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Pair $(\mathcal{F}, h)$ upper class and (α, μ)-generalized multivalued rational type contractions

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Abstract

In this paper, we introduce notions of (α, μ)-generalized rational contraction conditions and investigate the existence of the fixed point of such mappings on complete metric spaces. To illustrate our result we also construct an example.

Keywords: α-admissible, μ-subadmissible, fixed point, (α, μ)-generalized multivalued rational contraction, pair $(\mathcal{F}, h)$ upper class condition.
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1. Introduction

The notion of the pair $(\mathcal{F}, h)$ is an upper class which was introduced by Ansari et al. [6, 7]. He involved this pair in a contraction condition and proved a fixed point theorem which generalized many existing results. On the other hand, Samet et al. [19] introduced the notions of α-ψ-contractive and α-admissible mappings and proved fixed point theorems which also unify several existing fixed point results in the setting of complete metric spaces. Many authors were inspired by the work of Samet et al. [19] and generalized many other results by using the notion of α-admissible mappings, see for example [1–5, 8–10, 12–18]. Very recently, Karapinar et al. [11] gave a new type of rational contraction condition for multivalued mappings. In this paper, we combine the ideas of Karapinar et al. [11] and Ansari [6], to

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introduce contraction conditions which are even more general than the condition given by the Karapinar et al. [11]. Then by using our new contraction conditions we prove fixed point theorems which give us many other new results in different dimensions.

For the sake of completeness, we collect some necessary notions and basic results from the literature. We denote by $N(X)$ the class of all nonempty subsets of X and $CB(X)$ the class of all nonempty bounded and closed subsets of X. Let (X,d) be a metric space and $H : CB(X) \times CB(X) \to [0,\infty)$ be a mapping such that

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,B)\}$$

for every $A, B \in CB(X)$. Then, this mapping forms a metric and it is called as the Hausdorff metric with respect to d. Let $T : X \to N(X)$ be a multivalued mapping.

Lemma 1.1 ([10]). *Let (X,d) be a metric space, $\{A_k\}$ be a sequence in $CB(X)$, and $\{x_k\}$ be a sequence in X such that $x_k \in A_{k-1}$. Let $\phi : [0,\infty) \to [0,1)$ be a function satisfying $\limsup_{r\to t^+} \phi(r) < 1$ for every $t \in [0,\infty)$. Suppose $\{d(x_{k-1},x_k)\}$ is a nonincreasing sequence such that*

$$\begin{aligned} H(A_{k-1},A_k) &\leqslant \phi(d(x_{k-1},x_k))d(x_{k-1},x_k),\\ d(x_k,x_{k+1}) &\leqslant H(A_{k-1},A_k) + \phi^{n_k}(d(x_{k-1},x_k)),\end{aligned}$$

where $k, n_k \in \mathbb{N}$, and $n_1 < n_2 < \cdots$. Then $\{x_k\}$ is a Cauchy sequence in X.

Lemma 1.2. *If $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that*

$$d(a,b) \leqslant H(A,B) + \epsilon.$$

Definition 1.3 ([16]). Let $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to CB(X)$ is α-admissible if for each $x \in X$ and $y \in Tx$ such that $\alpha(x,y) \geqslant 1$, we have $\alpha(y,z) \geqslant 1$ for each $z \in Ty$.

Definition 1.4 ([17, 18]). Let $\mu : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is μ-subadmissible if

$$x,y \in X,\ \mu(x,y) \leqslant 1 \implies \mu(Tx,Ty) \leqslant 1.$$

We extend above definition to multivalued mappings in the following way.

Definition 1.5. Let $\eta : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to CB(X)$ is η-subadmissible if for each $x \in X$ and $y \in Tx$ such that $\eta(x,y) \leqslant 1$, we have $\eta(y,z) \leqslant 1$ for each $z \in Ty$.

Ansari and Shukla [7] introduced the following functions in order to unify some existing contraction conditions.

Definition 1.6 ([7]). We say that the function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geqslant 1 \Longrightarrow h(1,y) \leqslant h(x,y)$ for all $y \in \mathbb{R}^+$.

Example 1.7 ([7]). Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y) = (y+l)^x, l > 1$;
(b) $h(x,y) = (x+l)^y, l > 1$;
(c) $h(x,y) = x^n y,\ n \in \mathbb{N}$;
(d) $h(x,y) = y$;
(e) $h(x,y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y,\ n \in \mathbb{N}$;
(f) $h(x,y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1, n \in \mathbb{N}$,

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 1.8 ([7]). Let $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and: (i) $0 \leqslant s \leqslant 1 \Longrightarrow \mathcal{F}(s,t) \leqslant \mathcal{F}(1,t)$ for all $t \in \mathbb{R}^+$, (ii) for all $t, y \in \mathbb{R}^+$, $h(1,y) \leqslant \mathcal{F}(1,t) \Longrightarrow y \leqslant t$.

