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Some identities for umbral calculus associated with partially degenerate Bell numbers and polynomials

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Abstract

In this paper, we study partially degenerate Bell numbers and polynomials by using umbral calculus. We give some new identities for these numbers and polynomials which are associated with special numbers and polynomial.

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1. Introduction

The Bell polynomials (also called the exponential polynomials and denoted by $\phi_n(x)$) are defined by the generating function (see [4, 10, 16])

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x)\frac{t^n}{n!}. \tag{1.1}$$

From (1.1), we note that

$$\mathrm{Bel}_0(x) = 1,\ \mathrm{Bel}_1(x) = x,\ \mathrm{Bel}_2(x) = x^2 + x,\ \mathrm{Bel}_3(x) = x^3 + 3x^2 + x,$$
$$\mathrm{Bel}_4(x) = x^4 + 6x^3 + 7x^2 + x,\ \mathrm{Bel}_5(x) = x^5 + 10x^4 + 25x^3 + 15x^2 + x,\ \cdots.$$

When $x = 1$, $\mathrm{Bel}_n = \mathrm{Bel}_n(1)$ are called the Bell numbers. As is well-known, the Stirling numbers of the second kind are defined by the generating function (see [12, 14, 16])

$$\left(e^t - 1\right)^m = m! \sum_{n=m}^{\infty} S_2(n, m)\frac{t^n}{n!},\ (m \geqslant 0).$$

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The Stirling numbers of the first kind are defined as (see [1–17])

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \ (n \geqslant 0),$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geqslant 1)$. From (1.1), we note that (see [10, 12, 16])

$$\mathrm{Bel}_n(x) = \sum_{l=0}^{n} S_2(n,l)x^l, \ (n \geqslant 0).$$

The Bernoulli polynomials are given by the generating function as follows (see [1–17])

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $B_n = B_n(x)$ are called Bernoulli numbers. From (1.2), we note that (see [7, 11, 12, 16])

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}.$$

Note that

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x), \ (n \geqslant 1).$$

Recently, Kim-Kim considered the partially degenerate Bell polynomials which are given by the generating function (see [13])

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.3}$$

When $x=1$, $\mathrm{Bel}_{n,\lambda} = \mathrm{Bel}_{n,\lambda}(1)$ are called the partially degenerate Bell numbers. Note that $\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x) = \mathrm{Bel}_n(x)$, $(n \geqslant 0)$. Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of a linear functional $L \in \mathbb{P}^*$on a polynomial $p(x)$ is denoted by $\langle L \,|\, p(x)\rangle$, which is linearly extended by the rule $\langle cL + c'L' \,|\, p(x)\rangle = c\langle L \,|\, p(x)\rangle + c'\langle L' \,|\, p(x)\rangle$, where $c, c' \in \mathbb{C}$. For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, the action $\langle f(t) \,|\, \cdot\rangle$ of the linear functional $f(t)$ on $\mathbb{P}$ is defined by (see [11, 16])

$$\langle f(t) \,|\, x^n\rangle = a_n \text{ for all } n \geqslant 0. \tag{1.4}$$

By (1.4), we easily get (see [11, 14, 16])

$$\langle t^k \,|\, x^n\rangle = n!\delta_{n,k}, \ (n,k \geqslant 0), \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{k=0}^{\infty}\langle L \,|\, x^k\rangle \frac{t^k}{k!}$. Then, from (1.5), we have

$$\langle f_L(t) \,|\, x^n\rangle = \langle L \,|\, x^n\rangle.$$

Additionally, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ umbral algebra. The umbral calculus is the study of umbral algebra (see [11, 14, 16]).

Let $f(t)(\neq 0) \in \mathcal{F}$. Then the order of $f(t)$ is the smallest positive integer k for which the coefficient of t^k does not vanish. The order of $f(t)$ is denoted by $o\big(f(t)\big)$. Let $f(t), g(t) \in \mathcal{F}$ with $o\big(f(t)\big) = 1$ and $o\big(g(t)\big) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that

$$\left\langle g(t)f(t)^k \,\middle|\, S_n(x)\right\rangle = \left\langle g(t) \,\middle|\, f(t)^k S_n(x)\right\rangle = n!\delta_{n,k}, \;\; (n,k \geqslant 0), \quad \text{(see [16])}.$$

The sequence $S_n(x)$ is called the Sheffer sequence for $\big(g(t), f(t)\big)$, for which we write $S_n(x) \sim \big(g(t), f(t)\big)$.

