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Hardy type estimates for commutators of fractional integrals associated with Schrödinger operators

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Abstract

We consider the Schrödinger operator $\mathcal{L} = -\Delta + V$ on $\mathbb{R}^n$, where $n \geqslant 3$ and the nonnegative potential V belongs to reverse Hölder class RH_{q_1} for some $q_1 > \frac{n}{2}$. Let $\mathbb{I}_\alpha$ be the fractional integral associated with $\mathcal{L}$, and let b belong to a new Campanato space $\Lambda_\beta^\theta(\rho)$. In this paper, we establish the boundedness of the commutators $[b, \mathbb{I}_\alpha]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ whenever $1/q = 1/p - (\alpha+\beta)/n, 1 < p < n/(\alpha+\beta)$. When $\frac{n}{n+\beta} < p \leqslant 1, 1/q = 1/p - (\alpha+\beta)/n$, we show that $[b, \mathbb{I}_\alpha]$ is bounded from $H^p_{\mathcal{L}}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, we also prove that $[b, \mathbb{I}_\alpha]$ maps $H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)$ continuously into weak $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$.
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1. Introduction and results

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n, n \geqslant 3$. The function V is nonnegative, $V \neq 0$, and belongs to a reverse Hölder class RH_{q_1} for some $q_1 > \frac{n}{2}$, that is, there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(y)^{q_1} dy\right)^{1/q_1} \leqslant \frac{C}{|B|}\int_B V(y) dy$$

for every ball $B \subset \mathbb{R}^n$.

Suppose $V \in RH_{q_1}$ with $q_1 > n/2$. The fractional integral associated with $\mathcal{L}$ is defined by

$$\mathbb{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}}(f)(x)\frac{dt}{t^{-\alpha/2+1}}$$

for $0 < \alpha < n$. If $\mathcal{L} = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $\mathbb{I}_\alpha$ is the Riesz potential I_α, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

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As in [10], for a given potential $V \in RH_{q_1}$ with $q_1 > n/2$, we define the auxiliary function

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leqslant 1\right\}, \quad x \in \mathbb{R}^n.$$

It is well-known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Let $\theta > 0$ and $0 < \beta < 1$, according to [7], the new Campanato class $\Lambda_\beta^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}} \int_{B(x,r)} |b(y) - b_B|dy \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

holds for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\beta^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda_\beta^\theta(\rho)$ is the classical Campanato space. If $\beta = 0, \Lambda_\beta^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [1].

We recall the Hardy space associated with Schrödinger operator $\mathcal{L}$ which had been studied by Dziubański and Zienkiewicz in [4] and [5]. Because $V \in L_{loc}^{q_1}(\mathbb{R}^n)$, the Schrödinger operator $\mathcal{L}$ generates a (C_0) contraction semigroup $\{T_s^\mathcal{L} : s > 0\} = \{e^{-s\mathcal{L}} : s > 0\}$. The maximal function associated with $\{T_s^\mathcal{L} : s > 0\}$ is defined by $M^\mathcal{L} f(x) = \sup_{s>0} |T_s^\mathcal{L} f(x)|$. We always denote $\eta = 2 - n/q_1$ and $\delta' = \min\{1, \eta\}$. For $\frac{n}{n+\delta'} < p \leqslant 1$, the Hardy space $H_\mathcal{L}^p(\mathbb{R}^n)$ associated with Schrödinger operator $\mathcal{L}$ is defined as follows.

Definition 1.1. We say that f is an element of $H_\mathcal{L}^p(\mathbb{R}^n)$ if the maximal function $M^\mathcal{L} f$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm of f is defined by $\|f\|_{H_\mathcal{L}^p(\mathbb{R}^n)} = \|M^\mathcal{L} f\|_{L^p(\mathbb{R}^n)}$.

We introduce the concept of $H_\mathcal{L}^{p,q}$-atom.

Definition 1.2. Let $\frac{n}{n+\delta'} < p \leqslant 1 \leqslant q \leqslant \infty$. A function $a \in L^2(\mathbb{R}^n)$ is called an $H_\mathcal{L}^{p,q}$-atom if $r < \rho(x_0)$ and the following conditions hold:

(i) $\operatorname{supp} a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leqslant |B(x_0, r)|^{1/q-1/p}$,

(iii) if $r < \rho(x_0)/4$, then $\int_{B(x_0,r)} a(x)dx = 0$.

We have the following atomic characterization of Hardy space.

Proposition 1.3 ([5])**.** *Let* $\frac{n}{n+\delta'} < p \leqslant 1 \leqslant q \leqslant \infty$*. Then* $f \in H_\mathcal{L}^p(\mathbb{R}^n)$ *if and only if* f *can be written as* $f = \sum_j \lambda_j a_j$*, where* a_j *are* $H_\mathcal{L}^{p,q}$*-atoms,* $\sum_j |\lambda_j|^p < \infty$*, and the sum converges in the* $H_\mathcal{L}^p(\mathbb{R}^n)$ *quasi-norm. Moreover*

$$\|f\|_{H_\mathcal{L}^p(\mathbb{R}^n)} \approx \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all atomic decompositions of f *into* $H_\mathcal{L}^{p,q}$*-atoms.*

The above atomic decomposition of $H_\mathcal{L}^p(\mathbb{R}^n)$ implies that the Hardy space $H_\mathcal{L}^p(\mathbb{R}^n)$ is larger than the classical Hardy space $H^p(\mathbb{R}^n)$. Especially, the Hardy space $H_\mathcal{L}^p(\mathbb{R}^n)$ is exactly the local Hardy space $h^p(\mathbb{R}^n)$ introduced by Goldberg in [6] when the potential V is a positive constant.

