)$ such that

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x)) \leqslant 0. \tag{2.4}$$

Put

$$\nu_n = \max\{0, \sup_{p\in F(T), x\in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x))\},$$

which follows that $\nu_n \to 0$ as $n \to \infty$. Then, (2.4) is reduced to the following:

$$\phi(p, T^n x) \leqslant (1 + \mu_n)\phi(p, x) + k\phi(x, T^n x) + \nu_n, \quad \forall\, p \in F(T), \ \forall\, x \in C, \ \forall\, n \geqslant 1.$$

(9) T is said to be a total asymptotically strict quasi-ϕ-pseudo-contraction [23], if $F(T) \neq \emptyset$, and there exist two sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\phi(p, T^n x) \leqslant \phi(p, x) + \kappa\phi(x, T^n x) + \mu_n \varphi(\phi(p, x)) + \nu_n, \quad \forall\, x \in C, \ p \in F(T),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\varphi(0) = 0$.

Remark 2.2. According to the comparison with the definition above, the following facts can be obtained easily.

(a) The class of hemi-relatively mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. In fact, hemi-relatively nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require $F(T) = \widehat{F}(T)$.

(b) The class of generalized asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of asymptotically quasi-ϕ-nonexpansive mappings.

(c) If the sequence $\mu_n \equiv 0$, the class of asymptotically strict quasi-ϕ-pseudo-contractions is reduced to the class of strict quasi-ϕ-pseudo-contractions.

(d) If $k = 0$, the class of asymptotically strict quasi-ϕ-pseudo-contractions is reduced to the class of asymptotically quasi-ϕ-nonexpansive mappings.

(e) The class of asymptotically strict quasi-ϕ-pseudo-contractions in the intermediate sense is a generalization of the class of asymptotically strict quasi-ϕ-pseudo-contractions. In fact, if $k = 0$ and $\mu \equiv 0$, the class of asymptotically strict quasi-ϕ-pseudo-contractions in the intermediate sense is reduced to the class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.

(f) The class of total asymptotically strict quasi-ϕ-pseudo-contractions is reduced to the class of asymptotically strict quasi-ϕ-pseudo-contractions in the intermediate sense if $\varphi(x) \equiv x$ for all $x \in [0, \infty)$ and

$$\nu_n = \max\{0, \sup_{p\in F(T), x\in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x))\}.$$

The following example which is a total asymptotically strict quasi-ϕ-pseudo-contraction can be found in [23].

Example 2.3. Let C be a closed unit ball in $E = l^2 := \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$, and let $T : C \to C$ be a mapping defined by

$$T : (x_1, x_2, x_2, \cdots) \to (0, x_1^2, a_2x_2, a_3x_3, \cdots), \quad (x_1, x_2, x_3, \cdots) \in l^2,$$

where $\{a_i\}$ is a sequence in (0,1) such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Then, T is a total asymptotically strict quasi-ϕ-pseudo-contraction.

In order to prove our main results, we also need the following lemmas:

Lemma 2.4 ([11]). *Let E be a uniformly convex and smooth Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If $\phi(x_n, y_n) \to 0$ and $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.5 ([2]). *Let E be a reflexive, strictly convex, and smooth Banach space. Let C be a nonempty, closed, and convex subset of E, and $x \in E$ then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x), \quad \forall\, y \in C.$$

Lemma 2.6 ([2]). *Let C be a nonempty, closed, and convex subset of a smooth Banach space E and $x \in E$ then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0, \quad \forall\, y \in C.$$

Lemma 2.7 ([23]). *Let E be a uniformly convex and smooth Banach space, let C be a nonempty, closed and convex subset of E. Suppose $T : C \to C$ is a closed and total asymptotically strict quasi-ϕ-pseudo-contraction. Then, $F(T)$ is closed and convex.*