Let $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then, by (1.5), we get

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) \,|\, x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k \,|\, p(x) \rangle \frac{x^k}{k!}. \tag{1.6}$$

From (1.6), we have

$$p^{(k)}(0) = \langle t^k \,|\, p(x) \rangle = \langle 1 \,|\, p^{(k)}(x) \rangle, \; p^{(k)}(x) = \left(\frac{d}{dx}\right)^k p(x). \tag{1.7}$$

By (1.7), we easily get

$$t^k p(x) = p^{(k)}(x), \; e^{yt} p(x) = p(x+y), \text{ and } \langle e^{yt} \,|\, p(x) \rangle = p(y). \tag{1.8}$$

It is well-known that

$$S_n(x) \sim \big(g(t), f(t)\big) \Longleftrightarrow \frac{1}{g\big(\bar{f}(t)\big)} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \quad \text{(see [16])}, \tag{1.9}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $\bar{f}\big(f(t)\big) = f\big(\bar{f}(t)\big) = t$. Let $f_1(t), f_2(t), \cdots, f_m(t) \in \mathcal{F}$. Then we have

$$\langle f_1(t) \cdots f_m(t) \,|\, x^n \rangle = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1, \cdots, i_m} \langle f_1(t) \,|\, x^{i_1} \rangle \cdots \langle f_m(t) \,|\, x^{i_m} \rangle.$$

Let $f(t)$ be the linear functional such that

$$\langle f(t) \,|\, p(x) \rangle = \int_0^y p(u)\,du \tag{1.10}$$

for all polynomials $p(x)$. Then, from (1.6), we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \,|\, x^k \rangle}{k!} t^k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} t^k = \frac{1}{t}\big(e^{yt} - 1\big). \tag{1.11}$$

Thus, by (1.10) and (1.11), we get

$$\left\langle \frac{e^{yt}-1}{t} \,\middle|\, p(x) \right\rangle = \int_0^y p(u)\,du, \quad \frac{e^{yt}-1}{t} p(x) = \int_x^{x+y} p(u)\,du. \tag{1.12}$$

For $S_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, we have (see [11, 14, 16])

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{1.13}$$

where

$$C_{n,k} = \frac{1}{k!}\left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)}\left(l\left(\bar{f}(t)\right)\right)^{k} \;\middle|\; x^{n}\right\rangle. \tag{1.14}$$

In this paper, we study partially degenerate Bell numbers and polynomials by using umbral calculus. We give some new identities for these numbers and polynomials which are associated with special numbers and polynomials.

2. Some identities for the partially degenerate Bell numbers and polynomials

From (1.3), we recall that

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}.$$

Note that

$$\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!} = \lim_{\lambda\to 0} e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = e^{x(e^t-1)} = \sum_{n=0}^{\infty}\mathrm{Bel}_n(x)\frac{t^n}{n!}. \tag{2.1}$$

By (2.1), we get

$$\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x) = \mathrm{Bel}_n(x),\ \ (n \geqslant 0).$$

From (1.9), we note that

$$\mathrm{Bel}_{n,\lambda}(x) \sim \left(1,\ \frac{(1+t)^{\lambda}-1}{\lambda}\right) \Longleftrightarrow e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!} = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} &= \sum_{m=0}^{\infty}\frac{x^m}{m!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^m \\
&= \sum_{m=0}^{\infty}\frac{x^m}{m!}\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-1\right)^m \\
&= \sum_{m=0}^{\infty}x^m\sum_{k=m}^{\infty}S_2(k,m)\lambda^{-k}\frac{1}{k!}\left(\log(1+\lambda t)\right)^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_2(k,m)x^m\lambda^{-k}\right)\frac{1}{k!}\left(\log(1+\lambda t)\right)^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_2(k,m)x^m\lambda^{-k}\right)\sum_{n=k}^{\infty}S_1(n,k)\lambda^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}S_2(k,m)S_1(n,k)x^m\lambda^{n-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Comparing the coefficients on both sides of (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geqslant 0$, we have*