Let us consider the commutator associated with the Riesz potential I_α and locally integrable function b, $[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x)$. When $b \in BMO$, Chanillo proved in [3] that $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/q = 1/p - \alpha/n, 1 < p < n/\alpha$. When b belongs to the Campanato space $\Lambda_\beta, 0 < \beta < 1$, Paluszyński in [9] showed that $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q =$

$1/p-(\alpha+\beta)/n, 1<p<n/(\alpha+\beta)$. Furthermore, Lu et al. in [8] considered the boundedness of $[b, I_\alpha]$ on the classical Hardy spaces when $b \in \Lambda_\beta(0<\beta \leqslant 1)$. They proved that if $\frac{n}{n+\beta}<p \leqslant 1$ and $1/q = 1/p-(\alpha+\beta)/n$, $[b, I_\alpha]$ maps $H^p(\mathbb{R}^n)$ continuously into $L^q(\mathbb{R}^n)$. At the endpoint $p=\frac{n}{n+\beta}$, they also showed that $[b, I_\alpha]$ maps $H^p(\mathbb{R}^n)$ continuously into weak $L^{n/(n-\alpha)}(\mathbb{R}^n)$.

When $b \in BMO_\theta(\rho)$, Bui in [2] obtained the boundedness of $[b, \mathbb{I}_\alpha]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/q=1/p-\alpha/n, 1<p<n/\alpha$.

Inspired by the above results, in this paper, we are interested in the boundedness of $[b, \mathbb{I}_\alpha]$ when b belongs to the new Campanato class $\Lambda_\beta^\theta(\rho)$. The results of this paper are as follows.

Theorem 1.4. *Let $0<\alpha<n$, and let $V \in RH_{q_1}$ with $q_1>n/2$. Then for any $b \in \Lambda_\beta^\theta(\rho), 0<\beta<1$, the commutator $[b, \mathbb{I}_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha+\beta}{n}, 1<p<\frac{n}{\alpha+\beta}$.*

Theorem 1.5. *Let $0<\alpha<n$, and let $V \in RH_{q_1}$ with $q_1>n/2$. Suppose $b \in \Lambda_\beta^\theta(\rho), 0<\beta<\delta'$. If $\frac{n}{n+\beta}<p \leqslant 1$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha+\beta}{n}$, then the commutator $[b, \mathbb{I}_\alpha]$ is bounded from $H_{\mathcal{L}}^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Theorem 1.6. *Let $0<\alpha<n$, and let $V \in RH_{q_1}$ with $q_1>n/2$. Suppose $b \in \Lambda_\beta^\theta(\rho), 0<\beta<\delta'$. Then the commutator $[b, \mathbb{I}_\alpha]$ is bounded from $H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)$ into weak $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$.*

Finally, we make some conventions on the notation. Throughout the whole paper, we always use C to denote a positive constant, that is independent of the main parameters involved but whose value may differ from line to line. We shall use the symbol $A \lesssim B$ to indicate that there exists a constant C such that $A \leqslant CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

We would like to recall some important properties concerning the auxiliary function which will play an important role to obtain the main results.

Proposition 2.1 ([10]). *Let $V \in RH_{n/2}$. For the auxiliary function ρ there exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

A ball $B(x, \rho(x))$ is called critical. Assume that $Q=B(x_0, \rho(x_0))$, for $x \in Q$, the inequality above tells us that $\rho(x) \approx \rho(y)$ if $|x-y|<C\rho(x)$.

It is easy to get the following result from Proposition 2.1.

Lemma 2.2. *Let $k \in \mathbb{N}$ and $x \in 2^{k+1}B(x_0, r) \setminus 2^kB(x_0, r)$. Then we have*

$$\frac{1}{\left(1+\frac{2^k r}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}.$$

Proposition 2.3 ([4]). *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k)), k \geqslant 1$, satisfies*

(i) $\bigcup_k Q_k=\mathbb{R}^n$;

(ii) *there exists $N=N(\rho)$ such that for every $k \in N$,* $\mathrm{card}\{j : 4Q_j \cap 4Q_k\} \leqslant N$.

Given $\alpha>0$, we define the following maximal functions for $g \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_{\rho,\alpha}g(x)=\sup_{x\in B\in\mathcal{B}_{\rho,\alpha}}\frac{1}{|B|}\int_B|g(y)|dy,$$

$$M^\sharp_{\rho,\alpha}g(x)=\sup_{x\in B\in\mathcal{B}_{\rho,\alpha}}\frac{1}{|B|}\int_B|g(y)-g_B|dy,$$

where $\mathcal{B}_{\rho,\alpha}=\{B(z,r) : z \in \mathbb{R}^n \text{ and } r \leqslant \alpha\rho(y)\}$.

We have the following Fefferman-Stein type inequality.

Proposition 2.4 ([1]). *For $1 < p < \infty$, there exist η and γ such that if $\{Q_k\}_k$ is a sequence of balls as in Proposition 2.3, then*

$$\int_{\mathbb{R}^n} |M_{\rho,\eta} g(x)|^p \, dx \lesssim \int_{\mathbb{R}^n} |M^{\sharp}_{\rho,\gamma} g(x)|^p \, dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p$$

for all $g \in L^1_{loc}(\mathbb{R}^n)$.

Let us recall the property of $b \in \Lambda^{\theta}_{\beta}(\rho)$.

Lemma 2.5 ([7]). *Let $1 \leqslant s < \infty$, $b \in \Lambda^{\theta}_{\beta}(\rho)$, and $B = B(x, r)$. Then*

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B|^s \, dy \right)^{1/s} \leqslant [b]^{\theta}_{\beta} (2^k r)^{\beta} \left(1 + \frac{2^k r}{\rho(x)} \right)^{\theta'}$$

for all $k \in \mathbb{N}$, where $\theta' = (k_0 + 1)\theta$, and k_0 is the constant appearing in Proposition 2.1.

Proposition 2.6 ([5]). *Let $p_t(x, y)$ be the kernels associated with the semigroups $\{e^{-t\mathcal{L}}\}_{t>0}$. If $V \in RH_{q_1}$ with $q_1 > n/2$, then for every $0 < \delta < \delta'$ and every $N > 0$ there exists a constant $C > 0$ such that for $|y - z| < \frac{1}{2}|x - y|$, we have*

$$|p_t(x, y) - p_t(x, z)| + |p_t(y, x) - p_t(z, x)| \lesssim \frac{1}{t^{n/2}} \left(\frac{|y - z|}{\sqrt{t}} \right)^{\delta} \exp\left(-\frac{|x - y|^2}{5t} \right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right).$$

Let K_α be the kernel of $\mathbb{I}_\alpha$. The following results give the estimates on the kernel $K_\alpha(x, y)$.