Lemma 2.8. *Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E. Suppose $T : C \to C$ is a total asymptotically strict quasi-ϕ-pseudo-contraction. For arbitrary $x \in C$, $p \in F(T)$, then*

$$\phi(x, T^n x) \leqslant \frac{2}{1-k}\langle x - p, Jx - JT^n x\rangle + \frac{\mu_n}{1-k}\varphi(\phi(p, x)) + \frac{\nu_n}{1-k}.$$

Proof. For arbitrary $x \in C$, $p \in F(T)$, from the definition of T, one has

$$\phi(p, T^n x) \leqslant \phi(p, x) + k\phi(x, T^n x) + \mu_n\varphi(\phi(p, x)) + \nu_n. \tag{2.5}$$

On the other hand, from (2.3) one has

$$\phi(p, T^n x) = \phi(p, x) + \phi(x, T^n x) + 2\langle p - x, Jx - J^n x\rangle. \tag{2.6}$$

Combining (2.5) with (2.6), one arrives at

$$\phi(x, T^n x) \leqslant \frac{2}{1-k}\langle x - p, Jx - JT^n x\rangle + \frac{\mu_n}{1-k}\varphi(\phi(p, x)) + \frac{\nu_n}{1-k}.$$

This completes the proof. □

3. Main results

In this section, we state and prove our main theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E. Let $T_i : C \to C$, where $i = 1, 2, 3, \cdots, N$, be a closed and total asymptotically strict quasi-ϕ-pseudo-contraction with two sequences $\{\mu_n\} \subset [0, \infty)$, $\{\nu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$, $\nu_n \to 0$ as $n \to \infty$, and a constant $\kappa \in [0, 1)$.*

Assume that T_i *is asymptotically regular on* C *and* $F = \bigcap_{i=1}^N F(T_i)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_0^i = C,\ i = 1,2,\cdots,N, \quad C_0 = \bigcap_{i=1}^N C_0^i, \\ y_n^i = J^{-1}[\alpha_n J x_n + (1-\alpha_n) J T_i^n x_n], \\ C_{n+1}^i = \{u \in C_n : \phi(u, y_n^i) \leqslant \phi(u, x_n) + \frac{2\kappa}{1-\kappa}\langle x_n - u, Jx_n - JT_i^n x_n\rangle + \theta_n\}, \\ C_{n+1} = \bigcap_{i+1}^N C_{n+1}^i, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where $\theta_n = \mu_n \frac{M_n}{1-\kappa} + \frac{\nu_n}{1-\kappa}$, $M_n = \sup\{\varphi(\phi(p, x_n)) : p \in F\}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = \Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. The proof is split into six steps.

Step 1: Show that $\Pi_F x_0$ is well-defined for any $x_0 \in E$.
By Lemma 2.7, one knows that $F(T_i)$ is closed and convex for $i = 1,2,3,\cdots,N$. This implies that $F = \bigcap_{i=1}^N F(T_i)$ is also closed and convex. Furthermore, in view of the assumption of $F \neq \emptyset$, $\Pi_F x_0$ is well-defined for any $x_0 \in E$.

Step 2: Show that C_n is closed and convex for each $n \geqslant 0$.
It is obvious that $C_0 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \in \mathbb{N}$. For all $u \in C_m$, one sees from (3.1) that

$$\phi(u, y_m^i) \leqslant \phi(u, x_m) + \frac{2\kappa}{1-\kappa}\langle x_m - u, Jx_m - JT_i^m x_m\rangle + \theta_m,$$

is equivalent to

$$\langle u, \frac{1}{1-\kappa} Jx_m - Jy_m^i - \frac{\kappa}{1-\kappa} JT_i^m x_m\rangle \leqslant \langle x_m, Jx_m - JT_i^m x_m\rangle + \frac{\|x_m\|^2 + \|y_m^i\|^2}{2} + \frac{\theta_m}{2},$$

for each $1 \leqslant i \leqslant N$. It easily implies that C_{m+1}^i is closed and convex for each $1 \leqslant i \leqslant N$. Furthermore, one knows that C_{m+1} is closed and convex. Then, by the mathematical induction principle, for each $n \geqslant 0$, C_n is closed and convex.

