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Gohberg-Semencul type formula and application for the inverse of a conjugate-Toeplitz matrix involving imaginary circulant matrices

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Abstract

Gohberg-Semencul type inverse formula of conjugate-Toeplitz (CT) is obtained by constructing a kind of imaginary cyclic displacement transform. The stability of decomposition formula of inverse is investigated, and its algorithm is also given. Numerical example is provided to verify the feasibility of the inverse formula. How the analogue of our formula leads to a more efficient way to solve the conjugate-Toeplitz linear system of equations is proposed. The corresponding inverse, stability, and algorithm of conjugate-Hankel (CH) matrix are also considered.

Keywords: Conjugate-Toeplitz matrix, conjugate-Hankel matrix, stability, imaginary cyclic displacement, fast Fourier transform.
2010 MSC: 15A09, 15A15, 15A69, 65F05.

1. Introduction

Toeplitz matrix has become a satisfactory tool in restoration of signals and images [5, 26, 27]. In [4, 7], the authors introduced a generalization of Toeplitz matrices, called conjugate-Toeplitz (CT) matrices, and showed that certain properties of Toeplitz matrices could be extended to CT matrices. The explicit inverse of nonsingular conjugate-Toeplitz and conjugate-Hankel matrices are provided [12]. The inverses of conjugate-Toeplitz (CT) and conjugate-Hankel (CH) matrices can be expressed by the Gohberg-Semencul type formula [17]. Gover and Barnett [8] introduced a corresponding algorithm for any strongly nonsingular CT matrix (i.e., all of its leading principal minors are nonzero). Algorithms for inverting CT matrices and solving CT systems of equations, using $O(n^2)$ flops for matrices of order n, were given. The purpose of [9] is to show that some known properties of Toeplitz and CT matrices can be extended to rT matrices. In [22], an expression of the inverse of a conjugate Toeplitz matrix is obtained. The necessary conditions of applying the generalized Trench algorithm for CT matrices are discussed. It is shown that there exist strongly invertible CT matrices to which the algorithm may not be applied.

Gohberg-Semencul type formula for inverses of conjugate-Toeplitz and conjugate-Hankel matrix only mentioned in [4, 12, 17] have not been fully exploited.

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Let $T = [t_{j-k}]_{j,k=0}^{n-1}$ be a real symmetric positive-definite Toeplitz matrix of order n. There are several well-known $O(n^2)$ algorithms for solving the linear system of equations $Tx = b$, and more recently, several $O(n\log^2 n)$ algorithms have been developed. See, for example, [2, 3, 11, 29] and the references contained therein. Algorithms from both of these classes often rely, either implicitly or explicitly, on the Gohberg-Semencul formula [6], which provides a decomposition of T^{-1} into the sum of products of lower triangular and upper triangular Toeplitz matrices. Although we will consider T to be a real positive-definite Toeplitz matrix, formula presented by Gohberg and Semencul apply to the inverse of any invertible Toeplitz matrix.

In [18], the authors presented an innovative patterned matrix, RFPL-Toeplitz matrix, is neither the extension of Toeplitz matrix nor its special case. The group inverse of this new patterned matrix can be represented as the sum of products of lower and upper triangular Toeplitz matrices, then the explicit expression and the decomposition of the group inverse is given. The inverses of CUPL-Toeplitz and CUPL-Hankel matrices can be expressed by the Gohberg-Heinig type formula in [13]. Jiang and Hong [15] derived the formulas on representation of the inverses of the CUPL Toeplitz matrices in the form of sums of products of factor (1, 1)-circulants and (-1, -1)-circulants. The stability of the algorithms emerging from Toeplitz matrix inversion formulas is considered in [10, 30]. Xie and Wei [31] presented a stability analysis of Gohberg-Semencul-Trench type for Moore-Penrose and group inverses of Toeplitz matrices. Toeplitz inversion formula involving circulant matrices have also been presented in [1, 23, 24].

As far as we know, there is no more efficient algorithms for solving the linear system of equations $T_C x = b$ and $H_C x = b$. We hope that this paper will help in changing this to provide an algorithm for the effective application.

We remark that imaginary cyclic displacement structure plays a critical role in finding the inverse of CT matrix. At present, the articles about imaginary circulant matrix are very few, some interrelated basic knowledge is stated only in [14, 21, 25, 28].

We provide CT matrix inversion formula as a sum of two products of imaginary circulant matrix and upper triangular matrices. It will be shown the number of real arithmetic operations is less than the known results to solve the conjugate-Toeplitz and conjugate-Hankel linear system of equations.

In this paper, $\mathbb{R}$ denotes the set of real numbers and $i = \sqrt{-1}$. Now allow us to introduce some basic knowledges about CT and CH matrices, which are stated in [4, 7].

Definition 1.1. Let $c(x) = \bar{x}$ denote the complex conjugate of x. In particular note that $c^{2m}(x) = x$, and $c^{2m-1}(x) = \bar{x}$, for all positive integers m.