Lemma 2.7. *Suppose $V \in RH_{q_1}$ with $q_1 > \frac{n}{2}$.*

(i) *For every $N > 0$, there exists a constant C such that*

$$|K_\alpha(x, y)| \lesssim \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)} \right)^N} \frac{1}{|x - y|^{n-\alpha}}.$$

(ii) *For every $0 < \delta < \delta'$ there exists a constant C such that for every $N > 0$, we have*

$$|K_\alpha(x, y) - K_\alpha(x, z)| + |K_\alpha(y, x) - K_\alpha(z, x)| \lesssim \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)} \right)^N} \frac{|y - z|^{\delta}}{|x - y|^{n+\delta-\alpha}},$$

where $|y - z| \leqslant |x - y|/4$.

Proof. We observe that (i) is the result of Proposition 3.3 of [2]. By Proposition 2.4 and the methods used in Proposition 3.3 of [2], we can obtain (ii). □

Since $|\mathbb{I}_\alpha(f)(x)| \lesssim I_\alpha(|f|)(x)$, then we get the following.

Corollary 2.8. *Suppose $V \in RH_{q_1}$ with $q_1 > n/2$. Let $0 < \alpha < n$ and let $1 \leqslant p < q < \infty$ satisfy $1/q = 1/p - \alpha/n$. Then for all f in $L^p(\mathbb{R}^n)$ we have*

$$\|\mathbb{I}_\alpha f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

when $p > 1$, and also

$$\|\mathbb{I}_\alpha f\|_{WL^q(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}$$

when $p = 1$.

3. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemmas.

Lemma 3.1. *Let* $1 < s < p < n/(\alpha+\beta)$, $b \in \Lambda_\beta^\theta(\rho)$, *and* $Q = B(x_0, \rho(x_0))$. *Then*

$$\frac{1}{|Q|}\int_Q |[b, \mathbb{I}_\alpha]f(y)|dy \lesssim [b]_\beta^\theta \inf_{x\in Q} M_{\alpha+\beta,s}(f)(x),$$

where

$$M_{\alpha+\beta,s}(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-(\alpha+\beta)s/n}}\int_B |f(y)|^s dy\right)^{1/s}.$$

Proof. Since

$$[b, \mathbb{I}_\alpha]f(y) = (b(y) - b_Q)\mathbb{I}_\alpha f(y) - \mathbb{I}_\alpha((b - b_Q)f)(y),$$

we have

$$\frac{1}{|Q|}\int_Q |[b, \mathbb{I}_\alpha]f(y)|dy \leqslant \frac{1}{|Q|}\int_Q |(b(y) - b_Q)\mathbb{I}_\alpha f(y)|dy + \frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha((b - b_Q)f)(y)|dy = I_1 + I_2.$$

By Hölder's inequality and Lemma 2.5, for any $t > 1$ we get

$$\begin{aligned} I_1 &\leqslant \left(\frac{1}{|Q|}\int_Q |b(y) - b_Q|^{t'}dy\right)^{1/t'}\left(\frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha f(y)|^t dy\right)^{1/t} \\ &\lesssim [b]_\beta^\theta \rho(x_0)^\beta\left(\left(\frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha f_1(y)|^t dy\right)^{1/t} + \left(\frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha f_2(y)|^t dy\right)^{1/t}\right) = I_{11} + I_{12}, \end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$. Choose $t > 1$ such that $1/s - 1/t = \alpha/n$, then by the (L^s, L^t)-boundedness of $\mathbb{I}_\alpha$ (see Corollary 2.8), we have

$$\begin{aligned} I_{11} &\lesssim [b]_\beta^\theta \rho(x_0)^\beta \frac{1}{|Q|^{1/t}}\left(\int_{2Q} |f(y)|^s dy\right)^{1/s} \\ &\lesssim [b]_\beta^\theta\left(\frac{1}{|2Q|^{1-(\alpha+\beta)s/n}}\int_{2Q} |f(y)|^s dy\right)^{1/s} \lesssim [b]_\beta^\theta \inf_{x\in Q} M_{\alpha+\beta,s}(f)(x). \end{aligned}$$

Note that

$$|\mathbb{I}_\alpha f_2(y)| \leqslant \int_{(2Q)^c} |K_\alpha(y,z)f(z)|dz \lesssim \int_{(2Q)^c} \frac{|f(z)|}{\left(1 + \frac{|y-z|}{\rho(y)}\right)^N |y-z|^{n-\alpha}}dz.$$

In this situation, we have $\rho(y) \approx \rho(x_0)$, $|y - z| \approx |x_0 - z|$ for any $y \in Q$ and $z \in (2Q)^c$. So, decomposing $(2Q)^c$ into annuli $2^kQ \setminus 2^{k-1}Q$, $k \geqslant 2$, we get

$$|\mathbb{I}_\alpha f_2(y)| \lesssim \sum_{k\geqslant 2}\frac{2^{-kN}}{|2^kQ|^{1-\alpha/n}}\int_{2^kQ}|f(z)|dz.$$

Then, by Hölder's inequality we get

$$I_{12} \lesssim [b]_\beta^\theta \rho(x_0)^\beta \sum_{k\geqslant 2}\frac{2^{-kN}}{|2^kQ|^{1-\alpha/n}}\int_{2^kQ}|f(z)|dz$$

$$
\begin{aligned}
&\lesssim [b]_\beta^\theta \sum_{k\geqslant 2} \frac{2^{-kN}}{|2^kQ|^{1-(\alpha+\beta)/n}} \int_{2^kQ} |f(z)|dz \\
&\lesssim [b]_\beta^\theta \sum_{k\geqslant 2} 2^{-kN} \left(\frac{1}{|2^kQ|^{1-(\alpha+\beta)s/n}} \int_{2^kQ} |f(z)|^s dz \right)^{1/s} \\
&\lesssim [b]_\beta^\theta \inf_{x\in Q} M_{\alpha+\beta,s}(f)(x).
\end{aligned}
$$