Step 3: Show that $F = \bigcap_{i=1}^N F(T_i) \subset C_n$ for each $n \geqslant 0$.
It is obvious that $F \subset C = C_0$. Suppose that $F(T) \subset C_h$ for some $h \in \mathbb{N}$. One sees that $F \subset C_{h+1}$ for the same h. Indeed, For any $p \in F \subset C_h$, one learns from the definition of T_i, (3.1), and Lemma 2.8 that

$$\begin{aligned} \phi(p, y_h^i) &\leqslant \phi(p, J^{-1}[\alpha_h J x_h + (1-\alpha_h) JT_i^h x_h]) \\ &\leqslant \|p\|^2 - 2\langle p, \alpha_h J x_h + (1-\alpha_h) JT_i^h x_h\rangle + \|\alpha_h J x_h + (1-\alpha_h) JT_i^h x_h\|^2 \\ &\leqslant \|p\|^2 - 2\alpha_h\langle p, Jx_h\rangle - 2(1-\alpha_h)\langle p, JT_i^h x_h\rangle + \alpha_h\|Jx_h\|^2 + (1-\alpha_h)\|JT_i^h x_h\|^2 \\ &= \alpha_h \phi(p, x_h) + (1-\alpha_h)\phi(p, T_i^h x_h) \\ &\leqslant \alpha_h \phi(p, x_h) + (1-\alpha_h)[\phi(p, x_h) + \kappa\phi(x_h, T_i^h x_h) + \mu_h \varphi(\phi(p, x_h)) + \nu_h] \\ &= \phi(p, x_h) + (1-\alpha_h)[\kappa\phi(x_h, T_i^h x_h) + \mu_h \varphi(\phi(p, x_h)) + \nu_h] \\ &\leqslant \phi(p, x_h) + \kappa\phi(x_h, T_i^h x_h) + \mu_h \varphi(\phi(p, x_h)) + \nu_h \\ &\leqslant \kappa[\frac{2}{1-\kappa}\langle x_h - p, Jx_h - JT_i^h x_h\rangle + \frac{\mu_h}{1-\kappa}\varphi(\phi(p, x_h)) + \frac{\nu_h}{1-\kappa}] \\ &\quad + \phi(p, x_h) + \mu_h \varphi(\phi(p, x_h)) + \nu_h \\ &= \phi(p, x_h) + \frac{2\kappa}{1-\kappa}\langle x_h - p, Jx_h - JT_i^h x_h\rangle + \frac{\mu_h}{1-\kappa}\varphi(\phi(p, x_h)) + \frac{\nu_h}{1-\kappa} \end{aligned}$$

$$\begin{aligned}&\leqslant \phi(p,x_h)+\frac{2\kappa}{1-\kappa}\langle x_h-p, Jx_h-JT_i^h x_h\rangle+\mu_h\frac{M_h}{1-\kappa}+\frac{\nu_h}{1-\kappa}\\&=\phi(p,x_h)+\frac{2\kappa}{1-\kappa}\langle x_h-p, Jx_h-JT_i^h x_h\rangle+\theta_h,\end{aligned}$$

which implies that $p\in C_{h+1}^i$ for each $1\leqslant i\leqslant N$. Furthermore, one sees that $p\in C_{h+1}$ for the same h. By the mathematical induction principle, $F\subset C_n$ for each $n\geqslant 0$.

