Persistence and non-persistence of a stochastic food chain model with finite delay

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Communicated by A. Atangana
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Abstract

In this paper, we study a three species predator-prey time-delay chain model with stochastic perturbation. First, we analyze that this system has a unique positive solution. Then, we deduce the conditions that the system is persistent in time average. After that, conditions for the system going to be extinction in probability are established. At last, numerical simulations are carried out to support our results. ©2017 All rights reserved.

Keywords: Stochastic differential equation, persistent, non-persistent, extinction.
2010 MSC: 93E03.

1. Introduction

The most exciting modern application of mathematics is used in biology. As we know, two species systems such as predator-prey, plant-pest systems etcetera has been one of the dominant theme in the importance of ecology and mathematics because of its widespread. After that, the predator-prey chain model is the typical representative.

As we know, until the late 70s, some interest mathematics tritrophic food chain model appeared [5, 6]. One of the most famous population dynamics model on ecological system has received a lot of attention and extensive research which named Lotka-Volterra predator-prey system, refer to [3, 10, 20]. Especially the persistence and extinction of this model is very interesting topic.

The three species predator-prey chain model is described as follows:

$$
\begin{cases}
\dot{x}_1(t) = x_1(t)\left(a_1 - b_{11}x_1(t) - b_{12}x_2(t)\right), \\
\dot{x}_2(t) = x_2(t)\left(-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)\right), \\
\dot{x}_3(t) = x_3(t)\left(-a_3 + b_{32}x_2(t) - b_{33}x_3(t)\right),
\end{cases}
\tag{1.1}
$$

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where $x_i(t)(i,=1,2,3)$ represent the densities of prey, mid-level predator, and top predator species at time t, respectively. The parameters $a_1, a_2, a_3, b_{ii}(i,=1,2,3)$ are positive constants that stand for intrinsic growth rate, predator death rate of the second species, predator death rate of the third species, and coefficient of internal competition, respectively. b_{21}, b_{32} represent saturated rate of the second and the third predator, b_{12}, b_{23} represent the decrement rate of predator to prey. System (1.1) describes a three species predator-prey chain model in which the latter preys on the former. From a biological viewpoint, we not only require the positive solution of the system but also require its unexploded property in any finite time and stability.

We know that the global asymptotic stability of a positive equilibrium $x^* = (x_1^*, x_2^*, x_3^*)$ holds and is global stable if the following condition holds:

$$a_1 - \frac{b_{11}}{b_{21}}a_2 - \frac{b_{11}b_{22} + b_{12}b_{21}}{b_{21}b_{32}}a_3 > 0,$$

which could refer to [9].

In a recent period of time, it is easy to understand in the process of both natural and man-made involve the time delay, such as biology, medicine and so on. Kuang [19] mentioned that animals must take time to digest their food before further activities and responses take place. So, the dynamic model of any species lack of delay is an approximate model. Standard for three classes dynamic model of a single species with independent discrete delay to have global asymptotic stability equilibrium point was set up by friedman and Gopalsamy [4]. By constructing proper lyapunov functionals, the global stability of the time-delay systems is studied by Xiaoqing et al. [28]. Hence, we introduce time-delays in system (1.1) and suppose that the middle carnivore specie needs time τ to have the ability to hunt after birth and it just captures prey and τ mature adult. Similarly, suppose that the top carnivore specie needs time τ to have the ability to hunt after birth and it just captures mid-level and τ ([8, 14, 22]) mature adult predator. Then we get

$$\begin{cases} \dot{x_1}(t) = x_1(t)\left(a_1 - b_{11}x_1(t) - b_{12}x_2(t-\tau)\right), \\ \dot{x_2}(t) = x_2(t)\left(-a_2 + b_{21}x_1(t-\tau) - b_{22}x_2(t) - b_{23}x_3(t-\tau)\right), \\ \dot{x_3}(t) = x_3(t)\left(-a_3 + b_{32}x_2(t-\tau) - b_{33}x_3(t)\right). \end{cases} \quad (1.2)$$

However, population dynamic system in the real environment is unavoidable influenced by environmental noise (see, e.g., [7, 8]). The parameters in the system are not absolute constants, they are always in some near average. So we can not ignore the influence of noise on the system. Recently many authors have discussed population systems subject to white noise (see, e.g., [12, 13, 15–17, 23, 29]). May (see, e.g., [25]) pointed out that due to continuous fluctuation in the environment, the equilibrium distribution does not reach a stable value, but fluctuates randomly near the average value.

Therefore, Lotka-Volterra predator-prey chain models in random environments are becoming popular. Ji et al. [15, 16] investigated the asymptotic behavior of the stochastic predator-prey system with perturbation. Polansky [26] and Barra et al. [1] have given some special systems of their invariant distribution. After that, Gard [9] analysed that under some conditions the stochastic food chain model exists an invariant distribution. Mao et al. [24] have discussed nonexplosion, persistence, and asymptotic stability of the stochastic delay equations, they pointed out that the noise will not only suppress a potential population explosion in the delay model but will also make the population to be stochastically ultimately bounded. However, seldom people investigate the persistent and non-persistent of the food chain time-delay model with stochastic perturbation. Li et al. have studied the food chain model with stochastic perturbation in [21], and this paper is a continuation of the previous article.