The estimate for I_2 can be proceeded in the same way of I_1. The decomposition $f = f_1 + f_2$ gives

$$
I_2 \leqslant \frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha((b-b_Q)f_1)(y)|dy + \frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha((b-b_Q)f_2)(y)|dy = I_{21} + I_{22}.
$$

Choose r such that $1 < r < s < p$ and $1/r - 1/r_0 = \alpha/n$. By Hölder's inequality, Lemma 2.5 and (L^r, L^{r_0})-boundedness of $\mathbb{I}_\alpha$, for some $u > r$ we have

$$
\begin{aligned}
I_{21} &\lesssim \left(\frac{1}{|Q|}\int_Q |\mathbb{I}_\alpha((b-b_Q)f_1)(y)|^{r_0} dy \right)^{1/r_0} \\
&\lesssim \frac{1}{|Q|^{-\alpha/n}} \left(\frac{1}{|Q|}\int_{2Q} |((b-b_Q)f_1)(y)|^r dy \right)^{1/r} \\
&\lesssim \frac{1}{|Q|^{-\alpha/n}} \left(\frac{1}{|Q|}\int_{2Q} |f(y)|^s dy \right)^{1/s} \left(\frac{1}{|Q|}\int_{2Q} |b(y)-b_Q|^u dy \right)^{1/u} \\
&\lesssim [b]_\beta^\theta \inf_{x\in Q} M_{\alpha+\beta,s}(f)(x).
\end{aligned}
$$

The estimate $I_{22} \lesssim [b]_\beta^\theta \inf_{x\in Q} M_{\alpha+\beta,s}(f)(x)$ can be obtained by the similar approach to ones of I_{12} and I_{21}. Then we omit the details here. □

Lemma 3.2. *Let $B = B(x_0, r)$ with $r \leqslant \gamma\rho(x_0)$ and let $x \in B$, then for any $y, z \in B$ we have*

$$
\int_{(2B)^c} |K_\alpha(y,u) - K_\alpha(z,u)||b(u)-b_B||f(u)|du \lesssim [b]_\beta^\theta M_{\alpha+\beta,s}(f)(x).
$$

Proof. Setting $Q = B(x_0, \gamma\rho(x_0))$, due to the facts that $\rho(y) \approx \rho(z) \approx \rho(x_0)$ and $|y-u| \approx |z-u| \approx |x_0-u|$, then by Lemma 2.7 we get

$$
\int_{(2B)^c} |K_\alpha(y,u) - K_\alpha(z,u)||b(u)-b_B||f(u)|du \lesssim K_1 + K_2,
$$

where

$$
K_1 = r^\delta \int_{Q\setminus 2B} \frac{|f(u)(b(u)-b_B)|}{|x_0-u|^{n+\delta-\alpha}} du
$$

and

$$
K_2 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(u)(b(u)-b_B)|}{|x_0-u|^{n+N+\delta-\alpha}} du.
$$

Let j_0 be the least integer such that $2^{j_0} \geqslant \gamma\rho(x_0)/r$. Splitting into annuli, we have

$$
K_1 \leqslant \sum_{j=2}^{j_0} 2^{-\delta j}(2^j r)^\alpha \frac{1}{|2^jB|} \int_{2^jB} |f(u)||b(u)-b_B|du.
$$

By Hölder's inequality, Lemma 2.5 and $2^j r \leqslant \gamma\rho(x_0)$ for $j < j_0$, we have

$$K_1 \lesssim \sum_{j=2}^{j_0} 2^{-j\delta}(2^j r)^{\alpha}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(u)|^s du\right)^{1/s}\left(\frac{1}{|2^jB|}\int_{2^jB}|b(u)-b_B|^{s'}du\right)^{1/s'}$$
$$\lesssim [b]_\beta^\theta \sum_{j=2}^{j_0} 2^{-\delta j}(2^j r)^{\alpha+\beta}\left(1+\frac{2^j r}{\rho(x_0)}\right)^{\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(u)|^s du\right)^{1/s}$$
$$\lesssim [b]_\beta^\theta M_{\alpha+\beta,s}(f)(x).$$

Note that

$$\frac{1}{|2^jB|}\int_{2^jB}|f(u)||b(u)-b_B|du \lesssim [b]_\beta^\theta (2^j r)^\beta \left(1+\frac{2^j r}{\rho(x_0)}\right)^{\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(u)|^s du\right)^{1/s}.$$

Since $\frac{2^j r}{\rho(x_0)} \geqslant \gamma$ for $j \geqslant j_0$, then, by choosing $N > \theta'$ we get

$$K_2 \lesssim \rho(x_0)^N \sum_{j\geqslant j_0} 2^{-j\delta}\frac{1}{(2^j r)^{N-\alpha}}\frac{1}{|2^jB|}\int_{2^jB}|f(u)||b(u)-b_B|du$$
$$\lesssim [b]_\beta^\theta \left(\frac{2^j r}{\rho(x_0)}\right)^{-(N-\theta')}\sum_{j=j_0}^{\infty} 2^{-j\delta}(2^j r)^{\alpha+\beta}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(u)|^s du\right)^{1/s}$$
$$\lesssim [b]_\beta^\theta M_{\alpha+\beta,s} f(x).$$

□

Lemma 3.3. *Let* $1 < s < p < n/(\alpha+\beta)$, $B = B(x_0, r)$ *with* $r \leqslant \gamma\rho(x_0)$, *and* $x \in B$. *Then*

$$M^\sharp_{\rho,\gamma}([b,\mathbb{I}_\alpha]f)(x) \lesssim [b]_\beta^\theta\left(M_{\alpha+\beta,s}(f)(x) + M_{\beta,s}(\mathbb{I}_\alpha f)(x)\right).$$