Step 4: Show that $\{x_n\}$ is a Cauchy sequence.
From $x_n=\Pi_{C_n}x_0$, one knows that

$$\langle x_n-u, Jx_0-Jx_n\rangle\geqslant 0,\quad \forall\, u\in C_n.$$

Since $F\subset C_n$ for all $n\geqslant 0$, one sees that

$$\langle x_n-p, Jx_0-Jx_n\rangle\geqslant 0,\quad \forall\, p\in F.$$

From Lemma 2.5, one has

$$\phi(x_n,x_0)=\phi(\Pi_{C_n}x_0,x_0)\leqslant\phi(p,x_0)-\phi(p,x_n)\leqslant\phi(p,x_0),$$

for each $w\in F$ and $n\geqslant 0$. Therefore, the sequence $\phi(x_n,x_0)$ is bounded. On the other hand, in view of $x_n=\Pi_{C_n}x_0$ and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, one has

$$\phi(x_n,x_0)\leqslant\phi(x_{n+1},x_0),$$

for all $n\geqslant 0$. Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing. It implies that the limit of $\{\phi(x_n,x_0)\}$ exists. By the construction of C_n, one learns that $C_m\subset C_n$ and $x_m=\Pi_{C_m}x_0\in C_n$ for any positive integer $m\geqslant n$. Therefore, one has that

$$\begin{aligned}\phi(x_m,x_n)&=\phi(x_m,\Pi_{C_n}x_0)\\&\leqslant\phi(x_m,x_0)-\phi(\Pi_{C_n}x_0,x_0)\\&=\phi(x_m,x_0)-\phi(x_n,x_0).\end{aligned}\tag{3.2}$$

Letting $m,\ n\to\infty$ in (3.2), one arrives at $\phi(x_m,x_n)\to 0$. It follows from Lemma 2.4 that $x_m-x_n\to 0$ as $m,\ n\to\infty$. Then $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n\to\bar{x}\in C$ as $n\to\infty$.

Step 5: Show that $\bar{x}\in F$.
By utilizing the construction of C_n and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, one sees that

$$\phi(x_{n+1},y_n^i)\leqslant\phi(x_{n+1},x_n)+\frac{2\kappa}{1-\kappa}\langle x_n-x_{n+1}, Jx_n-JT_i^n x_n\rangle+\theta_n.\tag{3.3}$$

Since $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty}\theta_n=0$, one has from (3.3) that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n^i)=0,\quad\forall\, i=1,2,3,\cdots,N.$$

Due to Lemma 2.4, one knows that

$$\lim_{n\to\infty}\|x_{n+1}-y_n^i\|=0,\quad\forall\, i=1,2,3,\cdots,N.$$

Since J is uniformly norm-to-norm continuous on any bounded sets, one obtains that

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n^i\|=0,\quad\forall\, i=1,2,3,\cdots,N.\tag{3.4}$$

On the other hand, from $y_n^i = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n]$, one computes that

$$\begin{aligned}\|Jx_{n+1} - Jy_n^i\| &= \|Jx_{n+1} - [\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n]\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JT_i^n x_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JT_i^n x_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geqslant (1-\alpha_n)\|Jx_{n+1} - JT_i^n x_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|.\end{aligned}$$

Hence, one obtains that

$$\|Jx_{n+1} - JT_i^n x_n\| \leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n^i\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

Since $\{x_n\}$ is a Cauchy sequence and (3.4), one has that

$$\lim_{n\to\infty} \|Jx_{n+1} - JT_n^i x_n\| = 0, \quad \forall\, i = 1, 2, 3, \cdots, N.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, one gets that

$$\lim_{n\to\infty} \|x_{n+1} - T_n^i x_n\| = 0, \quad \forall\, i = 1, 2, 3, \cdots, N. \tag{3.5}$$

Note that (3.5) and $x_n \to \bar{x}$ as $n \to \infty$ and

$$\|T_i^n x_n - \bar{x}\| \leqslant \|T_i^n x_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \bar{x}\|.$$

It follows that

$$\lim_{n\to\infty} \|T_i^n x_n - \bar{x}\| = 0, \quad \forall\, i = 1, 2, 3, \cdots, N. \tag{3.6}$$

Observing that

$$\|T_i^{n+1} x_n - \bar{x}\| \leqslant \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - \bar{x}\|, \quad \forall\, i = 1, 2, 3, \cdots, N. \tag{3.7}$$

By using (3.6), (3.7) and the asymptotic regularity of T, one obtains that

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0,$$

that is, $T_i T_i^n x_n \to \bar{x}$ as $n \to \infty$ for each $i = 1, 2, 3, \cdots, N$. From the closedness of T_i, we obtain that $\bar{x} = T_i\bar{x}$ for each $i = 1, 2, 3, \cdots, N$.