Example 1.9 ([7]). Define $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y) = (y+l)^x, l > 1$ and $\mathcal{F}(s,t) = st + l$;
(b) $h(x,y) = (x+l)^y, l > 1$ and $\mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y) = x^m y$, $m \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;
(d) $h(x,y) = y$ and $\mathcal{F}(s,t) = t$;
(d) $h(x,y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;
(e) $h(x,y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1, n \in \mathbb{N}$ and $\mathcal{F}(s,t) = (1+l)^{st}$,

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 1.10 ([7]). We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II, if $x, y \geqslant 1 \Longrightarrow h(1,1,z) \leqslant h(x,y,z)$ for all $z \in \mathbb{R}^+$.

Example 1.11 ([7]). Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(b) $h(x,y,z) = (xy+l)^z, l > 1$;
(c) $h(x,y,z) = z$;
(d) $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}$;
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$,

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 1.12 ([7]). Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and: (i) $0 \leqslant s \leqslant 1 \Longrightarrow \mathcal{F}(s,t) \leqslant \mathcal{F}(1,t)$ for all $t \in \mathbb{R}^+$, (ii) for all $s, t, z \in \mathbb{R}^+$, $h(1,1,z) \leqslant \mathcal{F}(s,t) \Longrightarrow z \leqslant st$.

Example 1.13 ([7]). Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1, \mathcal{F}(s,t) = st + l$;
(b) $h(x,y,z) = (xy+l)^z, l > 1, \mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y,z) = z, F(s,t) = st$;
(d) $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s,t) = s^p t^p$;
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, \mathcal{F}(s,t) = s^k t^k$,

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

2. Main results

We begin this section by introducing the notion of (α, μ)-generalized multivalued rational contraction condition.

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \to CB(X)$ is said to be an (α, μ)-generalized multivalued rational contraction mapping of type I, if there exist four functions $\alpha, \mu : X \times X \to [0, \infty)$, $\phi : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$ and $\varphi : [0, \infty) \to [0, \infty)$, such that

$$h(\alpha(x,y), H(Tx, Ty)) \leqslant \mathcal{F}(\mu(x,y), \phi(M(x,y))M(x,y) + \varphi(N(x,y))N(x,y)), \quad \forall\, x, y \in X, \tag{2.1}$$

where $(\mathcal{F}, h)$ is an upper class of type I, and

$$M(x,y) = \max\left\{d(x,y), \frac{d(y,Tx)d(x,Ty)}{1+d(x,Tx)}\right\},$$

and

$$N(x,y) = \min\left\{d(x,y), \frac{d^2(x,Tx)}{1+d(x,Tx)}, \frac{d^2(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)d(y,Tx)}{1+d(y,Ty)}, \frac{d(y,Tx)d(x,Ty)}{1+d(x,Tx)}\right\}.$$

Before moving towards our main results, we shall prove the following auxiliary lemma.

Lemma 2.2. *Let* (X, d) *be a metric space and* $T : X \to CB(X)$ *be an* (α, μ)*-generalized multivalued rational contraction of type I with the four functions* α, μ, ϕ, *and* φ *as in Definition 2.1. Let* $\mathcal{O}_T(x_0) = \{x_k\}$ *be an orbit of* T *at* x_0 *such that* $\alpha(x_{k-1}, x_k) \geqslant 1$, $\mu(x_{k-1}, x_k) \leqslant 1$ *and*

$$d(x_k, x_{k+1}) \leqslant H(Tx_{k-1}, Tx_k) + \phi^{n_k}(M(x_{k-1}, x_k)), \tag{2.2}$$

where $n_1 < n_2 < \cdots$ *with* $k, n_k \in \mathbb{N}$ *and* $\{d(x_{k-1}, x_k)\}$ *is non-increasing sequence. Then* $\{x_k\}$ *is a Cauchy sequence in* X.