Definition 1.2. An $n \times n$ matrix $T_C = (t_{jk})$ is conjugate-Toeplitz (CT) if $t_{j+1,k+1} = c(t_{jk})$ for all j, k, that is,

$$T_C = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ c(t_1) & c(t_0) & c(t_{-1}) & \cdots & c(t_{-(n-2)}) \\ c^2(t_2) & c^2(t_1) & c^2(t_0) & \ddots & c^2(t_{-(n-3)}) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c^{n-1}(t_{n-1}) & c^{n-1}(t_{n-2}) & c^{n-1}(t_{n-3}) & \cdots & c^{n-1}(t_0) \end{pmatrix}. \tag{1.1}$$

Definition 1.3. An $n \times n$ matrix $H_C = (h_{jk})$ is conjugate-Hankel (CH) if $h_{j+1,k} = c(h_{j,k+1})$ for all j, k, that is,

$$H_C = \begin{pmatrix} h_0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ c(h_1) & c(h_2) & \cdots & c(h_{n-1}) & c(h_n) \\ c^2(h_2) & c^2(h_3) & \iddots & c^2(h_n) & c^2(h_{n+1}) \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ c^{n-1}(h_{n-1}) & c^{n-1}(h_n) & \cdots & c^{n-1}(h_{2n-3}) & c^{n-1}(h_{2n-2}) \end{pmatrix}.$$

2. Conjugate-Toeplitz inversion formula

Constructing a kind of imaginary cyclic displacement, we propose inversion formula for CT matrix as a sum of two products of imaginary circulant and upper triangular matrices.

Lemma 2.1. *Let $T_C = (t_{jk})$ be an $n \times n$ conjugate-Toeplitz matrix defined in equation (1.1) with T_C invertible and $t_{jk} \in i\mathbb{R}$. Then T_C satisfies the formula*

$$\Pi T_C + T_C \Pi = v e_n^T - e_1 v^T \widehat{J}, \tag{2.1}$$

where

$$\Pi = \begin{pmatrix} 0 & \cdots & 0 & i \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}, \qquad \widehat{J} = \begin{pmatrix} & & & (-1)^0 \\ & & (-1)^1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ c^0(t_{1-n} - it_1) \\ \vdots \\ c^{n-3}(t_{-2} - it_{n-2}) \\ c^{n-2}(t_{-1} - it_{n-1}) \end{pmatrix}.$$

We do not give a detail proof for Lemma 2.1, its technical skill can be calculated directly.

Theorem 2.2. *Let $T_C = (t_{jk})$ be an $n \times n$ conjugate-Toeplitz matrix defined with T_C invertible and $t_{jk} \in i\mathbb{R}$. If each of the systems of equations $T_C x = v$, $T_C y = e_1$ are solvable,*

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T, \quad y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T,$$

then T_C is invertible and

$$T_C^{-1} = \mathbb{I}_1^T U_1 + \mathbb{I}_2^T U_2, \tag{2.2}$$

where

$$\mathbb{I}_1 = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ iy_n & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_2 \\ iy_2 & \cdots & iy_n & y_1 \end{pmatrix}, \qquad \mathbb{I}_2 = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ ix_n & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_2 \\ ix_2 & \cdots & ix_n & x_1 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} (-1)^0 & (-1)^1 x_n & \cdots & (-1)^{n-1} x_2 \\ & (-1)^1 & \cdots & (-1)^{n-1} x_3 \\ & & \ddots & \vdots \\ & & \ddots & (-1)^{n-1} x_n \\ & & & (-1)^{n-1} \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & (-1)^2 y_n & (-1)^3 y_{n-1} & \cdots & (-1)^n y_2 \\ & 0 & (-1)^3 y_n & \ddots & \vdots \\ & & 0 & \ddots & (-1)^n y_{n-1} \\ & & & \ddots & (-1)^n y_n \\ & & & & 0 \end{pmatrix},$$

and $\mathbb{I}_1$, $\mathbb{I}_2$ are both imaginary circulant matrices [14, 16, 19–21, 25].

Proof. From equation (2.1) and $T_C x = v, T_C y = e_1$, we have

$$\Pi T_C = -T_C \Pi + v e_n^T - e_1 v^T \widehat{J} = T_C(-\Pi + x e_n^T - y v^T \widehat{J}).$$

Then

$$\Pi^j T_C = \Pi^{j-1} T_C(-\Pi + x e_n^T - y v^T \widehat{J}) = T_C(-\Pi + x e_n^T - y v^T \widehat{J})^j.$$

Therefore,

$$\Pi^j e_1 = \Pi^j T_C y = T_C(-\Pi + x e_n^T - y v^T \widehat{J})^j y.$$

Let

$$\mu_j = (-\Pi + x e_n^T - y v^T \widehat{J})^{j-1} y \quad \text{and} \quad \hat{T}_C = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{pmatrix}.$$

Then

$$\begin{aligned} T_C \mu_j &= T_C(-\Pi + x e_n^T - y v^T \widehat{J})^{j-1} y = \Pi^{j-1} e_1 = e_j, \\ T_C \hat{T}_C &= T_C \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} = I_n, \end{aligned}$$

where I_n denotes the $n \times n$ identity matrix. So matrix T_C is invertible, and T_C^{-1} is $\hat{T}_C$. It is easy to get

$$\begin{aligned} &\mu_1 = y, \ \ \mu_j = (-\Pi + x e_n^T - y v^T \widehat{J})\mu_{j-1} (j = 1, 2, \cdots, n), \\ &\mu_j = T_C^{-1} e_j, \ \ \widehat{J} e_j = (-1)^j e_{n-j+1}, \ \ \widehat{J} T_C \widehat{J} = T_C^T, \ \ \widehat{J}^{-1} = (-1)^{n+1} \widehat{J}. \end{aligned}$$