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}S_2(k,m)S_1(n,k)\lambda^{n-k}x^m = \sum_{m=0}^{n}\left(\sum_{k=m}^{n}S_2(k,m)S_1(n,k)\lambda^{n-k}\right)x^m.$$

By (1.12), we get

$$\begin{aligned}\frac{e^t-1}{t}\mathrm{Bel}_{n,\lambda}(x) &= \int_x^{x+1}\mathrm{Bel}_{n,\lambda}(u)du\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}S_2(k,m)S_1(n,k)\lambda^{n-k}\int_x^{x+1}u^m du\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}\left((x+1)^{m+1}-x^{m+1}\right)\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}\frac{1}{m+1}S_2(k,m)S_1(n,k)\lambda^{n-k}x^l\\ &= \sum_{l=0}^{n}\left(\sum_{k=l}^{n}\sum_{m=l}^{k}\binom{m+1}{l}\frac{1}{m+1}S_2(k,m)S_1(n,k)\lambda^{n-k}\right)x^l.\end{aligned} \tag{2.4}$$

From (1.2) and (1.9), we have

$$B_n(x)\sim\left(\frac{e^t-1}{t},t\right)\Longleftrightarrow\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{2.5}$$

Thus, by (2.5), we get

$$B_n(x)=\frac{t}{e^t-1}x^n,\ (n\geqslant 0). \tag{2.6}$$

From (2.4) and (2.6), we can derive the following equation (2.7):

$$\begin{aligned}\mathrm{Bel}_{n,\lambda}(x) &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}\frac{t}{e^t-1}x^l\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}B_l(x).\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For* $n\geqslant 0$, *we have*

$$\mathrm{Bel}_{n,\lambda}(x)=\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}B_l(x).$$

By (1.8) and (2.7), we get

$$\begin{aligned}t\mathrm{Bel}_{n,\lambda}(x) &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}tB_l(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=1}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{l}{m+1}B_{l-1}(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=0}^{m-1}\binom{m+1}{l+1}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{l+1}{m+1}B_l(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=0}^{m-1}\binom{m}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}B_l(x).\end{aligned}$$

From (1.12), we have

$$\begin{aligned}\frac{e^{yt}-1}{t}\mathrm{Bel}_{n,\lambda}(x) &= \int_x^{x+y}\mathrm{Bel}_{n,\lambda}(u)du\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{m+1}\int_x^{x+y}B_l(u)du\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{m+1}{l}S_2(k,m)S_1(n,k)\lambda^{n-k}\frac{1}{(m+1)(l+1)}\\ &\quad\times\Big(B_{l+1}(x+y)-B_{l+1}(x)\Big)\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\sum_{j=0}^{l}\binom{m+1}{l}\binom{l+1}{j}S_2(k,m)S_1(n,k)\frac{\lambda^{n-k}}{(m+1)(l+1)}y^{l+1-j}B_j(x).\end{aligned} \tag{2.8}$$

Thus, by (2.8), we get

$$\begin{aligned}&\mathrm{Bel}_{n,\lambda}(x+y)-\mathrm{Bel}_{n,\lambda}(x)\\ &= \Big(e^{yt}-1\Big)\mathrm{Bel}_{n,\lambda}(x)\\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{m}\sum_{j=0}^{l}\binom{m+1}{l}\binom{l+1}{j}S_2(k,m)S_1(n,k)\frac{\lambda^{n-k}}{(m+1)(l+1)}y^{l+1-j}tB_j(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=1}^{m}\sum_{j=1}^{l}\binom{m+1}{l}\binom{l+1}{j}S_2(k,m)S_1(n,k)\frac{j\lambda^{n-k}}{(m+1)(l+1)}y^{l+1-j}B_{j-1}(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=1}^{m}\sum_{j=0}^{l-1}\binom{m+1}{l}\binom{l+1}{j+1}S_2(k,m)S_1(n,k)\frac{(j+1)\lambda^{n-k}}{(m+1)(l+1)}y^{l-j}B_j(x)\\ &= \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=1}^{m}\sum_{j=0}^{l-1}\binom{m+1}{l}\binom{l}{j}S_2(k,m)S_1(n,k)\frac{\lambda^{n-k}}{m+1}y^{l-j}B_j(x).\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \geqslant 1$, we have*

$$\mathrm{Bel}_{n,\lambda}(x+y)-\mathrm{Bel}_{n,\lambda}(x) = \sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{l=1}^{m}\sum_{j=0}^{l-1}\binom{m+1}{l}\binom{l}{j}S_2(k,m)S_1(n,k)\frac{\lambda^{n-k}}{m+1}y^{l-j}B_j(x).$$