Proof. We write

$$\frac{1}{|B|}\int_B |[b,\mathbb{I}_\alpha]f(y) - ([b,\mathbb{I}_\alpha]f)_B|dy \leqslant \frac{2}{|B|}\int_B |(b(y)-b_B)\mathbb{I}_\alpha f(y)|dy + \frac{2}{|B|}\int_B |\mathbb{I}_\alpha((b-b_B)f_1)(y)|dy$$
$$+\frac{1}{|B|}\int_B |\mathbb{I}_\alpha((b-b_B)f_2)(y) - (\mathbb{I}_\alpha((b-b_B)f_2))_B|dy$$
$$= J_1 + J_2 + J_3,$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

Since $r \leqslant \gamma\rho(x_0)$ and $\rho(x) \approx \rho(x_0)$, by Hölder's inequality and Lemma 2.5, we get

$$J_1 \leqslant \left(\frac{1}{|B|}\int_B |b(y)-b_B|^{s'}dy\right)^{1/s'}\left(\frac{1}{|B|}\int_B |\mathbb{I}_\alpha f(y)|^s dy\right)^{1/s}$$
$$\lesssim [b]_\beta^\theta r^\beta \left(\frac{1}{|B|}\int_B |\mathbb{I}_\alpha f(y)|^s dy\right)^{1/s} \lesssim [b]_\beta^\theta M_{\beta,s}(\mathbb{I}_\alpha f)(x).$$

For some $1 < r < s$, and $1/r - 1/r_0 = \alpha/n$, by Hölder's inequality and Lemma 2.5, we have

$$J_2 \lesssim \left(\frac{1}{|B|}\int_B |\mathbb{I}_\alpha((b-b_B)f_1)(y)|^{r_0}dy\right)^{1/r_0}$$
$$\lesssim \frac{1}{|B|^{-\alpha/n}}\left(\frac{1}{|B|}\int_{2B}|(b(y)-b_B)f(y)|^r dy\right)^{1/r}$$

$$\begin{aligned}
&\lesssim \frac{1}{|B|^{-\alpha/n}} \left(\frac{1}{|B|} \int_{2B} |b(y) - b_B|^u dy \right)^{1/u} \left(\frac{1}{|B|} \int_{2B} |f(y)|^s dy \right)^{1/s} \\
&\lesssim [b]^\theta_\beta \left(\frac{1}{|2B|^{1-(\alpha+\beta)s/n}} \int_{2B} |f(y)|^s dy \right)^{1/s} \\
&\lesssim [b]^\theta_\beta M_{\alpha+\beta,s}(f)(x).
\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}
J_3 &\leqslant \frac{1}{|B|^2} \int_B \int_B \int_{(2B)^c} |K_\alpha(y,u) - K_\alpha(z,u)||b(u) - b_B||f(u)| du dz dy \\
&\lesssim \int_{(2B)^c} |K_\alpha(y,u) - K_\alpha(z,u)||b(u) - b_B||f(u)| du \\
&\lesssim [b]^\theta_\beta M_{\alpha+\beta,s}(f)(x).
\end{aligned}$$

□

We now come to prove Theorem 1.4. By proposition 2.4, Lemma 3.1, and Lemma 3.3 we have

$$\begin{aligned}
\|[b,\mathbb{I}_\alpha]f\|^q_{L^q(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} |M_{\rho,\eta}([b,\mathbb{I}_\alpha]f)(x)|^q \, dx \\
&\leqslant \int_{\mathbb{R}^n} \left|M^\sharp_{\rho,\gamma}([b,\mathbb{I}_\alpha]f)(x)\right|^q dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |[b,\mathbb{I}_\alpha]f(x)| \, dx \right)^q \\
&\lesssim \int_{\mathbb{R}^n} \left|M^\sharp_{\rho,\gamma}([b,\mathbb{I}_\alpha]f)(x)\right|^q dx + \sum_k |Q_k| \left(\inf_{y \in 2Q_k} M_{\alpha+\beta,s}(f)(y) \right)^q \\
&\lesssim ([b]^\theta_\beta)^q \int_{\mathbb{R}^n} \left|M_{\alpha+\beta,s}(f)(x) + M_{\beta,s}(\mathbb{I}_\alpha f)(x)\right|^q dx + ([b]^\theta_\beta)^q \sum_k \int_{2Q_k} |M_{\alpha+\beta,s}(f)(x)|^q dx \\
&\lesssim ([b]^\theta_\beta)^q \left(\int_{\mathbb{R}^n} |M_{\alpha+\beta,s}(f)(x)|^q dx + \int_{\mathbb{R}^n} \left|M_{\beta,s}(\mathbb{I}_\alpha f)(x)\right|^q dx \right) \\
&\lesssim ([b]^\theta_\beta)^q \|f\|^q_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where we have used the finite overlapping property given by Proposition 2.3.

4. Proofs of Theorems 1.5 and 1.6

Let us first prove Theorem 1.5.

Choosing $\tau > 1$, we only need to show that for any $H^{p,\tau}_{\mathcal{L}}$-atom a,

$$\|[b,\mathbb{I}_\alpha]a\|_{L^q(\mathbb{R}^n)} \leqslant C$$

holds, where C is a constant independent of a. Suppose supp $a \subset B = B(x_0, r)$ with $r < \rho(x_0)$. Then

$$\|[b,\mathbb{I}_\alpha]a\|_{L^q(\mathbb{R}^n)} \leqslant \left(\int_{2B} |[b,\mathbb{I}_\alpha]a(x)|^q dx \right)^{1/q} + \left(\int_{(2B)^c} |[b,\mathbb{I}_\alpha]a(x)|^q dx \right)^{1/q} = A_1 + A_2.$$