Proof. By hypothesis of lemma, we have $\mathcal{O}_T(x_0) = \{x_k\}$ an orbit of T at x_0 such that $\alpha(x_k, x_{k+1}) \geqslant 1$ and $\mu(x_k, x_{k+1}) \leqslant 1$ for each $k \in \mathbb{N} \cup \{0\}$. Since T is an (α, μ)-generalized multivalued rational contraction mapping of type I, we have

$$\begin{aligned}
h(1, H(Tx_{k-1}, Tx_k)) &\leqslant h(\alpha(x_{k-1}, x_k), H(Tx_{k-1}, Tx_k)) \\
&\leqslant \mathcal{F}\left(\mu(x_{k-1}, x_k), \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k)\right) \\
&\leqslant \mathcal{F}\left(1, \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k)\right) \\
&= \mathcal{F}\Big(1, \phi\Big(\max\Big\{d(x_{k-1}, x_k), \frac{d(x_k, Tx_{k-1})d(x_{k-1}, Tx_k)}{1 + d(x_{k-1}, Tx_{k-1})}\Big\}\Big) \\
&\quad \times \max\Big\{d(x_{k-1}, x_k), \frac{d(x_k, Tx_{k-1})d(x_{k-1}, Tx_k)}{1 + d(x_{k-1}, Tx_{k-1})}\Big\} \\
&\quad + \varphi\Big(\min\Big\{d(x_{k-1}, x_k), \frac{d^2(x_{k-1}, Tx_{k-1})}{1 + d(x_{k-1}, Tx_{k-1})}, \frac{d^2(x_k, Tx_k)}{1 + d(x_k, Tx_k)}, \\
&\quad \frac{d(x_{k-1}, Tx_k)d(x_k, Tx_{k-1})}{1 + d(x_k, Tx_k)}, \frac{d(x_k, Tx_{k-1})d(x_{k-1}, Tx_k)}{1 + d(x_{k-1}, Tx_{k-1})}\Big\}\Big) \\
&\quad \times \min\Big\{d(x_{k-1}, x_k), \frac{d^2(x_{k-1}, Tx_{k-1})}{1 + d(x_{k-1}, Tx_{k-1})}, \frac{d^2(x_k, Tx_k)}{1 + d(x_k, Tx_k)}, \\
&\quad \frac{d(x_{k-1}, Tx_k)d(x_{k-1}, Tx_{k-1})}{1 + d(x_k, Tx_k)}, \frac{d(x_k, Tx_{k-1})d(x_{k-1}, Tx_k)}{1 + d(x_{k-1}, Tx_{k-1})}\Big\}\Big) \\
&= \mathcal{F}(1, \phi(d(x_{k-1}, x_k))d(x_{k-1}, x_k) + \varphi(0)0) \\
&= \mathcal{F}(1, \phi(d(x_{k-1}, x_k))d(x_{k-1}, x_k)).
\end{aligned}$$

So,

$$H(Tx_{k-1}, Tx_k)) \leqslant \phi(d(x_{k-1}, x_k))d(x_{k-1}, x_k).$$

Regarding that $\{d(x_{k-1}, x_k)\}$ is non-increasing, (2.2) yields that

$$\begin{aligned}
d(x_k, x_{k-1}) &\leqslant H(Tx_{k-1}, Tx_k) + \phi^{n_k}(M(x_{k-1}, x_k)) \\
&= H(Tx_{k-1}, Tx_k) + \phi^{n_k}\Big(\max\Big\{d(x_{k-1}, x_k), \frac{d(x_k, Tx_{k-1})d(x_{k-1}, Tx_k)}{1 + d(x_{k-1}, Tx_{k-1})}\Big\}\Big) \\
&= H(Tx_{k-1}, Tx_k) + \phi^{n_k}(d(x_{k-1}, x_k)).
\end{aligned}$$

Thus, we conclude that the sequence $\{x_k\}$ is Cauchy in X due to Lemma 1.1. □

Theorem 2.3. *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be an* (α, μ)*-generalized multivalued rational contraction of type I such that the following conditions hold:*

(i) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geqslant 1$ *and* $\mu(x_0, x_1) \leqslant 1$;

(ii) T *is* α*-admissible and* μ*-subadmissible mapping;*

(iii) *either*

a) T *is continuous; or*

b) *for each sequence* $\{x_k\}$ *such that* $\alpha(x_k, x_{k+1}) \geqslant 1$, $\mu(x_k, x_{k+1}) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$ *and* $x_k \to x$ *as* $k \to \infty$, *we have* $\alpha(x_k, x) \geqslant 1$, $\mu(x_k, x) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$, $\mu(x_0, x_1) \leqslant 1$. Further, we can choose a positive integer n_1 such that

$$\phi^{n_1}(M(x_0, x_1)) \leqslant [1 - \phi(M(x_0, x_1))]M(x_0, x_1). \tag{2.3}$$

By Lemma 1.2, we have $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leqslant H(Tx_0, Tx_1) + \phi^{n_1}(M(x_0, x_1)). \tag{2.4}$$