Step 6: Show that $\bar{x} = \Pi_{F(T)}x_0$.
Noticing that (3.2), that is,

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geqslant 0, \quad \forall\, p \in F.$$

Taking the limit in the above inequality yields

$$\langle \bar{x} - p, Jx_0 - J\bar{x}\rangle \geqslant 0, \quad \forall\, w \in F.$$

Hence, we obtain from Lemma 2.6 that $\bar{x} = \Pi_F x_0$. This completes the proof. □

Remark 3.2. In view of Definition 2.1, one knows the class of total asymptotically strict quasi-ϕ-pseudo-contractions includes many nonlinear mappings as special cases, for instance, asymptotically strict quasi-ϕ-pseudo-contractions in the intermediate sense, asymptotically strict quasi-ϕ-pseudo-contractions, strict quasi-ϕ-pseudo-contractions, generalized asymptotically quasi-ϕ-nonexpansive mappings, asymptotically quasi-ϕ-nonexpansive mappings, relatively asymptotically nonexpansive mappings, hemi-relatively nonexpansive mappings and so on. So, Theorem 3.1 improves many current results, for further details, see Agarwal et al. [1], Qin and Cho [15], Matsushita and Takahashi [13], Qin et al. [17]. Su et al. [21], Wang et al. [22], Wu and Wang [25], Zhou et al. [28], Zhou and Gao [27].

Remark 3.3. By way of comparison with the main result in Wang and Yang [23], Theorem 3.1 improves the main result of Wang and Yang [23] in the following senses:

(1) The iterative algorithm (3.1) is more general than the one given in Wang and Yang [23]. In fact, by taking $\alpha_n \equiv 0$, the algorithm (3.1) is reduced to the analogous iterative algorithm in Wang and Yang [23].

(2) Theorem 3.1 mainly focuses on a finite family of total asymptotically strict quasi-ϕ-pseudo-contractions, but the main result given in Wang and Yang [23] is concerned only with one single total asymptotically strict quasi-ϕ-pseudo-contraction.

4. Applications

In this section, we consider the problem for finding the common solution of a system of generalized mixed equilibrium problems. Let C be a nonempty, closed, and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\{f_i\}_{i\in I}$ be a family of bifunctions from $C\times C$ into $\mathbb{R}$, $A_i : C \to E^*$ be a nonlinear mapping, and $\varphi_i : C \to \mathbb{R}$ be a real-valued function, where I and $\mathbb{R}$ denote the set of an arbitrary index set, and the set of real numbers, respectively. The "so-called" system of generalized mixed equilibrium problems is to find $x \in C$ such that

$$f_i(x,y) + \langle y-x, A_i x\rangle + \varphi_i(y) - \varphi_i(x) \geqslant 0, \quad \forall\, y \in C,\ i \in I. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $SGMEP(f_i, A_i, \varphi_i)$, where $i \in I$. A mapping $A : C \to E^*$ is called monotone if

$$\langle x-y, Ax-Ay\rangle \geqslant 0, \quad \forall\, x,\ y \in C.$$

A mapping A is called L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax-Ay\| \leqslant L\|x-y\|, \quad \forall\, x,\ y \in C.$$

For solving the generalized mixed equilibrium problem, let us assume that $\varphi : C \to \mathbb{R}$ is a convex and lower semi-continuous function, $A : C \to E^*$ is a continuous and monotone mapping, and $f : C\times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leqslant 0$ for all $x,\ y \in C$;

