



Faber polynomial coefficient estimates for certain classes of bi-univalent functions defined by using the Jackson (p, q) -derivative operator

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Abstract

In this work, we introduce a new subclass of bi-univalent functions under the $D_{p,q}$ operator. By using the Faber polynomial expansions, we obtain upper bounds for the coefficients of functions belonging to this analytic and bi-univalent function class. ©2017 All rights reserved.

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1. Introduction, definitions, and notations

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

By \mathcal{S} we denote the subclass of \mathcal{A} consisting of functions of the form (1.1) which are also univalent in \mathbb{U} . Further, \mathcal{P} be the class of functions consisting of φ , such that

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n,$$

which are regular in the open unit disc \mathbb{U} and satisfy the condition $\Re(\varphi(z)) > 0$ in \mathbb{U} . By the Carathéodory's Lemma (e.g., see [12]) we have $|\varphi_n| \leq 2$.

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The Koebe One-Quarter Theorem [12] states that the image of \mathbb{U} under every function f in the normalized univalent function class \mathcal{S} contains a disc of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse f^{-1} which satisfies the following condition:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{U} if both f and f^{-1} are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions defined in the unit disk \mathbb{U} . For a brief history of functions in the class, see [27] (see also [9, 10, 20, 22]). Recently, Srivastava et al. [27], Altınkaya and Yalçın [6], and Magesh and Yamini [21] made an effort to introduce various subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [28]). But the coefficient problem for each one of the following Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, \dots\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([7, 16, 18]).

The Faber polynomials introduced by Faber [13] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [15] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalence of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [25] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (see, for details, [24]).

If the functions f and F are analytic in \mathbb{U} , then f is said to be subordinate to F , written as

$$f(z) \prec F(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $u(z) = \sum_{n=1}^{\infty} c_n z^n$, analytic in \mathbb{U} , with

$$u(0) = 0 \quad \text{and} \quad |u(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = F(u(z)) \quad (z \in \mathbb{U}).$$

For the Schwarz function $u(z)$ we note that $|c_n| < 1$ (e.g., see Duren [12]).

In the field of geometric function theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q -calculus is the important tool that is used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q -calculus in the context of geometric function theory was actually provided and the basic (or q -) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [26]). In fact, the theory of univalent functions can be described by using the theory of the q -calculus. Moreover, in recent years, such q -calculus operators as the fractional q -integral and fractional q -derivative operators were used to construct several subclasses of analytic functions (see, for example, [5, 8, 23]).

For the convenience, we provide some basic definitions and concept details of q -calculus which are used in this paper. We suppose throughout the paper that $0 < q < p \leq 1$. We recall the definitions of fractional q -calculus operators of complex-valued function $f(z)$. We shall follow the notation and terminology in [14].

Definition 1.1 ([11]). The (p, q) -derivative of the function f given by (1.1) is defined as

$$(D_{p,q}f)(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad z \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided $f'(0)$ exists.

From Definition 1.1, we deduce that

$$(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},$$

where the symbol $[n]_{p,q}$ denotes the so-called (p, q) -bracket or *twin-basic number*

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It happens clearly that $D_{p,q}z^n = [n]_{p,q} z^{n-1}$. Note also that for $p = 1$, the Jackson (p, q) -derivative reduces to the Jackson q -derivative given by (see [17])

$$(D_qf)(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \quad z \neq 0.$$

The twin-basic number is a natural generalization of the q -number, that is

$$\lim_{p \rightarrow 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

The object of this paper is to introduce a new subclasses of bi-univalent functions defined by using the Jackson (p, q) -derivative operator and use the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ for the functions in this class. We also obtain estimates for the initial coefficients $|a_2|$ and $|a_3|$ of these functions.

2. Bounds derivable by the Faber polynomial expansion techniques

We begin this section by introducing the function class $D_{\Sigma}(p, q; \lambda, \varphi)$ by means of the following definition.