Using the notion of an (α, μ)-generalized multivalued rational contraction mapping type I, we have

$$\begin{aligned} h(1, H(Tx_0, Tx_1)) &\leqslant h(\alpha(x_0, x_1), H(Tx_0, Tx_1)) \\ &\leqslant \mathcal{F}(\mu(x_0, x_1), \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1)) \\ &\leqslant \mathcal{F}(1, \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1)) \\ &\Longrightarrow \\ H(Tx_0, Tx_1) &\leqslant \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1). \end{aligned}$$

Using (2.4), (2.3), and above inequality, we have

$$\begin{aligned} d(x_1, x_2) &\leqslant H(Tx_0, Tx_1) + \phi^{n_1}(M(x_0, x_1)) \\ &\leqslant H(Tx_0, Tx_1) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\ &\leqslant \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\ &= \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1) + M(x_0, x_1) - \phi(M(x_0, x_1))M(x_0, x_1) \\ &= M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1) \\ &= \max\left\{d(x_0, x_1), \frac{d(x_1, Tx_0)d(x_0, Tx_1)}{1 + d(x_0, Tx_0)}\right\} \\ &\quad + \varphi\left(\min\left\{d(x_0, x_1), \frac{d^2(x_0, Tx_0)}{1 + d(x_0, Tx_0)}, \frac{d^2(x_1, Tx_1)}{1 + d(x_1, Tx_1)}, \frac{d(x_0, Tx_1)d(x_1, Tx_0)}{1 + d(x_1, Tx_1)}, \frac{d(x_1, Tx_0)d(x_0, Tx_1)}{1 + d(x_0, Tx_0)}\right\}\right) \\ &\quad \times \min\left\{d(x_0, x_1), \frac{d^2(x_0, Tx_0)}{1 + d(x_0, Tx_0)}, \frac{d^2(x_1, Tx_1)}{1 + d(x_1, Tx_1)}, \frac{d(x_0, Tx_1)d(x_1, Tx_0)}{1 + d(x_1, Tx_1)}, \frac{d(x_1, Tx_0)d(x_0, Tx_1)}{1 + d(x_0, Tx_0)}\right\} \\ &\Longrightarrow \\ d(x_1, x_2) &\leqslant d(x_0, x_1). \end{aligned}$$

Since T is α-admissible and μ-subadmissible, we have $\alpha(x_1, x_2) \geqslant 1, \mu(x_1, x_2) \leqslant 1$. Now, we can choose a positive integer n_2, $n_2 > n_1$ such that

$$\phi^{n_2}(M(x_1, x_2)) \leqslant [1 - \phi(M(x_1, x_2))]M(x_1, x_2). \tag{2.5}$$

Again using Lemma 1.2 and we have $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leqslant H(Tx_1, Tx_2) + \phi^{n_2}(M(x_1, x_2)). \tag{2.6}$$

From (2.1), we have

$$
\begin{aligned}
h(1,H(Tx_1,Tx_2)) &\leqslant h(\alpha(x_1,x_2),H(Tx_1,Tx_2))\\
&\leqslant \mathcal{F}(\mu(x_1,x_2),\phi(M(x_1,x_2))M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2)\\
&\leqslant \mathcal{F}(1,\phi(M(x_1,x_2))M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2)\\
&\Longrightarrow\\
H(Tx_1,Tx_2) &\leqslant \phi(M(x_1,x_2))M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2).
\end{aligned}
$$

Again by using (2.6), (2.5), and above inequality, we have

$$
\begin{aligned}
d(x_2,x_3) &\leqslant H(Tx_1,Tx_2)+\phi^{n_2}(M(x_1,x_2))\\
&\leqslant H(Tx_1,Tx_2)+[1-\phi(M(x_1,x_2))]M(x_1,x_2)\\
&\leqslant \phi(M(x_1,x_2))M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2)+[1-\phi(M(x_1,x_2))]M(x_1,x_2)\\
&= \phi(M(x_1,x_2))M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2)+M(x_1,x_2)-\phi(M(x_1,x_2))M(x_1,x_2)\\
&= M(x_1,x_2)+\varphi(N(x_1,x_2))N(x_1,x_2)\\
&= \max\left\{d(x_1,x_2),\frac{d(x_2,Tx_1)d(x_1,Tx_2)}{1+d(x_1,Tx_1)}\right\}\\
&\quad+\varphi\left(\min\left\{d(x_1,x_2),\frac{d^2(x_1,Tx_1)}{1+d(x_1,Tx_1)},\frac{d^2(x_2,Tx_2)}{1+d(x_2,Tx_2)},\frac{d(x_1,Tx_2)d(x_2,Tx_1)}{1+d(x_2,Tx_2)},\frac{d(x_2,Tx_1)d(x_1,Tx_2)}{1+d(x_1,Tx_1)}\right\}\right)\\
&\quad\times\min\left\{d(x_1,x_2),\frac{d^2(x_1,Tx_1)}{1+d(x_1,Tx_1)},\frac{d^2(x_2,Tx_2)}{1+d(x_2,Tx_2)},\frac{d(x_1,Tx_2)d(x_2,Tx_1)}{1+d(x_2,Tx_2)},\frac{d(x_2,Tx_1)d(x_1,Tx_2)}{1+d(x_1,Tx_1)}\right\}\\
&\Longrightarrow\\
d(x_2,x_3) &\leqslant d(x_1,x_2).
\end{aligned}
$$