(A3) for all $x,\ y,\ z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leqslant f(x,y)$;

(A4) for each $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

Lemma 4.1 ([26])**.** *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $f : C\times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *For $r > 0$ and $x \in E$, define a mapping $Res_r^{f,A,\varphi}(x) : E \to C$ as follows:*

$$Res_r^{f,A,\varphi}(x) = \{u \in C : f(u,y) + \langle y-u, Au\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, Ju-Jx\rangle \geqslant 0, \quad \forall\, y \in C\}.$$

Then, the $Res_r^{f,A,\varphi}$ has the following properties:

(1) *$Res_r^{f,A,\varphi}$ is single-valued;*

(2) $F(Res_r^{f,A,\varphi}) = SGMEP(f, A, \varphi)$;

(3) $\mathrm{SGMEP}(f, A, \varphi)$ *is closed and convex;*

(4) $\phi(p, \mathrm{Res}_r^{f,A,\varphi} z) + \phi(\mathrm{Res}_r^{f,A,\varphi} z, z) \leqslant \phi(p, x)$, $\forall p \in F(\mathrm{Res}_r^{f,A,\varphi})$, $z \in E$.

Theorem 4.2. *Let* C *be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space* E. *For* $i = 1, 2, 3, \cdots, N$, *let* $f_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4), $A_i : C \to E^*$ *be a continuous and monotone mapping,* $\varphi_i : C \to \mathbb{R}$ *be a lower semi-continuous and convex function. Assume that* $F = \bigcap_{i=1}^N \mathrm{SGMEP}(f_i, A_i, \varphi_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \textit{ chosen arbitrarily}, \\ C_0^i = C, \ i = 1, 2, \cdots, N, \quad C_0 = \bigcap_{i=1}^N C_0^i, \\ y_n^i = J^{-1}[\alpha_n J x_n + (1-\alpha_n) J \mathrm{Res}_{r_{n,i}}^{f_i, A_i, \varphi_i} x_n], \\ C_{n+1}^i = \{u \in C_n : \phi(u, y_n^i) \leqslant \phi(u, x_n)\}, \\ C_{n+1} = \bigcap_{i+1}^N C_{n+1}^i, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{r_{n,i}\}$ *be a sequence in* $(0, \infty)$ *with assumption* $\lim_{n\to\infty} r_{n,i} > 0$ *for every* $i = 1, 2, 3, \cdots, N$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = \Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. From Lemma 4.1, one easily sees that $\mathrm{Res}_{r_n}^{f_i, A_i, \varphi_i}$ a closed hemi-relatively mapping. So, $\mathrm{Res}_{r_n}^{f_i, A_i, \varphi_i}$ is also a closed total asymptotically strict quasi-ϕ-pseudo-contraction for each $i = 1, 2, \cdots, N$. By applying Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P_F^f(x_0)$. □

5. Numerical examples

In this section, we give a numerical example about the special form of algorithm (3.1) to verify its validity.

Example 5.1. Let $E = \mathbb{R}$, $C = [0, \pi]$, $Tx = \sin \frac{1}{2} x$. Then T is also a closed total asymptotically strict quasi-ϕ-pseudo-contraction with $F(T) = \{0\}$.

Proof. From the definition of T, it is obvious that 0 is the unique fixed point of T, that is, $F(T) = \{0\}$. On the other hand, we have

$$\phi(0, Tx) = |0|^2 - \langle 0, JTx \rangle + |Tx|^2 = (\sin \frac{1}{2} x)^2 \leqslant \frac{1}{4} x^2 \leqslant x^2 = |0|^2 - \langle 0, Jx \rangle + |x|^2 = \phi(0, x).$$

It implies that T is a closed hemi-relatively nonexpansive mapping. Therefore, T is also a closed total asymptotically strict quasi-ϕ-pseudo-contraction. □