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $D_{\Sigma}(p, q; \lambda, \varphi)$ if the following subordination conditions hold true:

$$f \in \Sigma \quad \text{and} \quad (1 - \lambda) \frac{f(z)}{z} + \lambda(D_{p,q}f)(z) \prec \varphi(z), \quad (z \in \mathbb{U}; \lambda \geq 1) \tag{2.1}$$

and

$$f \in \Sigma \quad \text{and} \quad (1 - \lambda) \frac{g(w)}{w} + \lambda(D_{p,q}g)(w) \prec \varphi(w), \quad (w \in \mathbb{U}; \lambda \geq 1), \tag{2.2}$$

where $g(w) = f^{-1}(w)$.

We note from Definition 2.1 that

$$\lim_{p \rightarrow 1^-} D_{\Sigma}(p, q; \lambda, \varphi) = \left\{ f : f \in \Sigma \quad \text{and} \quad \left\{ \begin{array}{l} \lim_{p \rightarrow 1^-} \left[(1 - \lambda) \frac{f(z)}{z} + \lambda(D_{p,q}f)(z) \right] \\ \lim_{p \rightarrow 1^-} \left[(1 - \lambda) \frac{g(w)}{w} + \lambda(D_{p,q}g)(w) \right] \end{array} \right\} \right\} = D_{\Sigma}(q; \lambda, \varphi).$$

Furthermore, we note that

$$\lim_{q \rightarrow 1^-} D_{\Sigma}(q; \lambda, \varphi) = \left\{ f : f \in \Sigma \text{ and } \left\{ \begin{array}{l} \lim_{q \rightarrow 1^-} \left[(1 - \lambda) \frac{f(z)}{z} + \lambda(D_q f)(z) \right] \\ \lim_{q \rightarrow 1^-} \left[(1 - \lambda) \frac{g(w)}{w} + \lambda(D_q g)(w) \right] \end{array} \right\} \right\} = R_{\sigma}(\lambda, \varphi),$$

where $R_{\sigma}(\lambda, \varphi)$ is the class of bi-univalent defined and studied by Kumar et al. [19].

Using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [3]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j ($7 \leq j \leq n$) is a homogeneous polynomial in the variables a_2, a_3, \dots, a_n (see, for details, [4]). In particular, the first three terms of K_{n-1}^{-n} are given below:

$$\frac{1}{2} K_1^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, an expansion of K_n^p is given by (see [3])

$$K_n^p = p a_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_n^3 + \dots + \frac{p!}{(p-n)!n!} E_n^n, \quad (p \in \mathbb{Z}),$$

where

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \quad \text{and} \quad E_n^p = E_n^p(a_2, a_3, \dots)$$

and, alternatively, by (see [1] and [2])

$$E_n^m(a_1, a_2, \dots, a_n) = \sum_{m=1}^{\infty} \frac{m! (a_1)^{\mu_1} \dots (a_n)^{\mu_n}}{\mu_1! \dots \mu_n!},$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers μ_1, \dots, μ_n satisfying the following conditions:

$$\mu_1 + \mu_2 + \dots + \mu_n = m, \mu_1 + 2\mu_2 + \dots + n\mu_n = n.$$

Evidently,

$$E_n^n(a_1, a_2, \dots, a_n) = a_1^n.$$

Our first main result is given by Theorem 2.2 below.

Theorem 2.2. *Let f given by (1.1) be in the class $D_{\Sigma}(p, q; \lambda, \varphi)$ ($\lambda \geq 1$). If $a_m = 0$ for $2 \leq m \leq n - 1$, then*

$$|a_n| \leq \frac{2}{\left| 1 + ([n]_{p,q} - 1) \lambda \right|}, \quad (n \geq 4).$$

Proof. For analytic functions f of the form (1.1), we have

$$(1 - \lambda) \frac{f(z)}{z} + \lambda(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} \left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] a_n z^{n-1} \tag{2.3}$$

and

$$\begin{aligned} (1 - \lambda) \frac{g(w)}{w} + \lambda(D_{p,q}g)(w) &= 1 + \sum_{n=1}^{\infty} \left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] b_n w^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] \times \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^{n-1}. \end{aligned} \tag{2.4}$$