Let $1/q_1 = 1/\tau - (\alpha+\beta)/n$. By Theorem 1.4 and the size condition of atom a, we have

$$A_1 \leqslant \left(\int_{2B} |[b,\mathbb{I}_\alpha]a(x)|^{q_1} dx \right)^{1/q_1} (2r)^{\frac{n}{q} - \frac{n}{q_1}} \leqslant C \left(\int_{2B} |a(x)|^\tau dx \right)^{1/\tau} (2r)^{\frac{n}{q} - \frac{n}{q_1}} \leqslant C(2r)^{\frac{n}{\tau} - \frac{n}{p}} (2r)^{\frac{n}{q} - \frac{n}{q_1}} = C.$$

For A_2, we consider two cases, that are $r < \rho(x_0)/4$ and $\rho(x_0)/4 \leqslant r < \rho(x_0)$.
Case I: When $r < \rho(x_0)/4$, by the vanishing condition of a, we have

$$|[b,\mathbb{I}_\alpha]a(x)| \leqslant |b(x)-b_B|\int_B |K_\alpha(x,y)-K_\alpha(x,x_0)||a(y)|dy + \int_B |K_\alpha(x,y)(b(y)-b_B)a(y)|dy = A_{21}+A_{22}.$$

Note that

$$\int_B |a(y)|dy \lesssim r^{n-\frac{n}{p}},$$

and

$$\frac{1}{|2^kB|}\int_{2^kB}|b(x)-b_B|^q dx \lesssim ([b]_\beta^\theta)^q(2^kr)^{\beta q}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\theta' q}.$$

When $x \in 2^{k+1}B(x_0,r)\setminus 2^kB(x_0,r)$, and $y \in B$, by Lemmas 2.7 and 2.2, we can take δ such that $0<\beta<\delta<\delta'$ and

$$|K_\alpha(x,y)-K_\alpha(x,x_0)| \lesssim \frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{N/(k_0+1)}}\frac{r^\delta}{(2^kr)^{n+\delta-\alpha}}.$$

Noticing $1/q = 1/p-(\alpha+\beta)/n$ and $p > \frac{n}{n+\beta} > \frac{n}{n+\delta}$, then we get

$$\begin{aligned}\int_{(2B)^c}(A_{21})^q dx \lesssim & r^{(n-\frac{n}{p})q}([b]_\beta^\theta)^q\sum_{k\geqslant 1}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{Nq/(k_0+1)}}\frac{r^{\delta q}}{(2^kr)^{(n+\delta-\alpha)q}}\int_{2^kB}|b(x)-b_B|^q dx\\ \lesssim & ([b]_\beta^\theta)^q\sum_{k\geqslant 1}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{Nq/(k_0+1)-\theta' q}}2^{kq(\frac{n}{p}-n-\delta)}\\ \lesssim & ([b]_\beta^\theta)^q\sum_{k\geqslant 1}2^{kq(\frac{n}{p}-n-\delta)}\\ \lesssim & ([b]_\beta^\theta)^q.\end{aligned}$$

For $x \in (2B)^c, y \in B$, we have $|x-y| \approx |x-x_0|$. By Lemmas 2.7 and 2.2,

$$\begin{aligned}\left(\int_{2^{k+1}B\setminus 2^kB}|K_\alpha(x,y)|^q dx\right)^{1/q} \lesssim & \frac{r^\delta}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\left(\int_{2^{k+1}B\setminus 2^kB}\frac{dx}{|x-x_0|^{q(n+\delta-\alpha)}}\right)^{1/q}\\ \lesssim & \frac{2^{-k\delta}}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\frac{1}{(2^kr)^{\frac{n}{q'}-\alpha}}.\end{aligned}$$

By Hölder's inequality and Lemma 2.5 we get

$$\begin{aligned}\int_B |b(y)-b_B||a(y)|dy \leqslant & \left(\int_B |a(y)|^\tau dy\right)^{1/\tau}\left(\int_B |b(y)-b_B|^{\tau'}dy\right)^{1/\tau'}\\ \lesssim & [b]_\beta^\theta r^{\frac{n}{\tau}-\frac{n}{p}}r^{\beta+\frac{n}{\tau'}}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta'}\\ \lesssim & [b]_\beta^\theta r^{n-\frac{n}{p}+\beta}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta'}.\end{aligned}$$

Then, by Minkowski's inequality we get

$$\begin{aligned}\left(\int_{(2B)^c}(A_{22})^q dx\right)^{1/q} &\lesssim \int_B |b(y)-b_B||a(y)|dy\left(\sum_{k\geqslant 1}\int_{2^{k+1}B\setminus 2^kB}|K_\alpha(x,y)|^q dx\right)^{1/q}\\ &\lesssim [b]_\beta^\theta r^{n-\frac{n}{p}+\beta}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta'}\sum_{k\geqslant 1}\frac{2^{-k\delta}}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\frac{1}{(2^kr)^{\frac{n}{q'}-\alpha}}\\ &\lesssim [b]_\beta^\theta\sum_{k\geqslant 1}\frac{1}{2^{k(n-\frac{n}{p}+\beta+\delta)}}\\ &\lesssim [b]_\beta^\theta.\end{aligned}$$

Case II: When $\rho(x_0)/4 \leqslant r < \rho(x_0)$, this is $\frac{r}{\rho(x_0)} \geqslant 1/4$. The atom a does not satisfy the vanishing condition. By Minkowski's inequality,

$$\begin{aligned}A_2 &\leqslant \left\{\int_{(2B)^c}|b(x)-b_B|^q\left|\int_B K_\alpha(x,y)a(y)dy\right|^q dx\right\}^{1/q}\\ &\quad+\left\{\int_{(2B)^c}\left|\int_B |K_\alpha(x,y)(b(y)-b_B)a(y)|dy\right|^q dx\right\}^{1/q}\\ &=A_{21}'+A_{22}'.\end{aligned}$$

When $y\in B, x\in 2^{k+1}B\setminus 2^kB$, we have

$$|K_\alpha(x,y)| \lesssim \frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\frac{1}{(2^kr)^{n-\alpha}},$$

$$\int_B |a(y)|dy \lesssim r^{n-\frac{n}{p}},$$

and

$$\int_{2^kB}|b(x)-b_B|^q dx \lesssim ([b]_\beta^\theta)^q(2^kr)^{n+\beta q}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\theta' q}.$$

Note that $\frac{r}{\rho(x_0)} \geqslant 1/4$, then

$$\begin{aligned}(A_{21}')^q &\lesssim ([b]_\beta^\theta)^q\sum_{k\geqslant 1}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{qN}{k_0+1}-q\theta'}}\frac{(2^kr)^{n+\beta q}}{(2^kr)^{(n-\alpha)q}}r^{(n-\frac{n}{p})q}\\ &\lesssim ([b]_\beta^\theta)^q\sum_{k\geqslant 1}\frac{1}{(2^k)^{\frac{qN}{k_0+1}-q\theta'}}(2^k)^{(\frac{n}{p}-n)q} \lesssim ([b]_\beta^\theta)^q.\end{aligned}$$

Since N can be chosen large enough, the last series converges.

