Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2976–2985
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

The split common fixed-point problem for demicontractive mappings and quasi-nonexpansive mappings

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Communicated by P. Kumam

Abstract

In this paper, we study a split common fixed-point problem for demicontractive mappings and quasi-nonexpansive mappings, and propose some cyclic iterative schemes. Moreover we prove some strong convergence theorems. The results obtained in this paper generalize and improve the recent ones announced by many others.

Keywords: Split common fixed-point problem, demicontractive mapping, strong convergence, cyclic iterative scheme.
2010 MSC: 47H05, 47H09, 47H20.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) originally introduced in Censor and Elfving [1] is to find a point $x^* \in C$ with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1}$$

It serves as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in these operator's ranges. There are a number of significant applications of the SFP in intensity-modulated radiation therapy, signal processing, image reconstruction and so on. Recently the SFP has been widely studied by many authors (see, e.g., [3, 12, 13, 14]).

In the case where C and Q in the SFP (1.1) are the intersections of finitely many fixed-point sets of nonlinear operators, the problem (1.1) is called by Censor and Segal [2] the split common fixed-point problem (SCFP). More precisely, the SCFP requires to seek an element $x^* \in H_1$ satisfying

$$x^* \in \cap_{i=1}^{p} \mathrm{Fix}(U_i) \quad \text{and} \quad Ax^* \in \cap_{j=1}^{s} \mathrm{Fix}(T_j), \tag{1.2}$$

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doi:10.22436/jnsa.010.06.12

Received 2017-03-07

where $p, s \geqslant 1$ are integers, $\mathrm{Fix}(U_i)$ and $\mathrm{Fix}(T_j)$ denote the fixed point sets of two classes of nonlinear operators $U_i : H_1 \to H_1$ $(i = 1, 2, \cdots, p)$, $T_j : H_2 \to H_2$ $(j = 1, 2, \cdots, s)$. In particular, if $p = s = 1$, the problem (1.2) is reduced to find a point x^* with the property:

$$x^* \in \mathrm{Fix}(U) \text{ and } Ax^* \in \mathrm{Fix}(T), \tag{1.3}$$

which is usually called the two-set SCFP. To solve the two-set SCFP (1.3), Censor and Segal [2] proposed the following iterative method: for any initial guess $x_1 \in H_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = U(x_n - \lambda A^*(I - T)Ax_n),$$

where U and T are directed operators. The further generalization of this algorithm was studied by Moudafi [8] for demicontractive operators. Under suitable conditions he proved that the sequence $\{x_n\}$ converges weakly to a point of the two-set SCFP (1.3).

Recently, Wang and Xu [10] proposed the following cyclic algorithm:

$$x_{n+1} = U_{[n]}(x_n - \lambda A^*(I - T_{[n]})Ax_n),$$

where U_i and T_i are directed operators for $i = 1, 2, \cdots, p$, $[n] = n \pmod p$. They proved that the sequence $\{x_n\}$ generated by this algorithm converges weakly to a solution of the problem (1.2) if $p = s$.

Since the existing algorithm for the SCFP (1.2) has only weak convergence in infinite-dimensional spaces (see [8, 10]), Cui et al. [3] proposed a new iterative scheme as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n U_{[n]}[(1 - \alpha_n)(x_n - \lambda_n A^*(I - T_{[n]})Ax_n)],$$

where U_i and T_i are directed operators for $i = 1, 2, \cdots, p$. They proved that the sequence $\{x_n\}$ converges strongly to a solution of the problem (1.2) if $p = s$.

Motivated by the above works, we propose two algorithms for solving the SCFP (1.2) in the more general case of mappings which are demicontractive and quasi-nonexpansive, including nonexpansive mappings and directed operators in infinite-dimensional spaces and establish some strong convergence theorems.

2. Preliminaries

Throughout this paper, let N and R be the set of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let T be a mapping of C into H. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

In order to facilitate our investigation in this paper, we recall some definitions as follows.