Then, for $j > 1$,

$$\begin{aligned} \mu_j &= -\Pi \mu_{j-1} + x e_n^T \mu_{j-1} - y v^T \widehat{J} \mu_{j-1} \\ &= -\Pi \mu_{j-1} + x e_n^T T_C^{-1} e_{j-1} - y v^T \widehat{J} T_C^{-1} e_{j-1} \\ &= -\Pi \mu_{j-1} + x e_n^T \widehat{J}^{-1} \widehat{J} T_C^{-1} \widehat{J} \widehat{J}^{-1} e_{j-1} - y v^T \widehat{J} T_C^{-1} \widehat{J} \widehat{J}^{-1} e_{j-1} \\ &= -\Pi \mu_{j-1} + x e_1^T (T_C^{-1})^T (-1)^j e_{n-j+2} - y v^T (T_C^{-1})^T (-1)^j e_{n-j+2} \\ &= -\Pi \mu_{j-1} + (-1)^j x y^T e_{n-j+2} - (-1)^j y x^T e_{n-j+2} \\ &= -\Pi \mu_{j-1} + (-1)^j y_{n-j+2} x + (-1)^{j+1} x_{n-j+2} y. \end{aligned} \tag{2.3}$$

According to $\mu_1 = y$ and equation (2.3), we have

$$\begin{aligned} T_C^{-1} &= \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{pmatrix} \\ &= \begin{pmatrix} y & (-1)^1 \Pi y & \cdots & (-1)^{n-1} \Pi^{n-1} y \end{pmatrix} \Psi_1 + \begin{pmatrix} x & (-1)^1 \Pi x & \cdots & (-1)^{n-1} \Pi^{n-1} x \end{pmatrix} \Psi_2 \\ &= \begin{pmatrix} y & \Pi y & \cdots & \Pi^{n-1} y \end{pmatrix} \mathbb{D} \Psi_1 + \begin{pmatrix} x & \Pi x & \cdots & \Pi^{n-1} x \end{pmatrix} \mathbb{D} \Psi_2 \\ &= \begin{pmatrix} y & \Pi y & \cdots & \Pi^{n-1} y \end{pmatrix} U_1 + \begin{pmatrix} x & \Pi x & \cdots & \Pi^{n-1} x \end{pmatrix} U_2 \\ &= \begin{pmatrix} y_1 & i y_n & \cdots & i y_2 \\ y_2 & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i y_n \\ y_n & \cdots & y_2 & y_1 \end{pmatrix} U_1 + \begin{pmatrix} x_1 & i x_n & \cdots & i x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i x_n \\ x_n & \cdots & x_2 & x_1 \end{pmatrix} U_2 \\ &= \mathbb{I}_1^T U_1 + \mathbb{I}_2^T U_2, \end{aligned}$$

where

$$\Psi_1 = \begin{pmatrix} 1 & (-1)^1 x_n & \cdots & (-1)^{n-1} x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & (-1)^1 x_n \\ & & & 1 \end{pmatrix}, \Psi_2 = \begin{pmatrix} 0 & (-1)^2 y_n & (-1)^3 y_{n-1} & \cdots & (-1)^n y_2 \\ & 0 & (-1)^2 y_n & \ddots & \vdots \\ & & 0 & \ddots & (-1)^3 y_{n-1} \\ & & & \ddots & (-1)^2 y_n \\ & & & & 0 \end{pmatrix},$$

$$\mathbb{D} = \begin{pmatrix} (-1)^0 & & & \\ & (-1)^1 & & \\ & & \ddots & \\ & & & (-1)^{n-1} \end{pmatrix}.$$

□

We remark that the formulas in the theorem are expressed by solving the systems of equations $T_C x = v$, $T_C y = e_1$. This provides an algorithm to compute the inverse of T_C.

3. Conjugate-Hankel inversion formula

In this section we provide inversion formula for CH matrix as a sum of two products of left imaginary circulant matrix and upper triangular matrices.

Lemma 3.1. *Let $H_C = (h_{jk})$ be an $n \times n$ conjugate-Hankel matrix with H_C invertible and $h_{jk} \in i\mathbb{R}$. Then H_C satisfies*

$$\Pi H_C + H_C \Pi^T = \hat{v} e_1^T - e_1 \hat{v}^T \mathbb{D},$$

where Π, $\mathbb{D}$, and e_1 are defined above and

$$\hat{v} = \begin{pmatrix} 0 \\ c^0(h_0 - ih_n) \\ c^1(h_1 - ih_{n+1}) \\ \vdots \\ c^{n-2}(h_{n-2} - ih_{2n-2}) \end{pmatrix}.$$