Let

$$\mathbb{P}_n = \Big\{p(x)\in\mathbb{C}[x]\;\Big|\;\deg p(x)\leqslant n\Big\},\ (n\geqslant 0).$$

For $p(x)\in\mathbb{P}_n$, we assume that

$$p(x) = \sum_{l=0}^{n} a_l \mathrm{Bel}_{l,\lambda}(x). \tag{2.10}$$

From (2.2), we have

$$\Big\langle \Big(\tfrac{(t+1)^\lambda-1}{\lambda}\Big)^m \;\Big|\; \mathrm{Bel}_{n,\lambda}(x)\Big\rangle = n!\delta_{m,n},\ (m,n\geqslant 0). \tag{2.11}$$

By (2.10) and (2.11), we get

$$\left\langle \left(\tfrac{(t+1)^\lambda-1}{\lambda}\right)^m \,\middle|\, p(x)\right\rangle = \sum_{l=0}^{n} a_l \left\langle \left(\tfrac{(t+1)^\lambda-1}{\lambda}\right)^m \,\middle|\, \mathrm{Bel}_{l,\lambda}(x)\right\rangle = \sum_{l=0}^{n} a_l l! \delta_{l,m} = m! a_m. \tag{2.12}$$

From (2.12), we have

$$a_m = \frac{1}{m!}\left\langle \left(\tfrac{(t+1)^\lambda-1}{\lambda}\right)^m \,\middle|\, p(x)\right\rangle, \ (m \geqslant 0). \tag{2.13}$$

Therefore, by (2.10) and (2.13), we obtain the following theorem.

Theorem 2.4. *For* $p(x) \in \mathbb{P}_n$, $(n \geqslant 0)$, *we have*

$$p(x) = \sum_{m=0}^{n} a_m \mathrm{Bel}_{m,\lambda}(x),$$

where

$$a_m = \frac{1}{m!}\left\langle \left(\tfrac{(t+1)^\lambda-1}{\lambda}\right)^m \,\middle|\, p(x)\right\rangle.$$

Let us take $p(x) = B_n(x) \in \mathbb{P}_n$, $(n \geqslant 0)$. Then, by Theorem 2.4, we get

$$B_n(x) = \sum_{m=0}^{n} a_m \mathrm{Bel}_{m,\lambda}(x), \tag{2.14}$$

where

$$\begin{aligned} a_m &= \frac{1}{m!}\left\langle \left(\tfrac{(1+t)^\lambda-1}{\lambda}\right)^m \,\middle|\, B_n(x)\right\rangle = \frac{\lambda^{-m}}{m!}\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}\left\langle (1+t)^{\lambda l} \middle| B_n(x)\right\rangle \\ &= \frac{\lambda^{-m}}{m!}\sum_{l=0}^{m}\sum_{k=0}^{\infty}\binom{m}{l}\binom{\lambda l}{k}(-1)^{m-l}\left\langle t^k \,\middle|\, B_n(x)\right\rangle \\ &= \frac{\lambda^{-m}}{m!}\sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{\lambda l}{k}(-1)^{m-l}(n)_k B_{n-k} \\ &= \frac{\lambda^{-m}}{m!}\sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{\lambda l}{k}\binom{n}{k}(-1)^{m-l}B_{n-k}k!. \end{aligned} \tag{2.15}$$

We have another expression for a_m.