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) nonexpansive if
$$\|Tx - Ty\| \leqslant \|x - y\|, \ \forall\, x, y \in H;$$

(ii) quasi-nonexpansive if
$$\|Tx - q\| \leqslant \|x - q\|, \ \forall\, (x, q) \in H \times \mathrm{Fix}(T);$$

(iii) firmly nonexpansive if
$$\|Tx - Ty\|^2 \leqslant \langle x - y, Tx - Ty \rangle, \ \forall\, x, y \in H;$$

(iv) directed if
$$\|Tx - q\|^2 \leqslant \|x - q\|^2 - \|x - Tx\|^2, \ \forall\, (x, q) \in H \times \mathrm{Fix}(T);$$

(v) μ-demicontractive if there exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \leqslant \|x - q\|^2 + \mu\|x - Tx\|^2, \ \ \forall\, (x, q) \in H \times Fix(T),$$

which is equivalent to

$$\langle x - Tx, x - q\rangle \geqslant \frac{1-\mu}{2}\|x - Tx\|^2.$$

It is worth noting that the class of demicontractive mappings contains important mappings such as quasi-nonexpansive mappings and directed operators.

Remark 2.2. Notice that 0-demicontractive is exactly quasi-nonexpansive. In particular, we say that it is quasi-strictly pseudo-contractive [7], if $0 \leqslant \mu < 1$. Moreover, if $\mu \leqslant 0$, every μ-demicontractive mapping becomes quasi-nonexpansive. So, it seems to be sufficient to only take $\mu \in (0, 1)$ in (v) of Definition 2.1 in Hilbert spaces. However, as seen in (iv) of Definition 2.1, every directed operator is obvious (-1)-demicontractive.

Recall that the metric (or nearest point) projection from H onto C is the mapping $P : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\|.$$

It is well-known that $P_C x$ is characterized by the inequality

$$\langle x - P_C x, y - P_C x\rangle \leqslant 0, \ \ \forall\, y \in C. \tag{2.1}$$

Let us also recall that $I - T$ is said to be demiclosed at origin, if for any sequence $\{x_k\} \subset H$ and $x^* \in H$, we have

$$\left.\begin{array}{l} x_k \rightharpoonup x^* \\ (I - T)x_k \to 0 \end{array}\right\} \Rightarrow x^* = Tx^*.$$

As a special case of the demicloseness principle on uniformly convex Banach spaces given by [4], we know that if C is a nonempty closed convex subset of a Hilbert space H, and $T : C \to H$ is a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C. Now the following question is naturally raised: If $T : C \to H$ is quasi-nonexpansive, is $I - T$ still demiclosed on C? The answer is negative even at 0 as follows.

Example 2.3 (see [9, Example 2.11]). *The mapping* $T : [0, 1] \to [0, 1]$ *is defined by*

$$Tx = \begin{cases} \frac{x}{5}, & x \in [0, \frac{1}{2}], \\ x \sin \pi x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T *is a quasi-nonexpansive mapping, but* $I - T$ *is not demiclosed at 0.*

Remark 2.4. Notice that a demicontractive mapping could enjoy the demiclosedness property at origin, for example, let $H = \ell_2$ and let $T : C \to H$ be defined by $Tx = -kx$, for arbitrary $x \in \ell_2$, where $k > 1$ (see [9, Example 2.5]). Then T is not quasi-nonexpansive but μ-demicontactive, where $\mu = \frac{k-1}{k+1}$. However, $I - T$ is obviously demiclosed at 0. For, whenever $\{x_n\}$ is any sequence in ℓ_2 such that $x_n \rightharpoonup x \in \ell_2$ and $\|x_n - Tx_n\| \to 0$, we readily see that $x = 0 \in F(T)$.

In what follows, we give some lemmas needed for the convergence analysis of our algorithms. Let H_1 and H_2 be two real Hilbert spaces.

Lemma 2.5 ([11]). *Assume that* $\{a_n\}$ *is a sequence of non-negative real numbers such that*

$$a_{n+1} \leqslant (1 - \gamma_n)a_n + \gamma_n b_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{b_n\}$ *is a sequence in* R *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leqslant 0$ *or* $\Sigma_{n=1}^{\infty}\gamma_n|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([10])**.** *Assume that* $A : H_1 \to H_2$ *is a bounded linear operator and* $T : H_2 \to H_2$ *is a demicontractive operator. Let* $V_\lambda = I - \lambda A^*(I-T)A$ *with* $\lambda > 0$*. Then*

$$\mathrm{Fix}(V_\lambda) = A^{-1}(\mathrm{Fix}(T)),$$

whenever $A^{-1}(\mathrm{Fix}(T)) = \{x \in H_1 : Ax \in \mathrm{Fix}(T)\}$.