Next, we consider a simple case of the algorithm (3.1) which only contains a single nonlinear operator T. By using Example 5.1, The algorithm (3.1) can be simplified as

$$\begin{cases} x_0 \in \mathbb{R} \text{ chosen arbitrarily}, \\ C_0 = C = [0, \pi], \\ y_n = \alpha_n x_n + (1-\alpha_n) \sin \frac{1}{2} x_n, \\ C_{n+1} = \{u \in C_n : u \leqslant \frac{x_n + y_n}{2}\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{5.1}$$

In the following, for the three initial points $x_0 = 0.5,\ 1,\ 1.5$, and $\alpha_n = \frac{1}{n}$, we test the effectiveness and convergence of the algorithm (5.1) by MATLAB 7.0 software.

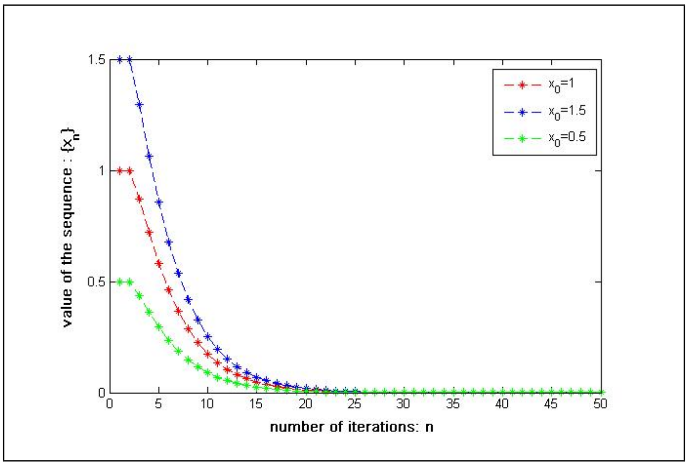

Figure 1: the convergence process of the sequence $\{x_n\}$ with different initial points.

From Figure 1 above, we see that for different initial points, each sequence $\{x_n\}$ converges to the same fixed point by using the algorithm (5.1).

Table 1: partial values of the sequence $\{x_n\}$ in the experiment.

n	$x_0 = 0.5$	$x_0 = 1$	$x_0 = 1.5$
0	0.5000	1.0000	1.5000
5	0.2357	0.4643	0.6795
10	0.0689	0.1352	0.1966
15	0.0186	0.0365	0.0530
20	0.0048	0.0095	0.0138
25	0.0012	0.0024	0.0035
30	0.0003	0.0006	0.0009
35	0.0001	0.0002	0.0002
40	0.0000	0.0000	0.0001
41	0.0000	0.0000	0.0000

Some values of the sequence $\{x_n\}$ in the numerical experiments of Figure 1 are shown on Table 1. Table 1 clearly indicates that each sequence $\{x_n\}$ converges to 0 for different initial points. In a word, the results of numerical simulations demonstrate that the algorithm of Theorem 3.1 is effective, realizable and convergent.

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References

[1] R. P. Agarwal, Y. J. Cho, X.-L. Qin, *Generalized projection algorithms for nonlinear operators*, Numer. Funct. Anal. Optim., **28** (2007), 1197–1215. 2.1, 3.2

[2] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178** (1996), 15–50. 2, 2.5, 2.6

[3] D. Butnariu, S. Reich, A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal., **7** (2001), 151–174. 2.1

[4] S. Y. Cho, B. A. Bin Dehaish, X.-L. Qin, *Weak convergence of a splitting algorithm in Hilbert spaces*, J. Appl. Anal. Comput., **7** (2017), 427–438. 1

[5] Y. J. Cho, X-L. Qin, *Systems of generalized nonlinear variational inequalities and its projection methods*, Nonlinear Anal., **69** (2008), 4443–4451. 1

[6] Y. J. Cho, X.-L. Qin, J. I. Kang, *Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems*, Nonlinear Anal., **71** (2009), 4203–4214. 1