Theorem 3.2. *Let $H_C = (h_{jk})$ be a conjugate-Hankel matrix with H_C invertible and $h_{jk} \in i\mathbb{R}$. If each of the systems of equations $H_C \xi = \hat{v}$, $H_C \eta = e_1$ are solvable, $\xi = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix}^T$, $\eta = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \end{pmatrix}^T$, then H_C is invertible and*

$$H_C^{-1} = \mathbb{L}_1 \Phi_1 + \mathbb{L}_2 \Phi_2, \tag{3.1}$$

where

$$\mathbb{L}_1 = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_n \\ \eta_2 & \eta_3 & \cdots & \eta_n & i\eta_1 \\ \eta_3 & & \iddots & i\eta_1 & i\eta_2 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \eta_n & i\eta_1 & i\eta_2 & \cdots & i\eta_{n-1} \end{pmatrix}, \qquad \mathbb{L}_2 = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_n \\ \xi_2 & \xi_3 & \cdots & \xi_n & i\xi_1 \\ \xi_3 & & \iddots & i\xi_1 & i\xi_2 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \xi_n & i\xi_1 & i\xi_2 & \cdots & i\xi_{n-1} \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} (-1)^0 & (-1)^1\xi_1 & \cdots & (-1)^{n-1}\xi_{n-1} \\ & (-1)^1 & \cdots & (-1)^{n-1}\xi_{n-2} \\ & & \ddots & \vdots \\ & & \ddots & (-1)^{n-1}\xi_1 \\ & & & (-1)^{n-1} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 0 & (-1)^2\eta_1 & (-1)^3\eta_2 & \cdots & (-1)^n\eta_{n-1} \\ & 0 & (-1)^3\eta_1 & \ddots & (-1)^n\eta_{n-2} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & (-1)^n\eta_1 \\ & & & & 0 \end{pmatrix},$$

and $\mathbb{L}_1, \mathbb{L}_2$ are both left imaginary circulant matrices [21].

Proof. The proof is similar to Theorem 2.2. □

4. Stability analysis of decomposition formulas (2.2) and (3.1)

In this section, we will analyze the stability of the inversion formulas given in Theorems 2.2 and 3.2. Now we show the error analysis of the explicit inversion formulas for CT matrices and CH matrices in terms of the 1-norm, ∞-norm, and 2-norm, respectively. Denote

$$\begin{aligned}
x &= (x_1, x_2 \dots, x_{n-1}, x_n)^T, & \overline{x} &= (ix_n \dots, ix_3, ix_2)^T,\\
y &= (y_1, y_2 \dots, y_{n-1}, y_n)^T, & \overline{y} &= (iy_n \dots, iy_3, iy_2)^T,\\
\xi &= (\xi_1, \xi_2 \dots, \xi_{n-1}, \xi_n)^T, & \overline{\xi} &= (i\xi_1 \dots, i\xi_{n-2}, i\xi_{n-1})^T,\\
\eta &= (\eta_1, \eta_2 \dots, \eta_{n-1}, \eta_n)^T, & \overline{\eta} &= (i\eta_1 \dots, i\eta_{n-2}, i\eta_{n-1})^T.
\end{aligned}$$

Theorem 4.1. *Let T_C be an $n \times n$ conjugate-Toeplitz matrix and well conditioned. Let $\sigma > 0$ and let $\hat{x}, \hat{y}, \hat{\overline{x}}, \hat{\overline{y}}$ be the corresponding numerical least squares solutions of the linear systems for deriving the formula (2.2). Denote by $\hat{T}_C^{-1}$ the inverse of $\hat{T}_C$. If $\frac{\|\hat{x}-x\|_1}{\|x\|_1} \leqslant \sigma$, $\frac{\|\hat{y}-y\|_1}{\|y\|_1} \leqslant \sigma$, $\frac{\|\hat{\overline{x}}-\overline{x}\|_1}{\|\overline{x}\|_1} \leqslant \sigma$, $\frac{\|\hat{\overline{y}}-\overline{y}\|_1}{\|\overline{y}\|_1} \leqslant \sigma$, then,*

$$\|T_C^{-1} - \hat{T}_C^{-1}\|_1 \leqslant (2+\sigma)\sigma(2\|y\|_1\|x\|_1 + \|\overline{y}\|_1\|x\|_1 + \|\overline{x}\|_1\|y\|_1) + \sigma(\|y\|_1 + \|\overline{y}\|_1). \tag{4.1}$$

Proof. Rewrite the inverse formula for T_C as

$$T_C^{-1} = (L_y + R_{\overline{y}})R_x^{(1)}\mathbb{D} - (L_x + R_{\overline{x}})R_y^{(0)}\mathbb{D},$$

where

$$L_y = \begin{pmatrix} y_1 & & & \\ y_2 & y_1 & & \\ \vdots & \ddots & \ddots & \\ y_n & \cdots & y_2 & y_1 \end{pmatrix}, \qquad R_{\overline{y}} = \begin{pmatrix} 0 & iy_n & \cdots & iy_2 \\ & 0 & \ddots & \vdots \\ & & \ddots & iy_n \\ & & & 0 \end{pmatrix},$$

$$L_x = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \cdots & x_2 & x_1 \end{pmatrix}, \qquad R_{\overline{x}} = \begin{pmatrix} 0 & ix_n & \cdots & ix_2 \\ & 0 & \ddots & \vdots \\ & & \ddots & ix_n \\ & & & 0 \end{pmatrix},$$

$$R_x^{(1)} = \begin{pmatrix} 1 & x_n & \cdots & x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & x_n \\ & & & 1 \end{pmatrix}, \qquad R_y^{(0)} = \begin{pmatrix} 0 & y_n & y_{n-1} & \cdots & y_2 \\ & 0 & y_n & \ddots & \vdots \\ & & \ddots & \ddots & y_{n-1} \\ & & & \ddots & y_n \\ & & & & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned}
\|T_C^{-1} - \hat{T}_C^{-1}\|_1 &\leqslant \|L_y R_x^{(1)} - \hat{L}_y \hat{R}_x^{(1)}\|_1 + \|R_{\overline{y}} R_x^{(1)} - \hat{R}_{\overline{y}} \hat{R}_x^{(1)}\|_1 + \|L_x R_y^{(0)} - \hat{L}_x \hat{R}_y^{(0)}\|_1 + \|R_{\overline{x}} R_y^{(0)} - \hat{R}_{\overline{x}} \hat{R}_y^{(0)}\|_1\\
&\triangleq \tau_1 + \tau_2 + \tau_3 + \tau_4.
\end{aligned}$$