Lemma 2.7 ([8])**.** *Assume that* $A : H_1 \to H_2$ *is a bounded linear operator and* $T : H_2 \to H_2$ *is a* μ*-demicontractive operator. Let* $V_\lambda = I - \lambda A^*(I-T)A$, $\lambda \in (0, (1-\mu)/\rho)$ *with* ρ *being the spectral radius of the operator* A^*A*. Then*

(i)
$$\|V_\lambda x - z\|^2 \leqslant \|x - z\|^2 - \lambda(1-\mu-\rho\lambda)\|(I-T)Ax\|^2, \quad \forall\, z \in A^{-1}(\mathrm{Fix}(T)),$$

consequently,

(ii)
$$\|V_\lambda x - z\| \leqslant \|x - z\|, \quad \forall\, z \in A^{-1}(\mathrm{Fix}(T)).$$

Lemma 2.8 ([5])**.** *For any* $x, y \in H$ *and* $\lambda \in \mathbb{R}$, *the following hold:*

(a) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$;

(b) $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.9 ([7, Proposition 2.1])**.** *Assume* C *is a closed convex subset of a Hilbert space* H*. Let* $T : C \to C$ *be a self-mapping of* C*. If* T *is a* μ*-demicontractive mapping (which is also called* μ*-quasi-strict pseudo-contraction in [7]), then the fixed point set* $F(T)$ *is closed and convex.*

3. Main results

In this section, let H_1 and H_2 be two real Hilbert spaces. We consider the SCFP (1.2) with $p = s$ to find an element $x^* \in H_1$ satisfying

$$x^* \in \cap_{i=1}^{p}\mathrm{Fix}(U_i) \;\text{ and }\; Ax^* \in \cap_{i=1}^{p}\mathrm{Fix}(T_i), \tag{3.1}$$

where p is a positive integer. Denote the solution set of the SCFP (3.1) by Ω, i.e.,

$$\Omega = (\cap_{i=1}^{p}\mathrm{Fix}(U_i)) \bigcap A^{-1}(\cap_{i=1}^{p}\mathrm{Fix}(T_i)).$$

Note that the problem (3.1) is a special case of the problem (1.2). However, this is not restrictive. Because following an idea in [10], one can easily extend the results to the general case.

For fixed positive integer p and each $n \geqslant 1$, the p-mod function $[n]$ is defined by

$$[n] = \begin{cases} p, & \text{if } r = 0, \\ r, & \text{if } 0 < r < p, \end{cases}$$

whenever $n = kp + r$ for some $k \geqslant 0$.

Lemma 3.1. *Let* $\{u_k\}$ *be a bounded sequence of a Hilbert space* H*. Let* p *be a positive integer and* $I = \{1, 2, \cdots, p\}$*. If* $\lim_{k\to\infty}\|u_{k+1} - u_k\| = 0$ *and* $x^* \in \omega_w(u_k)$, *then for any* $i \in I$, *there exists a subsequence* $\{u_{k_m}\}$ *of* $\{u_k\}$ *such that* $[k_m] = i$ *and* $u_{k_m} \rightharpoonup x^*$.

Proof. Obviously, $\omega_w(u_k) \neq \emptyset$ from boundedness of $\{u_k\}$. Now for any $i \in I$, since $\lim_{k\to\infty} \|u_{k+1} - u_k\| = 0$, we have

$$\|u_{k+i} - u_k\| \leqslant \|u_{k+i} - u_{k+i-1}\| + \|u_{k+i-1} - u_{k+i-2}\| + \cdots + \|u_{k+1} - u_k\| \to 0.$$

It follows from $x^* \in \omega_w(u_k)$ that there exists a subsequence $\{u_{t_m}\}$ of $\{u_k\}$ such that $u_{t_m} \rightharpoonup x^*$. So due to $\|u_{k+i} - u_k\| \to 0$ we obtain $u_{t_m+i} \rightharpoonup x^*$ for all $i \in I$. For any $i \in I$, there exists $t_1 + i_1 \in \{t_1 + 1, t_1 + 2, \cdots, t_1 + p\}$ such that $[t_1 + i_1] = i$. We choose $k_1 = t_1 + i_1$. And there exists $t_2 + i_2 \in \{t_2 + 1, t_2 + 2, \cdots, t_2 + p\}$ such that $[t_2 + i_2] = i$. If $t_2 + i_2 > k_1$, we choose $k_2 = t_2 + i_2$; if $t_2 + i_2 \leqslant k_1$, we skip it and go to the t_3. Repeating this process continuously, we can choose a subsequence $\{k_m\}$ such that $[k_m] = i$ for all $m \geqslant 1$ and $u_{k_m} \rightharpoonup x^*$ too. □