[7] N.-N. Fang, Y.-P. Gong, *Viscosity iterative methods for split variational inclusion problems and fixed point problems of a nonexpansive mapping*, Commun. Optim. Theory, **2016** (2016), 15 pages. 1

[8] A. Genel, J. Lindenstrauss, *An example concerning fixed points*, Israel J. Math., **22** (1975), 81–86. 1

[9] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., **29** (1991), 403–419. 1

[10] Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes*, Thése, Université de Paris, Paris, France, (1968). 1

[11] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13** (2002), 938–945. 2.4

[12] J. K. Kim, S. Y. Cho, X.-L. Qin, *Some results on generalized equilibrium problems involving strictly pseudocontractive mappings*, Acta Math. Sci. Ser. B Engl. Ed., **31** (2011), 2041–2057. 1

[13] S.-Y. Matsushita, W. Takahashi, *Approximating fixed points of nonexpansive mappings in a Banach space by metric projections*, Appl. Math. Comuput., **196** (2008), 422–425. 1, 3.2

[14] H. Piri, *Strong convergence of the CQ method for fixed points of semigroups of nonexpansive mappings*, J. Nonlinear Funct. Anal., **2015** (2015), 21 pages. 1

[15] X.-L. Qin, S. Y. Cho, *Convergence analysis of a monotone projection algorithm in reflexive Banach spaces*, Acta Math. Sci. Ser. B Engl. Ed., **37** (2017), 488–502. 3.2

[16] X.-L. Qin, S. Y. Cho, S. K. Kang, *On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings*, Appl. Math. Comput., **215** (2010), 3874–3883. 2.1

[17] X.-L. Qin, L. Wang, S. M. Kang, *Some results on fixed points of asymptotically strict quasi-φ-pseudocontractions in the intermediate sense*, Fixed Point Theory Appl., **2012** (2012), 18 pages. 1, 2.1, 3.2

[18] X.-L. Qin, J.-C. Yao, *Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators*, J. Inequal. Appl., **2016** (2016), 9 pages. 1

[19] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178** (1996), 313–318. 2

[20] O. Scherzer, *Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems*, J. Math. Anal. Appl., **194** (1995), 911–933. 1

[21] Y.-F. Su, Z.-M. Wang, H.-K. Xu, *Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings*, Nonlinear Anal., **71** (2009), 5616–5628. 2.1, 3.2

[22] Z.-M. Wang, Y.-F. Su, D.-X. Wang, Y.-C. Dong, *A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings and systems of equilibrium problems in Banach spaces*, J. Comput. Appl. Math., **235** (2011), 2364–2371. 2.1, 3.2

[23] Z.-M. Wang, J.-G. Yang, *Hybrid projection algorithms for total asymptotically strict quasi-ϕ-pseudo-contractions*, J. Nonlinear Sci. Appl., **8** (2015), 1032–1047. 1, 2.1, 2, 2.7, 3.3

[24] Z.-M. Wang, X.-M. Zhang, *Shrinking projection methods for systems of mixed variational inequalities of Browder type, systems of mixed equilibrium problems and fixed point problems*, J. Nonlinear Funct. Anal., **2014** (2014), 25 pages. 1

[25] C. Wu, G. Wang, *Hybrid projection algorithms for asymptotically quasi-phi-nonexpansive mappings*, Commun. Optim. Theory, **2013** (2013), 10 pages. 3.2

[26] S.-S Zhang, *Generalized mixed equilibrium problem in Banach spaces*, Appl. Math. Mech. (English Ed.), **30** (2009), 1105–1112. 4.1

[27] H.-Y. Zhou, X.-H. Gao, *An iterative method of fixed points for closed and quasi-strict pseudo-contractions in Banach spaces*, J. Appl. Math. Comput., **33** (2010), 227–237. 1, 2.1, 3.2

[28] H.-Y. Zhou, G.-L. Gao, B. Tan, *Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings*, J. Appl. Math. Comput., **32** (2010), 453–464. 3.2