$$\begin{aligned} a_m &= \lambda^{-m}\left\langle \frac{1}{m!}\left((1+t)^\lambda-1\right)^m \middle| \, B_n(x)\right\rangle \\ &= \lambda^{-m}\left\langle \frac{1}{m!}\left(e^{\lambda\log(1+t)}-1\right)^m \middle| \, B_n(x)\right\rangle \\ &= \lambda^{-m}\sum_{k=m}^{n} S_2(k,m)\lambda^k\left\langle \frac{1}{k!}(\log(1+t))^k \middle| \, B_n(x)\right\rangle \\ &= \sum_{k=m}^{n} S_2(k,m)\lambda^{k-m}\left\langle \sum_{j=k}^{n} S_1(j,k)\frac{t^j}{j!} \middle| \, B_n(x)\right\rangle \\ &= \sum_{k=m}^{n}\sum_{j=k}^{n} S_2(k,m)S_1(j,k)\lambda^{k-m}\binom{n}{j}B_{n-j}. \end{aligned} \tag{2.16}$$

Therefore, by (2.14), (2.15), and (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geqslant 0$, we have*

$$\begin{aligned} B_n(x) &= \sum_{m=0}^{n} \left(\sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{\lambda l}{k} \binom{n}{k} \lambda^{-m} (-1)^{m-l} \frac{k!}{m!} B_{n-k} \right) \mathrm{Bel}_{m,\lambda}(x) \\ &= \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \sum_{j=k}^{n} S_2(k,m) S_1(j,k) \lambda^{k-m} \binom{n}{j} B_{n-j} \right) \mathrm{Bel}_{m,\lambda}(x). \end{aligned}$$

Let us consider the following two Sheffer sequences:

$$\mathrm{Bel}_{n,\lambda}(x) \sim \left(1, \frac{(1+t)^{\lambda}-1}{\lambda}\right), \quad x^n \sim (1,t), \ (n \geqslant 0). \tag{2.17}$$

Then, by (1.13), (1.14), and (2.17), we get

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n} C_{n,m} x^m, \tag{2.18}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left((\lambda t + 1)^{\frac{1}{\lambda}} - 1 \right)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \left\langle (\lambda t + 1)^{\frac{l}{\lambda}} \middle| x^n \right\rangle \\ &= \frac{1}{m!} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \sum_{k=0}^{\infty} \binom{\frac{l}{\lambda}}{k} \lambda^k \langle t^k | x^n \rangle \\ &= \frac{n!}{m!} \sum_{l=0}^{m} \binom{m}{l} \binom{\frac{l}{\lambda}}{n} (-1)^{m-l} \lambda^n. \end{aligned} \tag{2.19}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.6. *For $n \geqslant 0$, we have*

$$\mathrm{Bel}_{n,\lambda}(x) = n! \lambda^n \sum_{m=0}^{n} \left(\sum_{l=0}^{m} \frac{1}{m!} \binom{m}{l} \binom{\frac{l}{\lambda}}{n} (-1)^{m-l} \right) x^m.$$

Remark 2.7. Note that

$$\lambda^n \binom{\frac{l}{\lambda}}{n} = \frac{l(l-\lambda)(l-2\lambda)\cdots(l-(n-1)\lambda)}{n!}, \ (n \geqslant 0). \tag{2.20}$$

By (2.20), we easily get

$$\lim_{\lambda \to 0} \lambda^n \binom{\frac{l}{\lambda}}{n} = \frac{l^n}{n!}, \ (n \geqslant 0).$$

From Theorem 2.6, we have

$$\begin{aligned} \lim_{\lambda \to 0} \mathrm{Bel}_{n,\lambda}(x) &= \sum_{m=0}^{n} \left(\sum_{l=0}^{m} \frac{1}{m!} \binom{m}{l} l^n (-1)^{m-l} \right) x^m \\ &= \sum_{m=0}^{n} \left(\frac{1}{m!} \Delta^m 0^n \right) x^m = \sum_{m=0}^{n} S_2(n,m) x^m = \mathrm{Bel}_n(x), \end{aligned}$$

where Δ is the difference operator with $\Delta f(x) = f(x+1) - f(x)$.