The estimate of A_{22}' is exactly the same as $\|A_{22}\|_{L^q((2B)^c)}$, we omit the detail of the proof. Then the proof of Theorem 1.5 is finished.

Finally, we proceed to prove Theorem 1.6.

Let $f\in H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)$, we write $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where each a_j is an $H_{\mathcal{L}}^{\frac{n}{n+\beta},l}$-atom, $1<l<\frac{n}{\alpha+\beta}$, and

$$\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^{\frac{n}{n+\beta}}\right)^{\frac{n+\beta}{n}} \leqslant 2\|f\|_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.$$

Suppose that $\operatorname{supp} a_j \subset B_j = B(x_j, r_j)$ with $r_j < \rho(x_j)$. Write

$$\begin{aligned}[b, \mathbb{I}_\alpha]f(x) &= \sum_{j=-\infty}^{\infty} \lambda_j [b, \mathbb{I}_\alpha]a_j(x)\chi_{8B_j}(x) + \sum_{j:r_j \geqslant \rho(x_j)/4} \lambda_j \big(b(x) - b_{B_j}\big)\mathbb{I}_\alpha a_j(x)\chi_{(8B_j)^c}(x) \\ &\quad + \sum_{j:r_j < \rho(x_j)/4} \lambda_j \big(b(x) - b_{B_j}\big)\mathbb{I}_\alpha a_j(x)\chi_{(8B_j)^c}(x) - \sum_{j=-\infty}^{\infty} \lambda_j \mathbb{I}_\alpha((b - b_{B_j})a_j)(x)\chi_{(8B_j)^c}(x) \\ &= \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} \lambda_j A_{ij}(x).\end{aligned}$$

In the following, we always let $q = \frac{n}{n-\alpha}$. Note that

$$\left(\int_{B_j} |a_j(x)|^l dx\right)^{1/l} \lesssim |B_j|^{\frac{1}{l} - \frac{n+\beta}{n}}.$$

Choose $t > \frac{n-\alpha}{n-\alpha-\beta}$ such that $\frac{1}{qt} = \frac{1}{l} - \frac{\alpha+\beta}{n}$. By Hölder's inequality and Theorem 1.4 we get

$$\begin{aligned}\|A_{1,j}\|_{L^q(\mathbb{R}^n)} &\lesssim \left(\int_{8B_j} \big|[b, \mathbb{I}_\alpha]a_j(x)\big|^{qt} dx\right)^{\frac{1}{qt}} r_j^{\frac{n}{qt'}} \\ &\lesssim [b]_\beta^\theta r_j^{\frac{n}{qt'}} \left(\int_{B_j} |a_j(x)|^l dx\right)^{1/l} \lesssim [b]_\beta^\theta |B_j|^{\frac{1}{qt'} + \frac{1}{l} - \frac{n+\beta}{n}} \lesssim [b]_\beta^\theta.\end{aligned}$$

Noticing $0 < \frac{n}{n+\beta} < 1$, we get

$$\begin{aligned}\left\|\sum_{j=-\infty}^{\infty} \lambda_j A_{1j}\right\|_{L^q(\mathbb{R}^n)} &\lesssim \sum_{j=-\infty}^{\infty} |\lambda_j| \, \|A_{1j}\|_{L^q(\mathbb{R}^n)} \lesssim [b]_\beta^\theta \sum_{j=-\infty}^{\infty} |\lambda_j| \lesssim [b]_\beta^\theta \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{\frac{n}{n+\beta}}\right)^{\frac{n+\beta}{n}} \\ &\lesssim [b]_\beta^\theta \|f\|_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.\end{aligned}$$

Then

$$\left|\left\{x \in \mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty} \lambda_j A_{1j}\right| > \frac{\lambda}{4}\right\}\right| \lesssim \frac{([b]_\beta^\theta)^q}{\lambda^q} \|f\|^q_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.$$

Since $x \in B_j, y \in 2^{k+1}B_j \setminus 2^k B_j$, we have $|x-y| \approx |x - x_j| \approx 2^k r_j$, and by Lemma 2.2 we get

$$\frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\frac{N}{k_0+1}}}.$$

Note that $\int_{B_j} |a_j(y)| dy \leqslant r_j^{-\beta}$, and $r_j/\rho(x_j) \geqslant 1/4$. Then

$$\begin{aligned}\|A_{2,j}(x)\|^q_{L^q(\mathbb{R}^n)} &= \sum_{k \geqslant 3} \int_{2^{k+1}B_j \setminus 2^k B_j} |b(x) - b_{B_j}|^q \left(\int_{B_j} \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-\alpha}} |a_j(y)| dy\right)^q dx \\ &\lesssim \sum_{k \geqslant 3} \frac{1}{\left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\frac{Nq}{k_0+1}}} \frac{1}{(2^k r_j)^{(n-\alpha)q}} \int_{2^{k+1}B_j} |b(x) - b_{B_j}|^q dx \left(\int_{B_j} |a_j(y)| dy\right)^q\end{aligned}$$

$$
\begin{aligned}
&\lesssim \big([b]_\beta^\theta\big)^q \sum_{k\geqslant 3} \frac{1}{\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{\left(\frac{N}{k_0+1}-\theta'\right)q}} (2^k B_j)^{\beta q} r_j^{-\beta q} \\
&\lesssim \big([b]_\beta^\theta\big)^q \sum_{k\geqslant 1} \frac{1}{2^{kq\left(\frac{N}{k_0+1}-\theta'-\beta\right)}} \\
&\lesssim \big([b]_\beta^\theta\big)^q.
\end{aligned}
$$