Theorem 3.2. *Let U_i be quasi-nonexpansive and T_i be μ_i-demicontractive such that $I - U_i$ and $I - T_i$ are demi-closed at origin for every $i = 1, 2, \cdots, p$. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $\Omega \neq \emptyset$ and ρ is as in Lemma 2.7. For any $x_1 \in H_1$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n U_n[(1 - \alpha_n)(x_n - \lambda_n A^*(I - T_n)Ax_n)], \tag{3.2}$$

where $U_n = U_{[n]}, T_n = T_{[n]}$ and $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, +\infty)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < \dfrac{1-\mu}{\rho}$, $\mu = \max_{1\leqslant i\leqslant p} \{\mu_i\}$.

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega(0)$.

Proof. From Lemma 2.9, for every $i \in \{1, 2, \cdots, p\}$, we notice that $\mathrm{Fix}(T_i)$ and $\mathrm{Fix}(U_i)$ are closed and convex. Thus $\bigcap_{i=1}^p \mathrm{Fix}(T_i)$ and $\bigcap_{i=1}^p F(U_i)$ are also closed and convex. Since A is bounded and linear, $A^{-1}(\cap_{i=1}^p \mathrm{Fix}(T_i))$ is closed and convex. Therefore, Ω is closed and convex.

Let $W_n = I - \lambda_n A^*(I - T_n)A$, $y_n = (1 - \alpha_n)W_n x_n$. Let $z = P_\Omega(0)$. Noting that for every $i(1 \leqslant i \leqslant p)$, $\mu_i \leqslant \mu$, so from Lemma 2.7 and the condition (iii) we have

$$\|W_n x_n - z\|^2 \leqslant \|x_n - z\|^2 - \lambda_n(1 - \mu - \lambda_n\rho)\|(I - T_n)Ax_n\|^2 \tag{3.3}$$
$$\leqslant \|x_n - z\|^2. \tag{3.4}$$

It follows from (3.4) that

$$\begin{aligned} \|y_n - z\| &= \|(1 - \alpha_n)(W_n x_n - z) - \alpha_n z\| \\ &\leqslant (1 - \alpha_n)\|W_n x_n - z\| + \alpha_n\|z\| \\ &\leqslant (1 - \alpha_n)\|x_n - z\| + \alpha_n\|z\|, \end{aligned}$$

then

$$\begin{aligned} \|x_{n+1} - z\| &\leqslant (1 - \beta_n)\|x_n - z\| + \beta_n\|U_n y_n - z\| \\ &\leqslant (1 - \beta_n)\|x_n - z\| + \beta_n\|y_n - z\| \\ &\leqslant (1 - \beta_n)\|x_n - z\| + \beta_n[(1 - \alpha_n)\|x_n - z\| + \alpha_n\|z\|] \\ &= (1 - \alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n\|z\| \\ &\leqslant \max\{\|x_1 - z\|, \|z\|\}. \end{aligned}$$

Thus $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{W_n x_n\}$. From (3.3), the quasi-nonexpansivity of U_n and Lemma 2.8 (b), we obtain

$$\begin{aligned}\|U_n y_n - z\|^2 &\leqslant \|y_n - z\|^2\\ &= \|(1-\alpha_n)(W_n x_n - z) - \alpha_n z\|^2\\ &\leqslant (1-\alpha_n)\|W_n x_n - z\|^2 + 2\alpha_n\langle z, z - y_n\rangle\\ &\leqslant (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle z, z-y_n\rangle\\ &\quad - \lambda_n(1-\alpha_n)(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2. \end{aligned} \tag{3.5}$$