By repeating this process, for all $k\in\mathbb{N}$, we may choose a positive integer n_k such that

$$\phi^{n_k}(M(x_{k-1},x_k))\leqslant[1-\phi(M(x_{k-1},x_k))]M(x_{k-1},x_k) \tag{2.7}$$

and $x_{k+1}\in Tx_k$ such that

$$d(x_k,x_{k+1})\leqslant H(Tx_{k-1},Tx_k)+\phi^{n_k}(M(x_{k-1},x_k)). \tag{2.8}$$

By using α-admissibility and μ-subadmissibility of T, we get $\alpha(x_k,x_{k+1})\geqslant 1$, $\mu(x_k,x_{k+1})\leqslant 1$ for each $k\in\mathbb{N}\cup\{0\}$. Now by last four inequalities together with the notion of an (α,μ)-generalized multivalued rational contraction mapping of type I, we have

$$
\begin{aligned}
h(1,H(Tx_{k-1},Tx_k)) &\leqslant h(\alpha(x_{k-1},x_k),H(Tx_{k-1},Tx_k))\\
&\leqslant \mathcal{F}(\mu(x_{k-1},x_k),\phi(M(x_{k-1},x_k))M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k))\\
&\leqslant \mathcal{F}(1,\phi(M(x_{k-1},x_k))M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k)),\ \forall k\in\mathbb{N},\\
&\Longrightarrow\\
H(Tx_{k-1},Tx_k) &\leqslant \phi(M(x_{k-1},x_k))M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k)),\quad \forall k\in\mathbb{N}.
\end{aligned}
$$

Thus, by using (2.8), (2.7), and above inequality, we have

$$
\begin{aligned}
d(x_k,x_{k+1}) &\leqslant H(Tx_{k-1},Tx_k)+\phi^{n_k}(M(x_{k-1},x_k))\\
&\leqslant H(Tx_{k-1},Tx_k)+[1-\phi(M(x_{k-1},x_k))]M(x_{k-1},x_k)\\
&\leqslant \phi(M(x_{k-1},x_k))M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k)+[1-\phi(M(x_{k-1},x_k))]M(x_{k-1},x_k)\\
&= \phi(M(x_{k-1},x_k))M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k)+M(x_{k-1},x_k)\\
&\quad-\phi(M(x_{k-1},x_k))M(x_{k-1},x_k)
\end{aligned}
$$

$$
\begin{aligned}
&= M(x_{k-1},x_k)+\varphi(N(x_{k-1},x_k))N(x_{k-1},x_k)\\
&= \max\left\{d(x_{k-1},x_k),\frac{d(x_k,Tx_{k-1})d(x_{k-1},Tx_k)}{1+d(x_{k-1},Tx_{k-1})}\right\}\\
&\quad+\varphi\Big(\min\Big\{d(x_{k-1},x_k),\frac{d^2(x_{k-1},Tx_{k-1})}{1+d(x_{k-1},Tx_{k-1})},\frac{d^2(x_k,Tx_k)}{1+d(x_k,Tx_k)},\\
&\quad\frac{d(x_{k-1},Tx_k)d(x_k,Tx_{k-1})}{1+d(x_k,Tx_k)},\frac{d(x_k,Tx_{k-1})d(x_{k-1},Tx_k)}{1+d(x_{k-1},Tx_{k-1})}\Big\}\Big)\\
&\quad\times\min\Big\{d(x_{k-1},x_k),\frac{d^2(x_{k-1},Tx_{k-1})}{1+d(x_{k-1},Tx_{k-1})},\\
&\quad\frac{d^2(x_k,Tx_k)}{1+d(x_k,Tx_k)},\frac{d(x_{k-1},Tx_k)d(x_k,Tx_{k-1})}{1+d(x_k,Tx_k)},\frac{d(x_k,Tx_{k-1})d(x_{k-1},Tx_k)}{1+d(x_{k-1},Tx_{k-1})}\Big\},\quad\forall k\in\mathbb{N}\\
&\Longrightarrow\\
&d(x_k,x_{k+1})\leqslant d(x_{k-1},x_k)\ \text{for each } k\in\mathbb{N}.
\end{aligned}
$$