For the first term τ_1, we have,

$$\begin{aligned}
\tau_1 &\leqslant \|L_y R_x^{(1)} - \hat{L}_y \hat{R}_x^{(1)}\|_1\\
&\leqslant \|L_y R_x^{(1)} - \hat{L}_y R_x^{(1)} + \hat{L}_y R_x^{(1)} - \hat{L}_y \hat{R}_x^{(1)}\|_1\\
&\leqslant \|L_y R_x^{(1)} - \hat{L}_y R_x^{(1)}\|_1 + \|\hat{L}_y R_x^{(1)} - \hat{L}_y \hat{R}_x^{(1)}\|_1
\end{aligned}$$

$$\begin{aligned}
&\leqslant \|L_y - \hat{L}_y\|_1 \|R_x^{(1)}\|_1 + \|\hat{L}_y\|_1 \|R_x^{(1)} - \hat{R}_x^{(1)}\|_1 \\
&\leqslant \|y - \hat{y}\|_1 (1 + \|x\|_1) + (1+\sigma)\|y\|_1 \sigma \|x\|_1 \\
&\leqslant \sigma\|y\|_1 (1 + \|x\|_1) + (1+\sigma)\|y\|_1 \sigma \|x\|_1 \\
&\leqslant (2+\sigma)\sigma\|y\|_1\|x\|_1 + \sigma\|y\|_1.
\end{aligned}$$

Then τ_2, τ_3, τ_4 can be derived similarly, which are

$$\tau_2 \leqslant (2+\sigma)\sigma\|\overline{y}\|_1\|x\|_1 + \sigma\|\overline{y}\|_1, \qquad \tau_3 \leqslant (2+\sigma)\sigma\|y\|_1\|x\|_1, \qquad \tau_4 \leqslant (2+\sigma)\sigma\|y\|_1\|\overline{x}\|_1.$$

We obtain the designed result by summing the above four inequalities. □

Remark 4.2. Under the assumptions and notations of Theorem 4.1, it can be obtained the same upper bound of $\|T_C^{-1} - \hat{T}_C^{-1}\|_\infty$ with the 1-norm in a similar way,

$$\|T_C^{-1} - \hat{T}_C^{-1}\|_\infty \leqslant (2+\sigma)\sigma(2\|y\|_1\|x\|_1 + \|\overline{y}\|_1\|x\|_1 + \|\overline{x}\|_1\|y\|_1) + \sigma(\|y\|_1 + \|\overline{y}\|_1). \tag{4.2}$$

So we are in a position to present the upper bound with respect to the 2-norm, since

$$\|T_C^{-1} - \hat{T}_C^{-1}\|_2^2 \leqslant \|T_C^{-1} - \hat{T}_C^{-1}\|_1 \|T_C^{-1} - \hat{T}_C^{-1}\|_\infty,$$

and we have from (4.1) and (4.2) that

$$\begin{aligned}
\|T_C^{-1} - \hat{T}_C^{-1}\|_2 \leqslant\ & (2+\sigma)\sigma\sqrt{n(n-1)}(\|\overline{y}\|_2\|x\|_2 + \|\overline{x}\|_2\|y\|_2) \\
&+ 2(2+\sigma)\sigma n\|y\|_2\|x\|_2 + \sigma\sqrt{n}(\|\overline{y}\|_2 + \|y\|_2),
\end{aligned}$$

as $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$, $\|\overline{x}\|_1 \leqslant \sqrt{n-1}\|\overline{x}\|_2$, $\|y\|_1 \leqslant \sqrt{n}\|y\|_2$, and $\|\overline{y}\|_1 \leqslant \sqrt{n-1}\|\overline{y}\|_2$. Therefore, the formula presented in Theorem 2.2 is forward stable.

Theorem 4.3. *Let H_C be an $n \times n$ conjugate-Hankel matrix and well conditioned. Let $\sigma > 0$ and let $\hat{\xi}, \hat{\eta}, \hat{\overline{\xi}}, \hat{\overline{\eta}}$ be the corresponding numerical least squares solutions of the linear systems for deriving the formula (3.1). Denote by $\hat{H}_C^{-1}$ the inverse of $\hat{H}_C$. If $\frac{\|\hat{\xi}-\xi\|_1}{\|\xi\|_1} \leqslant \sigma$, $\frac{\|\hat{\eta}-\eta\|_1}{\|\eta\|_1} \leqslant \sigma$, $\frac{\|\hat{\overline{\xi}}-\overline{\xi}\|_1}{\|\overline{\xi}\|_1} \leqslant \sigma$, $\frac{\|\hat{\overline{\eta}}-\overline{\eta}\|_1}{\|\overline{\eta}\|_1} \leqslant \sigma$, then*

$$\|H_C^{-1} - \hat{H}_C^{-1}\|_1 \leqslant (2+\sigma)\sigma(2\|\eta\|_1\|\xi\|_1 + \|\overline{\eta}\|_1\|\xi\|_1 + \|\overline{\xi}\|_1\|\eta\|_1) + \sigma(\|\overline{\eta}\|_1 + \|\eta\|_1). \tag{4.3}$$