It follows from (3.2), (3.5) and Lemma 2.8 (a) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\beta_n)(x_n - z) + \beta_n(U_n y_n - z)\|^2\\ &= (1-\beta_n)\|x_n - z\|^2 + \beta_n\|U_n y_n - z)\|^2 - \beta_n(1-\beta_n)\|U_n y_n - x_n\|^2\\ &\leqslant (1-\beta_n)\|x_n - z\|^2 - \beta_n(1-\beta_n)\|U_n y_n - x_n\|^2\\ &\quad + \beta_n[(1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle z, z-y_n\rangle\\ &\quad - \lambda_n(1-\alpha_n)(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2]\\ &= (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle z, z-y_n\rangle\\ &\quad - \beta_n(1-\beta_n)\|U_n y_n - x_n\|^2 - \lambda_n\beta_n(1-\alpha_n)(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2,\end{aligned}$$

i.e., we have the following inequality

$$s_{n+1} \leqslant (1-\alpha_n\beta_n)s_n + 2\alpha_n\beta_n\langle z, z-y_n\rangle - c_n \tag{3.6}$$

$$\qquad \leqslant (1-\alpha_n\beta_n)s_n + 2\alpha_n\beta_n\langle z, z-y_n\rangle, \tag{3.7}$$

where $s_n = \|x_n - z\|^2$ and

$$c_n = \lambda_n\beta_n(1-\alpha_n)(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2 + \beta_n(1-\beta_n)\|U_n y_n - x_n\|^2.$$

It follows from (3.6) that

$$c_n \leqslant M\alpha_n\beta_n + s_n - s_{n+1}, \tag{3.8}$$

where $M = 2\sup_{n\geqslant 1}\{\|z\|\cdot\|z-y_n\|\}$.

Finally we will prove $s_n \to 0$. To see this, let us consider two possible cases on such a sequence and employ an idea developed by Mainge [6].

Case I. Assume that there exists an integer N_1 such that $s_n \geqslant s_{n+1}$ for all $n \geqslant N_1$. In this case $\{s_n\}$ must be convergent. So due to (3.8) and the conditions (i)-(iii), we have both $\{\|(I-T_n)Ax_n\|\}$ and $\{\|U_n y_n - x_n\|\}$ converge to zero. Then we obtain

$$\|x_{n+1} - x_n\| = \beta_n\|U_n y_n - x_n\| \to 0, \tag{3.9}$$

and

$$\begin{aligned}\|y_n - x_n\| &= \|(1-\alpha_n)(x_n - \lambda_n A^*(I-T_n)Ax_n) - x_n\|\\ &= \|(1-\alpha_n)\lambda_n A^*(I-T_n)Ax_n + \alpha_n x_n\|\\ &\leqslant (1-\alpha_n)\lambda_n\|A^*\|\cdot\|(I-T_n)Ax_n\| + \alpha_n\|x_n\| \to 0.\end{aligned} \tag{3.10}$$

Therefore

$$\begin{aligned}\|U_n y_n - y_n\| &\leqslant \|U_n y_n - x_n\| + \|x_n - y_n\| \to 0,\\ \|y_{n+1} - y_n\| &\leqslant \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0.\end{aligned}$$

Take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle z, z-y_n\rangle = \lim_{k\to\infty}\langle z, z-y_{n_k}\rangle.$$

Without loss of generality, we assume that $\{y_{n_k}\}$ converges weakly to an element y^*, then by (3.10) we have $y^* \in \omega_w(x_n)$. Let an index $i \in \{1, 2, \cdots, p\}$ be fixed. Noting that the pool of indices is finite and (3.9), by Lemma 3.1 we can find a subsequence $\{x_{m_k}\} \subset \{x_n\}$ such that $x_{m_k} \rightharpoonup y^*$ and $[m_k] = i$ for all $k \geqslant 1$. So from (3.10) we obtain $y_{m_k} \rightharpoonup y^*$. Since

$$\|U_i y_{m_k} - y_{m_k}\| = \|U_{m_k} y_{m_k} - y_{m_k}\| \to 0,$$

and $U_i - I$ is demiclosed at origin, we obtain $y^* \in Fix(U_i)$. It follows from (3.10) and the weak continuity of A that $Ax_{m_k} \rightharpoonup Ay^*$. Furthermore, since $I - T_i$ is demiclosed at origin and $\|(I - T_i)Ax_{m_k}\| \to 0$, we have $Ay^* \in Fix(T_i)$. Since the index i is arbitrary, we have $y^* \in \Omega$. Thus by (2.1) and $z = P_\Omega(0)$, we obtain

$$\limsup_{n\to\infty}\langle z, z-y_n\rangle = \langle z, z-y^*\rangle \leqslant 0. \tag{3.11}$$

Now since all the hypotheses of Lemma 2.5 are fulfilled, we conclude that $s_n \to 0$.