Then

$$
\Big\| \sum_{j=-\infty}^{\infty} \lambda_j A_{2j} \Big\|_{L^q(\mathbb{R}^n)} \lesssim [b]_\beta^\theta \|f\|_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.
$$

Therefore

$$
\Big| \Big\{ x\in\mathbb{R}^n : \Big| \sum_{j=-\infty}^{\infty} \lambda_j A_{2j} \Big| > \frac{\lambda}{4} \Big\} \Big| \lesssim \frac{\big([b]_\beta^\theta\big)^q}{\lambda^q} \|f\|^q_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.
$$

When $x\in 2^{k+1}B_j\setminus 2^k B_j$, and $y\in B_j$, by Lemmas 2.7 and 2.2 we have

$$
|K_\alpha(x,y)-K_\alpha(x,x_j)| \lesssim \frac{1}{\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{N/(k_0+1)}} \frac{r_j^\delta}{(2^k r_j)^{n+\delta-\alpha}}.
$$

Thus, by the vanishing condition of a_j and $0<\beta<\delta<\delta'$ we have

$$
\begin{aligned}
\|A_{3,j}(x)\|^q_{L^q(\mathbb{R}^n)} &= \sum_{k\geqslant 3} \int_{2^{k+1}B_j\setminus 2^k B_j} |b(x)-b_{B_j}|^q \left(\int_{B_j} |K_\alpha(x,y)-K_\alpha(x,x_j)||a_j(y)|dy \right)^q dx \\
&\lesssim \sum_{k\geqslant 3} \frac{1}{\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{\frac{Nq}{k_0+1}}} \frac{r_j^{\delta q}}{(2^k r_j)^{(n+\delta-\alpha)q}} \int_{2^{k+1}B_j} |b(x)-b_{B_j}|^q dx \left(\int_{B_j} |a_j(y)|dy \right)^q \\
&\lesssim \big([b]_\beta^\theta\big)^q \sum_{k\geqslant 3} \frac{1}{\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{\left(\frac{N}{k_0+1}-\theta'\right)q}} \frac{r_j^{\delta q}}{(2^k r_j)^{(n+\delta-\alpha)q}} (2^k r_j)^{n+\beta q} r_j^{-\beta q} \\
&\lesssim \big([b]_\beta^\theta\big)^q \sum_{k\geqslant 1} \frac{1}{2^{k(\delta-\beta)}} \\
&\lesssim \big([b]_\beta^\theta\big)^q.
\end{aligned}
$$

Then

$$
\Big\| \sum_{j=-\infty}^{\infty} \lambda_j A_{3j} \Big\|_{L^q(\mathbb{R}^n)} \lesssim [b]_\beta^\theta \|f\|_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.
$$

Therefore

$$
\Big| \Big\{ x\in\mathbb{R}^n : \Big| \sum_{j=-\infty}^{\infty} \lambda_j A_{3j} \Big| > \frac{\lambda}{4} \Big\} \Big| \lesssim \frac{\big([b]_\beta^\theta\big)^q}{\lambda^q} \|f\|^q_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.
$$

Note that

$$
\|(b-b_{B_j})a_j\|_{L^1} \leqslant \left(\int_{B_j} |b(x)-b_{B_j}|^{l'} dx \right)^{1/l'} \left(\int_{B_j} |a_j(x)|^{l} dx \right)^{1/l}
$$

$$\lesssim [b]_\beta^\theta r_j^{\frac{n}{l}-n-\beta+\frac{n}{l'}+\beta}\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta'} \lesssim [b]_\beta^\theta,$$

and

$$|A_{4j}(x)| \leqslant \sum_{j=-\infty}^{\infty} |\lambda_j| \mathbb{I}_\alpha(|(b-b_{B_j})a_j|)(x)\chi_{(8B_j)^c}(x) \leqslant \mathbb{I}_\alpha\left(\sum_{j=-\infty}^{\infty} |\lambda_j(b-b_{B_j})a_j|\right)(x).$$

By the boundedness of $\mathbb{I}_\alpha$ from $L^1(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$ (see Corollary 2.8) we get

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty}\lambda_j A_{4j}\right| > \frac{\lambda}{4}\right\}\right| &\leqslant \left|\left\{x\in\mathbb{R}^n : \left|\mathbb{I}_\alpha\Big(\sum_{j=-\infty}^{\infty}|\lambda_j(b-b_{B_j})a_j|\Big)(x)\right| > \frac{\lambda}{4}\right\}\right| \\ &\lesssim \frac{1}{\lambda^q}\left\|\sum_{j=-\infty}^{\infty}|\lambda_j(b-b_{B_j})a_j|\right\|_{L^1(\mathbb{R}^n)}^q \\ &\lesssim \frac{1}{\lambda^q}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\|(b-b_{B_j})a_j\|_{L^1(\mathbb{R}^n)}\right)^q \\ &\lesssim \frac{([b]_\beta^\theta)^q}{\lambda^q}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\right)^q \lesssim \frac{([b]_\beta^\theta)^q}{\lambda^q}\|f\|^q_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{i=1}^{4}\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}\right| > \lambda\right\}\right| &\lesssim \sum_{i=1}^{4}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}\right| > \frac{\lambda}{4}\right\}\right| \\ &\lesssim \frac{([b]_\beta^\theta)^q}{\lambda^q}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\right)^q \lesssim \frac{([b]_\beta^\theta)^q}{\lambda^q}\|f\|^q_{H_{\mathcal{L}}^{\frac{n}{n+\beta}}(\mathbb{R}^n)},\end{aligned}$$

which completes the proof of Theorem 1.6.

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