Proof. Rewrite the inverse formula for H_C as

$$H_C^{-1} = (L_\eta + R_{\overline{\eta}})R_\xi^{(1)}\mathbb{D} - (L_\xi + R_{\overline{\xi}})R_\eta^{(0)}\mathbb{D},$$

where

$$L_\eta = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_n \\ \eta_2 & \eta_3 & \cdots & \eta_n & \\ \eta_3 & & \iddots & & \\ \vdots & \iddots & & & \\ \eta_n & & & & \end{pmatrix}, \qquad R_{\overline{\eta}} = \begin{pmatrix} & & & & 0 \\ & & & 0 & i\eta_1 \\ & & 0 & i\eta_1 & i\eta_2 \\ & \iddots & \iddots & \iddots & \vdots \\ 0 & i\eta_1 & i\eta_2 & \cdots & i\eta_{n-1} \end{pmatrix},$$

$$L_\xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_n \\ \xi_2 & \xi_3 & \cdots & \xi_n & \\ \xi_3 & & \iddots & & \\ \vdots & \iddots & & & \\ \xi_n & & & & \end{pmatrix}, \qquad R_{\overline{\xi}} = \begin{pmatrix} & & & & 0 \\ & & & 0 & i\xi_1 \\ & & 0 & i\xi_1 & i\xi_2 \\ & \iddots & \iddots & \iddots & \vdots \\ 0 & i\xi_1 & i\xi_2 & \cdots & i\xi_{n-1} \end{pmatrix},$$

$$R_{\xi}^{(1)} = \begin{pmatrix} 1 & \xi_1 & \cdots & \xi_{n-1} \\ & 1 & \ddots & \vdots \\ & & \ddots & \xi_1 \\ & & & 1 \end{pmatrix}, \qquad R_{\eta}^{(0)} = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \cdots & \eta_{n-1} \\ & 0 & \eta_1 & \ddots & \vdots \\ & & 0 & \ddots & \eta_2 \\ & & & \ddots & \eta_1 \\ & & & & 0 \end{pmatrix}.$$

We obtain

$$\begin{aligned} \|H_C^{-1} - \hat{H}_C^{-1}\|_1 &\leqslant \|L_\eta R_\xi^{(1)} - \hat{L}_\eta \hat{R}_\xi^{(1)}\|_1 + \|R_{\overline{\eta}} R_\xi^{(1)} - \hat{R}_{\overline{\eta}} \hat{R}_\xi^{(1)}\|_1 + \|L_\xi R_\eta^{(0)} - \hat{L}_\xi \hat{R}_\eta^{(0)}\|_1 + \|R_{\overline{\xi}} R_\eta^{(0)} - \hat{R}_{\overline{\xi}} \hat{R}_\eta^{(0)}\|_1 \\ &\triangleq \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4. \end{aligned}$$

For the first term κ_1, we have,

$$\kappa_1 \leqslant (2+\sigma)\sigma\|\eta\|_1\|\xi\|_1 + \sigma\|\eta\|_1.$$

Then $\kappa_2, \kappa_3, \kappa_4$ can be derived similarly, which are

$$\kappa_2 \leqslant (2+\sigma)\sigma\|\overline{\eta}\|_1\|\xi\|_1 + \sigma\|\overline{\eta}\|_1, \qquad \kappa_3 \leqslant (2+\sigma)\sigma\|\eta\|_1\|\xi\|_1, \qquad \kappa_4 \leqslant (2+\sigma)\sigma\|\eta\|_1\|\overline{\xi}\|_1.$$

We obtain the desired result by summing the above four inequalities. □

Remark 4.4. Under the assumptions and notations of Theorem 4.3, it can be obtained the similar upper bound for $\|H_C^{-1} - \hat{H}_C^{-1}\|_\infty$:

$$\|H_C^{-1} - \hat{H}_C^{-1}\|_\infty \leqslant (2+\sigma)\sigma(2\|\eta\|_1\|\xi\|_1 + \|\overline{\eta}\|_1\|\xi\|_1 + \|\overline{\xi}\|_1\|\eta\|_1) + \sigma(\|\overline{\eta}\|_1 + \|\eta\|_1). \tag{4.4}$$

So we are in a position to give the upper bound with respect to the 2-norm, since

$$\|H_C^{-1} - \hat{H}_C^{-1}\|_2^2 \leqslant \|H_C^{-1} - \hat{H}_C^{-1}\|_1 \|H_C^{-1} - \hat{H}_C^{-1}\|_\infty,$$

and we have from (4.3) and (4.4) that

$$\begin{aligned} \|H_C^{-1} - \hat{H}_C^{-1}\|_2 \leqslant\ & (2+\sigma)\sigma\sqrt{n(n-1)}(\|\overline{\eta}\|_2\|\xi\|_2 + \|\overline{\xi}\|_2\|\eta\|_2) \\ & + 2(2+\sigma)\sigma n\|\eta\|_2\|\xi\|_2 + \sigma\sqrt{n}(\|\overline{\eta}\|_2 + \|\eta\|_2), \end{aligned}$$

as $\|\xi\|_1 \leqslant \sqrt{n}\|\xi\|_2$, $\|\overline{\xi}\|_1 \leqslant \sqrt{n-1}\|\overline{\xi}\|_2$, $\|\eta\|_1 \leqslant \sqrt{n}\|\eta\|_2$, and $\|\overline{\eta}\|_1 \leqslant \sqrt{n-1}\|\overline{\eta}\|_2$. Therefore, the formula presented in Theorem 3.2 is forward stable.