Accordingly, $\{d(x_{k-1},x_k)\}$ is non-increasing sequence of non-negative numbers. Now it follows from Lemma 2.2 that $\{x_k\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z\in X$ such that $x_n\to z$ as $n\to\infty$. If T is continuous, then, clearly we have $z\in Tz$. If (iii-b) holds, by using the notion of an (α,μ)-generalized multivalued rational contraction mapping of type I, we get

$$
\begin{aligned}
h(1,H(Tx_{k-1},Tz))&\leqslant h(\alpha(x_{k-1},z),H(Tx_{k-1},Tz))\\
&\leqslant \mathcal{F}(\eta(x_{k-1},z),\phi(M(x_{k-1},z))M(x_{k-1},z)+\varphi(N(x_{k-1},z))N(x_{k-1},z))\\
&\leqslant \mathcal{F}(1,\phi(M(x_{k-1},z))M(x_{k-1},z)+\varphi(N(x_{k-1},z))N(x_{k-1},z))\\
&\Longrightarrow\\
H(Tx_{k-1},Tz)&\leqslant \phi(M(x_{k-1},z))M(x_{k-1},z)+\varphi(N(x_{k-1},z))N(x_{k-1},z)).
\end{aligned}
$$

Now, by using the triangular inequality and above inequality, we have

$$
\begin{aligned}
d(z,Tz)&\leqslant d(z,x_k)+d(x_k,Tz)\\
&\leqslant d(z,x_k)+H(Tx_{k-1},Tz)\\
&\leqslant d(z,x_k)+\phi(M(x_{k-1},z))M(x_{k-1},z)+\varphi(N(x_{k-1},z))N(x_{k-1},z)\\
&= d(z,x_k)+\phi\Big(\max\Big\{d(x_{k-1},z),\frac{d(z,Tx_{k-1})d(x_{k-1},Tz)}{1+d(x_{k-1},Tx_{k-1})}\Big\}\Big)\\
&\quad\times\max\Big\{d(x_{k-1},z),\frac{d(z,Tx_{k-1})d(x_{k-1},Tz)}{1+d(x_{k-1},Tx_{k-1})}\Big\}\\
&\quad+\varphi\Big(\min\Big\{d(x_{k-1},z),\frac{d^2(x_{k-1},Tx_{k-1})}{1+d(x_{k-1},Tx_{k-1})},\frac{d^2(z,Tz)}{1+d(z,Tz)},\frac{d(x_{k-1},Tz)d(z,Tx_{k-1})}{1+d(z,Tz)},\\
&\quad\frac{d(z,Tx_{k-1})d(x_{k-1},Tz)}{1+d(x_{k-1},Tx_{k-1})}\Big\}\Big)\times\min\Big\{d(x_{k-1},z),\frac{d^2(x_{k-1},Tx_{k-1})}{1+d(x_{k-1},Tx_{k-1})},\\
&\quad\frac{d^2(z,Tz)}{1+d(z,Tz)},\frac{d(x_{k-1},Tz)d(z,Tx_{k-1})}{1+d(z,Tz)},\frac{d(z,Tx_{k-1})d(x_{k-1},Tz)}{1+d(x_{k-1},Tx_{k-1})}\Big\},\quad\forall k\in\mathbb{N}.
\end{aligned}
$$

Letting $k\to\infty$ in the above inequality, the right hand side of above inequality tends to zero. Thus, we have $d(z,Tz)=0$. Therefore, T has a fixed point. □

Example 2.4. Let $X=[0,\infty)$ be endowed with the usual metric $d(x,y)=|x-y|$. Define $T:X\to CB(X)$ by

$$
Tx=\begin{cases}[0,\frac{x}{4}], & \text{if } x\in[0,2),\\ \{0\}, & \text{if } x=2,\\ \{2x\}, & \text{otherwise,}\end{cases}
$$

and $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 4, & \text{if } x,y \in [0,2], \\ \frac{1}{4}, & \text{if } x,y \in (2,3], \\ 0, & \text{otherwise}, \end{cases}$$

and $\mu : X \times X \to [0,\infty)$ by $\mu(x,y) = \frac{1}{2}$ for each $x,y \in X$. Consider $h(x,y) = (x)^{\frac{1}{2}}y$, $F(x,y) = y$ for each $x,y \geqslant 0$ and $\phi(t) = \frac{4}{5}$ for each $t \geqslant 0$,

$$\varphi(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{2}{t} & \text{if } t \neq 0. \end{cases}$$

It is easy to see that T is generalized multivalued rational contraction mapping of type I and all other conditions of Theorem 2.3 are satisfied. Thus T has a fixed point.