Case II. Assume that there exists a subsequence $\{s_{m_k}\}$ of $\{s_n\}$ such that $s_{m_k} < s_{m_k+1}$ for all $k \geqslant 1$. Employing [6, Lemma 3.1] in Maingé, we can take a nondecreasing sequence $\{\tau(n)\}_{n\geqslant n_1}$ of integers satisfying the following properties:

$$s_{\tau(n)} \leqslant s_{\tau(n)+1} \quad \text{and} \quad s_n \leqslant s_{\tau(n)+1},$$

for all $n \geqslant n_1$. Then from (3.8) and $\alpha_n \to 0$ we have

$$c_{\tau(n)} \leqslant s_{\tau(n)} - s_{\tau(n)+1} + \alpha_{\tau(n)}\beta_{\tau(n)}M \leqslant \alpha_{\tau(n)}\beta_{\tau(n)}M \to 0.$$

So it follows from the conditions (i)-(iii) that both $\{\|(I - T_{\tau(n)})Ax_{\tau(n)}\|\}$ and $\{\|U_{\tau(n)}y_{\tau(n)} - x_{\tau(n)}\|\}$ converge to zero. Being similar to the proof of (3.9) and (3.11) in Case I, we have

$$\begin{aligned} &\lim_{n\to\infty}\|x_{\tau(n)} - x_{\tau(n)+1}\| = 0,\\ &\limsup_{n\to\infty}\langle z, z-y_{\tau(n)}\rangle \leqslant 0. \end{aligned} \tag{3.12}$$

From (3.7) and $s_{\tau(n)} \leqslant s_{\tau(n)+1}$, we have

$$s_{\tau(n)} \leqslant 2\langle z, z-y_{\tau(n)}\rangle.$$

Hence from (3.12) we have $\limsup_{n\to\infty} s_{\tau(n)} \leqslant 0$, which implies that $s_{\tau(n)} \to 0$. Furthermore,

$$\begin{aligned} s_{\tau(n)+1} &\leqslant |s_{\tau(n)+1} - s_{\tau(n)}| + s_{\tau(n)}\\ &\leqslant \|x_{\tau(n)+1} - x_{\tau(n)}\|(\|x_{\tau(n)+1} - z\| + \|x_{\tau(n)} - z\|) + s_{\tau(n)} \to 0. \end{aligned}$$

Therefore, it follows from $s_n \leqslant s_{\tau(n)+1}$ that $s_n \to 0$. □

Remark 3.3. Compared with [3, Theorem 1], Theorem 3.2 relaxes the conditions on $\{T_i\}$ from directed mappings to demicontractive mappings and $\{U_n\}$ from directed mappings to quasi-nonexpansive mappings.

Theorem 3.4. *Let U_i be quasi-nonexpansive and T_i be μ_i-demicontractive such that $I - U_i$ and $I - T_i$ are demiclosed at origin for every $i = 1, 2, \cdots, p$. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $\Omega \neq \emptyset$ and ρ is as in Lemma 2.7. For any $x_1 \in H_1$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\beta_n)W_n x_n + \beta_n U_n[(1-\alpha_n)W_n x_n], \tag{3.13}$$

where $U_n = U_{[n]}$, $T_n = T_{[n]}$, $W_n x_n = x_n - \lambda_n A^*(I - T_n)Ax_n$ and $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, +\infty)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < \dfrac{1-\mu}{\rho}$, $\mu = \max_{1\leqslant i\leqslant p} \{\mu_i\}$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_\Omega(0)$.

Proof. Let $y_n = (1-\alpha_n)W_n x_n$ and $z = P_\Omega(0)$. It follows from (3.4) and (3.13) that

$$\begin{aligned}\|x_{n+1} - z\| &\leqslant (1-\beta_n)\|W_n x_n - z\| + \beta_n\|U_n[(1-\alpha_n)W_n x_n] - z\|\\ &\leqslant (1-\beta_n)\|W_n x_n - z\| + \beta_n\|(1-\alpha_n)W_n x_n - z\|\\ &\leqslant (1-\beta_n)\|W_n x_n - z\| + \beta_n(1-\alpha_n)\|W_n x_n - z\| + \alpha_n\beta_n\|z\|\\ &\leqslant (1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n\|z\|\\ &\leqslant \max\{\|x_1 - z\|, \|z\|\},\end{aligned}$$

which implies that $\{x_n\}$ is bounded, further, $\{W_n x_n\}$ is bounded too. Since

$$\begin{aligned}\|y_n - z\| &\leqslant (1-\alpha_n)\|W_n x_n - z\| + \alpha_n\|z\|\\ &\leqslant (1-\alpha_n)\|x_n - z\| + \alpha_n\|z\|,\end{aligned}$$