5. Two algorithms on finding T_C^{-1} and H_C^{-1}

In this section, two algorithms on finding T_C^{-1} and H_C^{-1} are given.

Algorithm 5.1. Using Theorem 2.2, we proceed with

Step 1. Compute $v = (0\ \ c^0(t_{1-n} - it_1)\ \ \cdots\ \ c^{n-3}(t_{-2} - it_{n-2})\ \ c^{n-2}(t_{-1} - it_{n-1}))^T$.

Step 2. Compute $x = (x_1\ \ x_2\ \ \ldots\ \ x_n)^T$ and $y = (y_1\ \ y_2\ \ \ldots\ \ y_n)^T$ by solving the systems of equations

$$T_C x = v \ \text{ and } \ T_C y = e_1.$$

Step 3. Compute T_C^{-1} via formula (2.2).

Algorithm 5.2. Using Theorem 3.2, we proceed with

Step 1. Compute $\hat{v} = (0\ \ c^0(h_0 - ih_n)\ \ c(h_1 - ih_{n+1})\ \ \cdots\ \ c^{n-2}(h_{n-2} - ih_{2n-2}))^T$.

Step 2. Compute $\xi = (\ \xi_1\ \ \xi_2\ \ \cdots\ \ \xi_n\ )^T$ and $\eta = (\ \eta_1\ \ \eta_2\ \ \cdots\ \ \eta_n\ )^T$ by solving the systems of equations

$$H_C \xi = \hat{v} \ \text{ and } \ H_C \eta = e_1.$$

Step 3. Compute H_C^{-1} via formula (3.1).

6. Numerical example

In this section we give two examples to demonstrate our main results.

Example 6.1. Give a 4×4 CT matrix

$$T_C = \begin{pmatrix} i & 2i & 3i & 4i \\ 5i & -i & -2i & -3i \\ i & -5i & i & 2i \\ 3i & -i & 5i & -i \end{pmatrix}.$$

It is obvious that T_C is invertible. By Algorithm 5.1, we have

Step 1. Compute v by Lemma 2.1:

$$v = \begin{pmatrix} 0 \\ -5+4i \\ -1-3i \\ -3+2i \end{pmatrix}.$$

Step 2. Compute x, y by the systems of equations $T_C x = v$, $T_C y = e_1$:

$$x = \begin{pmatrix} \frac{6462+8768i}{10000} \\ \frac{3685-3236i}{10000} \\ \frac{307+177i}{10000} \\ \frac{-4603-1851i}{10000} \end{pmatrix}, \quad y = \begin{pmatrix} -\frac{1229}{10000}i \\ -\frac{951}{10000}i \\ \frac{177}{10000}i \\ -\frac{1850}{10000}i \end{pmatrix}.$$

Step 3. Compute T_C^{-1} by using the equation (2.2):

$$\begin{aligned} T_C^{-1} = &\begin{pmatrix} -\frac{1229}{10000}i & \frac{1850}{10000} & -\frac{177}{10000} & \frac{951}{10000} \\ -\frac{951}{10000}i & -\frac{1229}{10000}i & \frac{1850}{10000} & -\frac{177}{10000} \\ \frac{177}{10000}i & -\frac{951}{10000}i & -\frac{1229}{10000}i & \frac{1850}{10000} \\ -\frac{1850}{10000}i & \frac{177}{10000}i & -\frac{951}{10000}i & -\frac{1229}{10000}i \end{pmatrix} \\ &\times \begin{pmatrix} 1 & \frac{4603+1851i}{10000} & \frac{307+177i}{10000} & \frac{-3685+3236i}{10000} \\ 0 & -1 & -\frac{4603+1851i}{10000} & -\frac{307+177i}{10000} \\ 0 & 0 & 1 & \frac{4603+1851i}{10000} \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{6462+8768i}{10000} & \frac{-4603i+1851}{10000} & \frac{307i-177}{10000} & \frac{3685i+3236}{10000} \\ \frac{3685-3236i}{10000} & \frac{6462+8768i}{10000} & \frac{-4603i+1851}{10000} & \frac{307i-177}{10000} \\ \frac{307+177i}{10000} & \frac{3685-3236i}{10000} & \frac{6462+8768i}{10000} & \frac{-4603i+1851}{10000} \\ \frac{-4603-1851i}{10000} & \frac{307+177i}{10000} & \frac{3685-3236i}{10000} & \frac{6462+8768i}{10000} \end{pmatrix} \\ &\times \begin{pmatrix} 0 & -\frac{1850}{10000}i & -\frac{177}{10000}i & -\frac{951}{10000}i \\ 0 & 0 & \frac{1850}{10000}i & \frac{177}{10000}i \\ 0 & 0 & 0 & -\frac{1850}{10000}i \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ = &\begin{pmatrix} -\frac{1229}{10000}i & -\frac{1761}{10000}i & -\frac{152}{10000}i & \frac{63}{10000}i \\ -\frac{951}{10000}i & -\frac{229}{10000}i & \frac{1635}{10000}i & \frac{152}{10000}i \\ \frac{177}{10000}i & \frac{976}{10000}i & \frac{229}{10000}i & -\frac{1762}{10000}i \\ -\frac{1850}{10000}i & -\frac{177}{10000}i & -\frac{951}{10000}i & \frac{1229}{10000}i \end{pmatrix}. \end{aligned}$$