Definition 2.5. Let (X,d) be a metric space. A mapping $T : X \to CB(X)$ is said to be an (α,μ)-generalized multivalued rational contraction mapping of type II, if there exist five functions $\alpha,\beta : X \to [0,\infty)$, $\mu : X \times X \to [0,\infty)$, $\phi : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \phi(r) < 1$, $\forall t \in [0,\infty)$, and $\varphi : [0,\infty) \to [0,\infty)$, such that

$$h(\alpha(x),\beta(y),H(Tx,Ty)) \leqslant \mathcal{F}(\mu(x,y),\phi(M(x,y))M(x,y) + \varphi(N(x,y))N(x,y)), \quad \forall\, x,y \in X,$$

where $(\mathcal{F},h)$ is an upper class of type II, and

$$M(x,y) = \max\left\{d(x,y), \frac{d(y,Tx)d(x,Ty)}{1+d(x,Tx)}\right\},$$

and

$$N(x,y) = \min\left\{d(x,y), \frac{d^2(x,Tx)}{1+d(x,Tx)}, \frac{d^2(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)d(y,Tx)}{1+d(y,Ty)}, \frac{d(y,Tx)d(x,Ty)}{1+d(x,Tx)}\right\}.$$

The proof of the following lemma and theorem can be obtained on the same lines as above is done.

Lemma 2.6. *Let (X,d) be a metric space and $T : X \to CB(X)$ be a (α,μ)-generalized multivalued rational contraction of type II with the five functions α, β, μ, ϕ, and φ as in Definition 2.5. Let $\mathcal{O}_T(x_0) = \{x_k\}$ be an orbit of T at x_0 such that $\alpha(x_{k-1}) \geqslant 1$, $\beta(x_{k-1}) \geqslant 1$, $\mu(x_{k-1},x_k) \leqslant 1$, and*

$$d(x_k,x_{k+1}) \leqslant H(Tx_{k-1},Tx_k) + \phi^{n_k}(M(x_{k-1},x_k)),$$

where $n_1 < n_2 < \cdots$ with $k,n_k \in \mathbb{N}$ and $\{d(x_{k-1},x_k)\}$ is non-increasing sequence. Then $\{x_k\}$ is a Cauchy sequence in X.

Let $\alpha : X \to [0,\infty)$ be a function and $T : X \to CB(X)$. We say that T is α-admissible, if for each $x \in X$ with $\alpha(x) \geqslant 1$, we have $\alpha(z) \geqslant 1$ for all $z \in Tx$.

Theorem 2.7. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be an (α,μ)-generalized multivalued rational contraction of type II such that the following conditions hold:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\mu(x_0,x_1) \leqslant 1$;*

(iii) *T is α-admissible, β-admissible, and μ-subadmissible mapping;*

(iv) *either*

  a) *T is continuous; or*
  b) *for each sequence $\{x_k\}$ such that $\alpha(x_k) \geqslant 1$, $\beta(x_k) \geqslant 1$ and $\mu(x_k,x_{k+1}) \leqslant 1$ for each $k \in \mathbb{N}\cup\{0\}$ and $x_k \to x$ as $k \to \infty$, we have either $\alpha(x_k) \geqslant 1$, $\beta(x) \geqslant 1$ and $\mu(x_k,x) \leqslant 1$ for each $k \in \mathbb{N}\cup\{0\}$ or $\alpha(x) \geqslant 1$, $\beta(x_k) \geqslant 1$ and $\mu(x,x_k) \leqslant 1$ for each $k \in \mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

3. Consequences

In this section, we present some fixed point theorems which are the direct consequences of our results.

Theorem 3.1. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping such that there exist four functions* $\alpha, \mu : X \times X \to [0, \infty)$, $\phi : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r\to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$, *and* $\varphi : [0, \infty) \to [0, \infty)$, *satisfying*

$$h(\alpha(x, y), d(Tx, Ty)) \leqslant \mathcal{F}(\mu(x, y), \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y)), \quad \forall\, x, y \in X,$$

where $(\mathcal{F}, h)$ *is an upper class of type I, and*

$$M(x, y) = \max\left\{d(x, y), \frac{d(y, Tx)d(x, Ty)}{1 + d(x, Tx)}\right\},$$

and

$$N(x, y) = \min\left\{d(x, y), \frac{d^2(x, Tx)}{1 + d(x, Tx)}, \frac{d^2(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty)d(y, Tx)}{1 + d(y, Ty)}, \frac{d(y, Tx)d(x, Ty)}{1 + d(x, Tx)}\right\}.$$

Further, assume that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\mu(x_0, Tx_0) \leqslant 1$;