On the other hand, the inequalities (2.1) and (2.2) imply the existence of two Schwartz functions $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = \sum_{n=1}^{\infty} d_n w^n$ so that

$$(1 - \lambda) \frac{f(z)}{z} + \lambda(D_{p,q}f)(z) = \varphi(u(z)) \tag{2.5}$$

and

$$(1 - \lambda) \frac{g(w)}{w} + \lambda(D_{p,q}g)(w) = \varphi(v(w)), \tag{2.6}$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n \varphi_k E_n^k(c_1, c_2, \dots, c_n) z^n \tag{2.7}$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n \varphi_k E_n^k(d_1, d_2, \dots, d_n) w^n. \tag{2.8}$$

Thus, from (2.3), (2.5), and (2.7) we have

$$\left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] a_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k(c_1, c_2, \dots, c_{n-1}), \quad (n \geq 2). \tag{2.9}$$

Similarly, by using (2.4), (2.6), and (2.8), we find that

$$\left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] b_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k(d_1, d_2, \dots, d_{n-1}), \quad (n \geq 2). \tag{2.10}$$

We note that, for $a_m = 0$ ($2 \leq m \leq n - 1$), we have

$$b_n = -a_n$$

and so

$$\left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] a_n = \varphi_1 c_{n-1}, \quad - \left[1 + \left([n]_{p,q} - 1 \right) \lambda \right] a_n = \varphi_1 d_{n-1}.$$

Now taking the absolute values of either of the above two equations and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$, and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| = \frac{|\varphi_1 c_{n-1}|}{\left| 1 + \left([n]_{p,q} - 1 \right) \lambda \right|} = \frac{|\varphi_1 d_{n-1}|}{\left| 1 + \left([n]_{p,q} - 1 \right) \lambda \right|} \leq \frac{2}{\left| 1 + \left([n]_{p,q} - 1 \right) \lambda \right|},$$

which evidently completes the proof of Theorem 2.2. □

By letting $p \rightarrow 1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.3. *Let f given by (1.1) be in the class $D_{\Sigma}(q; \lambda, \varphi)$ ($\lambda \geq 1$). If $a_m = 0$ for $2 \leq m \leq n - 1$, then*

$$|a_n| \leq \frac{2(1 - q)}{1 - q + (q - q^n)\lambda}, \quad (n \geq 4).$$

3. Estimates for the initial coefficients a_2 and a_3

In this section, we choose to relax the coefficient restrictions imposed in Theorem 2.2 and derive the resulting estimates for the initial coefficients a_2 and a_3 of functions $f \in D_{\Sigma}(p, q; \lambda, \varphi)$; given by the Taylor-Maclaurin series expansion (1.1).

Theorem 3.1. *Let f given by (1.1) be in the class $D_{\Sigma}(p, q; \lambda, \varphi)$ ($\lambda \geq 1$). Then*

$$|a_2| \leq \min \left\{ \frac{2}{|1 + (p + q - 1)\lambda|}, \frac{2}{\sqrt{|1 + (p^2 + pq + q^2 - 1)\lambda|}} \right\},$$

$$|a_3| \leq \min \left\{ \frac{4}{|1 + (p + q - 1)\lambda|^2} + \frac{2}{|1 + (p^2 + pq + q^2 - 1)\lambda|}, \frac{6}{|1 + (p^2 + pq + q^2 - 1)\lambda|} \right\},$$

and

$$|a_3 - 2a_2^2| \leq \frac{4}{|1 + (p^2 + pq + q^2 - 1)\lambda|}.$$

Proof. Replacing n by 2 and 3 in (2.9) and (2.10), respectively, we find that

$$\left[1 + \left([2]_{p,q} - 1\right)\lambda\right] a_2 = \varphi_1 c_1, \tag{3.1}$$

$$\left[1 + \left([3]_{p,q} - 1\right)\lambda\right] a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{3.2}$$

$$-\left[1 + \left([2]_{p,q} - 1\right)\lambda\right] a_2 = \varphi_1 d_1, \tag{3.3}$$

and

$$\left[1 + \left([3]_{p,q} - 1\right)\lambda\right] (2a_2^2 - a_3) = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{3.4}$$