Example 6.2. Give a 4×4 CH matrix

$$H_C = \begin{pmatrix} i & i & 2i & 3i \\ -i & -2i & -3i & -4i \\ 2i & 3i & 4i & 5i \\ -3i & -4i & -5i & -2i \end{pmatrix}.$$

It is obvious that H_C is invertible. By Algorithm 5.2, we have

Step 1. Compute $\hat{v}$ by Lemma 3.1:

$$\hat{v} = \begin{pmatrix} 0 \\ 4+i \\ -5-i \\ 2+2i \end{pmatrix}.$$

Step 2. Compute ξ, η by the systems of equations $H_C \xi = \hat{v}$, $H_C \eta = e_1$:

$$\xi = \begin{pmatrix} 1-3i \\ -\frac{3}{4}+6i \\ -\frac{1}{2}-3i \\ \frac{1}{4}+i \end{pmatrix}, \quad \eta = \begin{pmatrix} -i \\ 2i \\ -i \\ 0 \end{pmatrix}.$$

Step 3. Compute H_C^{-1} by using the equation (3.1):

$$\begin{aligned} H_C^{-1} &= \begin{pmatrix} -i & 2i & -i & 0 \\ 2i & -i & 0 & 1 \\ -i & 0 & 1 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+3i & -\frac{3}{4}+6i & \frac{1}{2}+3i \\ 0 & -1 & 1-3i & \frac{3}{4}-6i \\ 0 & 0 & 1 & -1+3i \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &+ \begin{pmatrix} 1-3i & -\frac{3}{4}+6i & -\frac{1}{2}-3i & \frac{1}{4}+i \\ -\frac{3}{4}+6i & -\frac{1}{2}-3i & \frac{1}{4}+i & 3+i \\ -\frac{1}{2}-3i & \frac{1}{4}+i & 3+i & -6-\frac{3}{4}i \\ \frac{1}{4}+i & 3+i & -6-\frac{3}{4}i & 3-\frac{1}{2}i \end{pmatrix} \begin{pmatrix} 0 & -i & -2i & -i \\ 0 & 0 & i & 2i \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -i & -2i & -i & 0 \\ 2i & -\frac{1}{4}i & -\frac{3}{2}i & -\frac{1}{4}i \\ -i & \frac{3}{2}i & 2i & \frac{1}{2}i \\ 0 & -\frac{1}{4}i & -\frac{1}{2}i & -\frac{1}{4}i \end{pmatrix}. \end{aligned}$$

7. Implications of application for decomposition formulas (2.2) and (3.1)

We now propose how the analogue of formulas (2.2) and (3.1) lead to a more efficient way to calculate $T_C^{-1}b$ and $H_C^{-1}b$.

Ammar and Gader [1] proposed that the circulant-vector product $z = C_x y$ is equal to the cyclic convolution of the vectors x and y, which we denote by $x * y$. Moreover, $z = x * y$ if and only if $F_n z = (F_n x) \cdot (F_n y)$, where $x \cdot y$ denotes the componentwise product of x and y, $nF_n = [\omega_n^{-jk}]_{j,k=0}^{n-1}$ and $\omega_n = e^{2\pi i/n}$ denotes the principal nth root of unity. Consequently, $z = W_n((F_n x) \cdot (F_n y))$, where $W_n = F_n^{-1} = [\omega_n^{jk}]_0^{n-1} = n\bar{F}_n$. So z can be computed in $\tau(n) + O(n)$ arithmetic operations, where $\tau(n)$ denotes the amount of computation required to perform one real FFT of order n.

According to computational implications of Ammar and Gader [1], we know that the computation of $x = T_C^{-1}b$ using our formula (2.2) requires at most $9\tau(n) + O(n)$ computations, as well as the computation of $x = H_C^{-1}b$ using our formula (3.1) requires at most $9\tau(n) + O(n)$ computations, too.

In Table 1, we list the number of real arithmetic operations (additions and multiplications) required by the algorithms of Gohberg-Semencul formula-like [12] and the Gohberg-Semencul formula [17] for $T_C^{-1}b$ and $H_C^{-1}b$, as well as operation counts for the implementations of formula (2.2), (3.1) described above.

The value of our formulas (2.2), (3.1) increase dramatically in situations in which $T_C^{-1}b_k$ and $H_C^{-1}b_k$ are to be obtained for several different vectors b_k. Instances of this situation are in the iterative improvement of solutions.

Table 1: Operation counts.

Algorithm	Number of real arithmetic operations
1. The Gohberg-Semencul formula-like for T_C	$42n\log_2 n$ [12]
2. The Gohberg-Semencul formula-like for H_C	$42n\log_2 n$ [12]
3. The Gohberg-Semencul formula for T_C	$28n\log_2 n$ [17]
4. The Gohberg-Semencul formula for H_C	$28n\log_2 n$ [17]
5. The imaginary circulant GS formula (2.2) for T_C	$18n\log_2 n$
6. The imaginary left circulant GS formula (3.1) for H_C	$18n\log_2 n$

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