(ii) T *is* α*-admissible and* μ*-subadmissible mapping;*

(iii) *either*

a) T *is continuous; or*

b) *for each sequence* $\{x_k\}$ *such that* $\alpha(x_k, x_{k+1}) \geqslant 1, \mu(x_k, x_{k+1}) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$ *and* $x_k \to x$ *as* $k \to \infty$, *we have* $\alpha(x_k, x) \geqslant 1$, $\mu(x_k, x) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

Theorem 3.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist four functions* $\alpha, \mu : X \times X \to [0, \infty)$, $\phi : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r\to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$, *and* $\varphi : [0, \infty) \to [0, \infty)$, *such that*

$$h(\alpha(x, y), d(Tx, Ty)) \leqslant \mathcal{F}(\mu(x, y), \phi(d(x, y))d(x, y) + \varphi(N(x, y))N(x, y)), \quad \forall\, x, y \in X,$$

where $(\mathcal{F}, h)$ *is an upper class of type I, and*

$$N(x, y) = \min\left\{d(x, y), \frac{d^2(x, Tx)}{1 + d(x, Tx)}, \frac{d^2(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty)d(y, Tx)}{1 + d(y, Ty)}, \frac{d(y, Tx)d(x, Ty)}{1 + d(x, Tx)}\right\}.$$

Suppose also that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\mu(x_0, Tx_0) \leqslant 1$;

(ii) T *is* α*-admissible and* μ*-subadmissible mapping;*

(iii) *either*

a) T *is continuous; or*

b) *for each sequence* $\{x_k\}$ *such that* $\alpha(x_k, x_{k+1}) \geqslant 1, \mu(x_k, x_{k+1}) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$ *and* $x_k \to x$ *as* $k \to \infty$, *we have* $\alpha(x_k, x) \geqslant 1$, $\mu(x_k, x) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

Theorem 3.3. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist three functions* $\alpha, \mu : X \times X \to [0, \infty)$, $\phi : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r \to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$ *such that*

$$h(\alpha(x, y), d(Tx, Ty)) \leqslant \mathcal{F}(\mu(x, y), \phi(d(x, y))d(x, y)), \ \ \forall\, x, y \in X,$$

where $(\mathcal{F}, h)$ *is an upper class of type I. Suppose also that the following conditions hold:*

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\mu(x_0, Tx_0) \leqslant 1$;

(ii) T *is* α*-admissible and* μ*-subadmissible mapping;*

(iii) *either*

a) T *is continuous; or*

b) *for each sequence* $\{x_k\}$ *such that* $\alpha(x_k, x_{k+1}) \geqslant 1, \mu(x_k, x_{k+1}) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$ *and* $x_k \to x$ *as* $k \to \infty$, *we have* $\alpha(x_k, x) \geqslant 1$, $\mu(x_k, x) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

If we take $h(x, y) = xy$ and $\mathcal{F}(r, t) = rt$ in Theorem 3.3, we have the following result.

Theorem 3.4. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist three functions* $\alpha, \mu : X \times X \to [0, \infty)$, $\phi : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r \to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$ *such that*

$$\alpha(x, y)d(Tx, Ty) \leqslant \mu(x, y)\phi(d(x, y))d(x, y), \ \ \forall\, x, y \in X.$$

Suppose also that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\mu(x_0, Tx_0) \leqslant 1$;

(ii) T *is* α*-admissible and* μ*-subadmissible mapping;*

(iii) *either*

a) T *is continuous; or*

b) *for each sequence* $\{x_k\}$ *such that* $\alpha(x_k, x_{k+1}) \geqslant 1, \mu(x_k, x_{k+1}) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$ *and* $x_k \to x$ *as* $k \to \infty$, *we have* $\alpha(x_k, x) \geqslant 1$, $\mu(x_k, x) \leqslant 1$ *for each* $k \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

By taking $\alpha(x, y) = 1$, $\mu(x, y) = 1$ for all $x, y \in X$, in Theorem 3.4 we get the following result.

Theorem 3.5. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist two functions* $\phi : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r \to t^+} \phi(r) < 1$, $\forall t \in [0, \infty)$ *and* $\varphi : [0, \infty) \to [0, \infty)$, *such that*

$$d(Tx, Ty) \leqslant \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y), \ \ \forall\, x, y \in X,$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(y, Tx)d(x, Ty)}{1 + d(x, Tx)}\right\},$$

and

$$N(x, y) = \min\left\{d(x, y), \frac{d^2(x, Tx)}{1 + d(x, Tx)}, \frac{d^2(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty)d(y, Tx)}{1 + d(y, Ty)}, \frac{d(y, Tx)d(x, Ty)}{1 + d(x, Tx)}\right\}.$$

Then T *has a fixed point.*

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