$\{y_n\}$ is also bounded. It follows from (3.4), (3.5), (3.13) and Lemma 2.8 (a) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\beta_n)(W_n x_n - z) + \beta_n(U_n y_n - z)\|^2\\ &= (1-\beta_n)\|W_n x_n - z\|^2 + \beta_n\|U_n y_n - z)\|^2 - \beta_n(1-\beta_n)\|U_n y_n - W_n x_n\|^2\\ &\leqslant (1-\beta_n)\|x_n - z\|^2 + \beta_n[(1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle z, z - y_n\rangle\\ &\quad - \lambda_n(1-\alpha_n)(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2] - \beta_n(1-\beta_n)\|U_n y_n - W_n x_n\|^2\\ &= (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle z, z - y_n\rangle - \beta_n(1-\beta_n)\|U_n y_n - W_n x_n\|^2\\ &\quad - \beta_n(1-\alpha_n)\lambda_n(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2,\end{aligned}$$

i.e.,

$$s_{n+1} \leqslant (1-\alpha_n\beta_n)s_n + 2\alpha_n\beta_n\langle z, z - y_n\rangle - c_n, \tag{3.14}$$

where

$$c_n = \beta_n(1-\beta_n)\|U_n y_n - W_n x_n\|^2 + \beta_n(1-\alpha_n)\lambda_n(1-\mu-\lambda_n\rho)\|(I-T_n)Ax_n\|^2,$$

and $s_n = \|x_n - z\|^2$. First, in a similar way to the proof of Case I in Theorem 3.2, we have both $\{\|(I - T_n)Ax_n\|\}$ and $\{\|U_n y_n - W_n x_n\|\}$ converge to zero. Since

$$\|W_n x_n - x_n\| \leqslant \lambda_n\|A^*\| \cdot \|(I-T_n)Ax_n\| \to 0, \tag{3.15}$$

we have

$$\|U_n y_n - x_n\| \leqslant \|U_n y_n - W_n x_n\| + \|W_n x_n - x_n\| \to 0, \tag{3.16}$$

$$\begin{aligned}\|y_n - x_n\| &\leqslant \|(1-\alpha_n)W_n x_n - x_n\|\\ &\leqslant \|W_n x_n - x_n\| + \alpha_n\|W_n x_n\| \to 0.\end{aligned} \tag{3.17}$$

It follows from (3.13), (3.15) and (3.16) that

$$\|x_{n+1} - x_n\| \leqslant (1-\beta_n)\|W_n x_n - x_n\| + \beta_n\|U_n y_n - x_n\| \to 0. \tag{3.18}$$

From (3.16), (3.17), (3.18) we have

$$\|U_n y_n - y_n\| \leqslant \|U_n y_n - x_n\| + \|x_n - y_n\| \to 0, \tag{3.19}$$

$$\|y_{n+1} - y_n\| \leqslant \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0. \tag{3.20}$$

By virtue of (3.14), (3.18), (3.19), (3.20), and $\|(I - T_n)Ax_n\| \to 0$, mimicking the proof of Case I and Case II in Theorem 3.2, we conclude that the sequence $\{x_n\}$ defined by (3.13) converges strongly to $z = P_\Omega(0)$. □

Now we shall give an example which satisfies all the conditions of the solution set Ω of the SCFP (3.1), the mappings $\{U_i\}_{i=1}^p$, and $\{T_i\}_{i=1}^p$ in Theorems 3.2 and 3.4.

Example 3.5. *Let* $H_1 = H_2 = H_3 = \ell_2$. *For each* $i \in \{1, 2, \cdots, p\}$, *let* U_i, $T_i : \ell_2 \to \ell_2$ *be defined by*

$$U_i x = (\overbrace{0, \cdots, 0}^{i}, x_1, x_2, \cdots),$$

and $T_i x = -(i+1)x$ *for all* $x = (x_1, x_2, \cdots) \in \ell_2$. *Then*

$$\Omega = (\cap_{i=1}^p \mathrm{Fix}(U_i)) \bigcap A^{-1}(\cap_{i=1}^p \mathrm{Fix}(T_i)) = \{0\}.$$