For $(x)_n \sim \left(1, e^t - 1\right)$, we have

$$e^{x\log(1+t)} = \sum_{m=0}^{\infty} x^m \frac{1}{m!}\Big(\log(1+t)\Big)^m = \sum_{m=0}^{\infty} x^m \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_1(n,m)x^m\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.$$

Let us consider the following two Sheffer sequences:

$$\mathrm{Bel}_{n,\lambda}(x) \sim \left(1,\ \frac{(1+t)^{\lambda}-1}{\lambda}\right), \quad (x)_n \sim (1,\ e^t - 1). \tag{2.21}$$

From (1.13), (1.14), and (2.21), we have

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n} C_{n,m}(x)_m, \ (n \geqslant 0). \tag{2.22}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)}-1\right)^m \,\middle|\, x^n\right\rangle \\ &= \sum_{k=m}^{\infty} S_2(k,m)\frac{1}{k!}\left\langle \left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)^k \,\middle|\, x^n\right\rangle \\ &= \sum_{k=m}^{n}\frac{S_2(k,m)}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\langle (\lambda t+1)^{\frac{l}{\lambda}} \,\middle|\, x^n\right\rangle \\ &= \sum_{k=m}^{n}\frac{S_2(k,m)}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{j=0}^{\infty}\binom{\frac{l}{\lambda}}{j}\lambda^j\langle t^j \mid x^n\rangle \\ &= \sum_{k=m}^{n}\frac{S_2(k,m)}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\binom{\frac{l}{\lambda}}{n}\lambda^n n! \\ &= n!\lambda^n\sum_{k=m}^{n}\sum_{l=0}^{k}\frac{S_2(k,m)}{k!}\binom{k}{l}\binom{\frac{l}{\lambda}}{n}(-1)^{k-l}. \end{aligned} \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geqslant 0$, we have*

$$\mathrm{Bel}_{n,\lambda}(x) = n!\lambda^n\sum_{m=0}^{n}\left(\sum_{k=m}^{n}\sum_{l=0}^{k}\frac{S_2(k,m)}{k!}\binom{k}{l}\binom{\frac{l}{\lambda}}{n}(-1)^{k-l}\right)(x)_m.$$

On the other hand,

$$C_{n,m} = \frac{1}{m!}\left\langle \left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)}-1\right)^m \,\middle|\, x^n\right\rangle, \tag{2.24}$$

$$\left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)}-1\right)^m = \underbrace{\left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)}-1\right)\times\cdots\times\left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)}-1\right)}_{m-\text{times}}$$
$$= \left(\sum_{l_1=1}^{\infty}\mathrm{Bel}_{l_1,\lambda}\frac{t^{l_1}}{l_1!}\right)\times\cdots\times\left(\sum_{l_m=1}^{\infty}\mathrm{Bel}_{l_m,\lambda}\frac{t^{l_m}}{l_m!}\right) \tag{2.25}$$

$$= \sum_{k=m}^{\infty} \left(\sum_{\substack{l_1+\cdots+l_m=k \\ l_1,\cdots,l_m \geqslant 1}} \binom{k}{l_1,\cdots,l_m} \mathrm{Bel}_{l_1,\lambda} \cdots \mathrm{Bel}_{l_m,\lambda} \right) \frac{t^k}{k!}.$$

From (2.24) and (2.25), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left(e^{\left((\lambda t+1)^{\frac{1}{\lambda}}-1\right)} - 1 \right)^m \,\middle|\, x^n \right\rangle \\ &= \frac{1}{m!} \sum_{k=m}^{n} \left(\sum_{\substack{l_1+\cdots+l_m=k \\ l_1,\cdots,l_m \geqslant 1}} \binom{k}{l_1,\cdots,l_m} \mathrm{Bel}_{l_1,\lambda} \cdots \mathrm{Bel}_{l_m,\lambda} \right) \frac{1}{k!} \langle t^k | x^n \rangle \\ &= \frac{1}{m!} \sum_{\substack{l_1+\cdots+l_m=n \\ l_1,\cdots,l_m \geqslant 1}} \binom{n}{l_1,\cdots,l_m} \mathrm{Bel}_{l_1,\lambda} \cdots \mathrm{Bel}_{l_m,\lambda}. \end{aligned} \tag{2.26}$$

Therefore, by (2.22) and (2.26), we get

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n} \left(\frac{1}{m!} \sum_{\substack{l_1+\cdots+l_m=n \\ l_1,\cdots,l_m \geqslant 1}} \binom{n}{l_1,\cdots,l_m} \mathrm{Bel}_{l_1,\lambda} \cdots \mathrm{Bel}_{l_m,\lambda} \right) (x)_m.$$

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