From (3.1) and (3.3) we obtain

$$d_1 = -c_1$$

and

$$|a_2| = \frac{|\varphi_1 c_1|}{\left|1 + \left([2]_{p,q} - 1\right)\lambda\right|} = \frac{|\varphi_1 d_1|}{\left|1 + \left([2]_{p,q} - 1\right)\lambda\right|} \leq \frac{2}{|1 + (p + q - 1)\lambda|}. \tag{3.5}$$

Now, by adding (3.2) to (3.4), we have

$$2 \left[1 + \left([3]_{p,q} - 1\right)\lambda\right] a_2^2 = \varphi_1 (c_2 + d_2) + \varphi_2 (c_1^2 + d_1^2)$$

or, equivalently,

$$|a_2| \leq \frac{2}{\sqrt{|1 + (p^2 + pq + q^2 - 1)\lambda|}}. \tag{3.6}$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.4) from (3.2). Thus, we get

$$2 \left[1 + \left([3]_{p,q} - 1\right)\lambda\right] (a_3 - a_2^2) = \varphi_1 (c_2 - d_2) + \varphi_2 (c_1^2 - d_1^2)$$

or

$$|a_3| \leq |a_2|^2 + \frac{|\varphi_1(c_2 - d_2)|}{2 \left| 1 + ([3]_{p,q} - 1)\lambda \right|}. \tag{3.7}$$

Upon substituting the value of a_2^2 from (3.5) and (3.6) into (3.7), it follows that

$$|a_3| \leq \frac{4}{[1 + (p + q - 1)\lambda]^2} + \frac{2}{|1 + (p^2 + pq + q^2 - 1)\lambda|}$$

and

$$|a_3| \leq \frac{6}{|1 + (p^2 + pq + q^2 - 1)\lambda|}.$$

Finally, from (3.4), we deduce (by the Carathéodory Lemma) that

$$|a_3 - 2a_2^2| = \frac{|\varphi_1 d_2 + \varphi_2 d_1^2|}{\left| 1 + ([3]_{p,q} - 1)\lambda \right|} \leq \frac{4}{|1 + (p^2 + pq + q^2 - 1)\lambda|}.$$

This evidently completes the proof of the above theorem. □

By letting $p \rightarrow 1$ in Theorem 3.1, we obtain the following consequence.

Corollary 3.2. *Let f given by (1.1) be in the class $D_\Sigma(q; \lambda, \varphi)$ ($\lambda \geq 1$). If $a_m = 0$ for $2 \leq m \leq n - 1$, then*

$$|a_2| \leq \frac{2}{1 + q\lambda}, \quad |a_3| \leq \frac{4}{(1 + q\lambda)^2} + \frac{2}{1 + (q^2 + q)\lambda}, \quad \text{and} \quad |a_3 - 2a_2^2| \leq \frac{4}{1 + (q^2 + q)\lambda}.$$

Corollary 3.3. *Let f given by (1.1) be in the class $R_\sigma(\lambda, \varphi)$ ($\lambda \geq 1$). If $a_m = 0$ for $2 \leq m \leq n - 1$, then*

$$|a_2| \leq \frac{2}{1 + \lambda} \tag{3.8}$$

and

$$|a_3| \leq \frac{4}{(1 + \lambda)^2} + \frac{2}{1 + 2\lambda}.$$

Remark 3.4. The above estimate for $|a_2|$ shows that the inequality (3.8) is an improvement of the estimate obtained by Kumar et al. ([19], Theorem 2.2).

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