Furthermore, for each $i \in \{1, 2, \cdots, p\}$, U_i *is quasi-nonexpansive,* T_i *is* μ*-demicontractive with* $\mu = \frac{i}{i+2}$, $I - U_i$ *and* $I - T_i$ *are demiclosed at 0.*

In fact, since $\cap_{i=1}^p \mathrm{Fix}(U_i) = \{0\} = \cap_{i=1}^p \mathrm{Fix}(T_i)$, it results that $\Omega = \{0\}$. Now we show the demicloseness property of $I - U_i$ at 0 ($i = 1, 2, \cdots p$). To this end, for any $i \in \{1, 2, \cdots p\}$, let $x_n \rightharpoonup z$ and $(I - U_i)x_n \to 0$, where $x_n = (x_1^{(n)}, x_2^{(n)}, \cdots) \in \ell_2$ and $z = (z_1, z_2, \cdots) \in \ell_2$. The weak convergence of $\{x_n\}$ to z implies that $x_j^{(n)} \to z_j$ for each $j \geqslant 1$. Since

$$\|(I - U_i)x_n\|^2 = \sum_{k=1}^{i} |x_k^{(n)}|^2 + \sum_{k=i+1}^{\infty} |x_{k-i}^{(n)} - x_k^{(n)}|^2 \to 0,$$

it follows that for each fixed $1 \leqslant k \leqslant i$, $x_k^{(n)} \to 0 = z_k$. Hence

$$z_1 = z_2 = \cdots = z_i = 0. \tag{3.21}$$

Also, for $k \geqslant i+1$, $x_{k-i}^{(n)} - x_k^{(n)} \to 0 = z_{k-i} - z_k$. Using (3.21) we see

$$z_{i+1} = z_{i+2} = \cdots = z_{2i} = 0. \tag{3.22}$$

Using (3.22) again, we have $z_{2i+1} = z_{2i+2} = \cdots = z_{3i} = 0$. Continuing this process, we get all $z_j = 0$ for all $j \geqslant 1$, which implies $z = (0, 0, \cdots) = 0 \in \mathrm{Fix}(U_i)$. Hence $I - U_i$ is demiclosed at 0.

Furthermore, it is obvious that each T_i is μ-demicontactive, where $\mu = \frac{i}{i+2}$; see [9, Example 2.5]. However, for each $i \in \{1, 2, \cdots p\}$, $I - T_i$ is obviously demiclosed at 0 by Remark 2.4.

If $U_i = U$, $T_i = T$, $i = 1, 2, \cdots, p$ in Theorem 3.2, we obtain the following conclusion.

Corollary 3.6. *Let* U *be quasi-nonexpansive and* T *be* μ*-demicontractive such that* $I - U$ *and* $I - T$ *are demiclosed at origin. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $\Omega = \mathrm{Fix}(U) \bigcap A^{-1}(\mathrm{Fix}(T)) \neq \emptyset$. *Let* ρ *be as in Lemma 2.7. For any* $x_1 \in H_1$, *define the sequence* $\{x_n\}$ *by*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n U[(1 - \alpha_n)(x_n - \lambda_n A^*(I - T)Ax_n)],$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, +\infty)$ *satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < \frac{1-\mu}{\rho}$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_\Gamma(0)$, *where* Γ *is the solution set of the two-set SCFP* (1.3).

If $U_i = U, T_i = T,\ i = 1, 2, \cdots, p$ in Theorem 3.4, we obtain the following conclusion.

Corollary 3.7. *Let* U *be quasi-nonexpansive and* T *be* μ*-demicontractive such that* $I - U$ *and* $I - T$ *are demiclosed at origin. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $\Omega \neq \emptyset$ *and* ρ *is as in Lemma* 2.7. *For any* $x_1 \in H_1$, *define the sequence* $\{x_n\}$ *by*

$$x_{n+1} = (1 - \beta_n)W_n x_n + \beta_n U[(1 - \alpha_n)W_n x_n],$$

where $W_n x_n = x_n - \lambda_n A^*(I - T)Ax_n$ *and* $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, +\infty)$ *satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < \frac{1-\mu}{\rho}$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_\Gamma(0)$, *where* Γ *is the solution set of the two-set SCFP* (1.3).

Acknowledgment

This research is supported by the Natural Science Foundation of China (no. 11401388, 11501431), the Natural Science Foundation of Zhejiang (no. LQ13A010007, LY14A010006).

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