In this paper, we introduce the white noise into the intrinsic growth rate of system (1.2), and suppose $a_i \to a_i + \sigma_i \dot{B}_i(t) \ (i=1,2,3)$, then we obtain the following stochastic system

$$\begin{cases} dx_1(t) = x_1(t)\left(a_1 - b_{11}x_1(t) - b_{12}x_2(t-\tau)\right)dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left(-a_2 + b_{21}x_1(t-\tau) - b_{22}x_2(t) - b_{23}x_3(t-\tau)\right)dt - \sigma_2 x_2(t)dB_2(t), \\ dx_3(t) = x_3(t)\left(-a_3 + b_{32}x_2(t-\tau) - b_{33}x_3(t)\right)dt - \sigma_3 x_3(t)dB_3(t), \end{cases} \quad (1.3)$$

where $B_i(t)$ $(i = 1, 2, 3)$ are independent white noises with $B_i(0) = 0, \sigma_i^2 > 0$ $(i = 1, 2, 3)$ representing the intensities of the noise.

The aim of this paper is to discuss the long time behavior of system (1.3) by stochastic comparison theorem which is different from Mao et al. [24]. We have mentioned that $x^* = (x_1^*, x_2^*, x_3^*)$ is also the positive equilibrium of system (1.2). But, when it is suffered stochastic perturbations, there is no positive equilibrium. Hence, it is impossible that the solution of system (1.3) will tend to a fixed point. In this paper, we show that system (1.3) is persistent in time average. Furthermore, under certain conditions, we prove the population of system (1.3) will die out in probability which will not happen in deterministic system and could reveal that large white noise may lead to extinction.

The rest of this paper is organized as follows. In Section 2, we show that there is a unique non-negative solution of system (1.3). In Section 3, we show that system (1.3) is persistent in time average. While in Section 4, we consider three situations when the population of the system will be extinction. In Section 5, numerical simulations are carried out to support our results.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t \geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let R_+^3 denote the positive cone of R^3, namely $R_+^3 = \{x \in R^3 : x_i > 0, \ 1 \leqslant i \leqslant 3\}$, $\bar{R}_+^3 = \{x \in R^3 : x_i \geqslant 0, \ 1 \leqslant i \leqslant 3\}$.

2. Existence and uniqueness of the nonnegative solution

To investigate the dynamical behavior, the first concern thing is whether the solution is global existence. Moreover, for a population model, whether the solution is nonnegative is also considered. Hence, in this section we show that the solution of system (1.3) is global and nonnegative. As we have known, in order for a stochastic differential equation to have a unique global (i.e., no explosion at a finite time) solution with any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (see, e.g., [22]). It is easy to see that the coefficients of system (1.3) are locally Lipschitz continuous, so system (1.3) has a local solution. By Theorem 2.2, we show the global existence of this solution.

Let $N(t)$ be the solution of the non-autonomous logistic equation with random perturbation

$$dN(t) = N(t)[(a(t) - b(t)N(t))dt + \alpha(t)dB(t)], \tag{2.1}$$

where $B(t)$ is one-dimensional standard Brownian motion, $N(0) = N_0 > 0$ and N_0 is independent of $B(t)$.

Lemma 2.1 ([18]). *There exists a unique continuous solution $N(t)$ of (2.1) for any initial value $N(0) = N_0 > 0$, which is global and represented by*

$$N(t) = \frac{\exp\{\int_0^t [a(s) - \frac{\sigma^2(s)}{2}]ds + \sigma(s)dB(s)\}}{1/N_0 + \int_0^t b(s) \exp\{\int_0^s [a(\tau) - \frac{\sigma^2(\tau)}{2}]d\tau + \sigma(\tau)dB(\tau)\}ds}, \quad t \geqslant 0. \tag{2.2}$$

In order to get the conclusion, we should introduce two systems first,

$$\begin{cases} d\Phi_1(t) = \Phi_1(t)(a_1 - b_{11}\Phi_1(t))\,dt + \sigma_1\Phi_1(t)dB_1(t), \\ d\Phi_2(t) = \Phi_2(t)(-a_2 + b_{21}\Phi_1(t-\tau) - b_{22}\Phi_2(t))\,dt - \sigma_2\Phi_2(t)dB_2(t), \\ d\Phi_3(t) = \Phi_3(t)(-a_3 + b_{32}\Phi_2(t-\tau) - b_{33}\Phi_3(t))\,dt - \sigma_3\Phi_3(t)dB_3(t), \\ \Phi_i(t) = \xi_i(t) \in C([-\tau, 0]; R_+), \quad i = 1, 2, 3, \end{cases} \tag{2.3}$$

and

$$\begin{cases} dI_1(t) = I_1(t)(a_1 - b_{11}I_1(t) - b_{12}\Phi_2(t-\tau))\,dt + \sigma_1 I_1(t)dB_1(t), \\ dI_2(t) = I_2(t)(-a_2 + b_{21}I_1(t-\tau) - b_{22}I_2(t) - b_{23}\Phi_3(t-\tau))\,dt - \sigma_2 I_2(t)dB_2(t), \\ dI_3(t) = I_3(t)(-a_3 + b_{32}I_2(t-\tau) - b_{33}I_3(t))\,dt - \sigma_3 I_3(t)dB_3(t), \\ I_i(t) = \xi_i(t) \in C([-\tau, 0]; R_+) \quad i = 1, 2, 3, \end{cases} \tag{2.4}$$

where

$$\Phi(t) = (\Phi_1(t), \Phi_2(t), \Phi_3(t))^\intercal, \qquad I(t) = (I_1(t), I_2(t), I_3(t))^\intercal,$$

are the solutions of the above stochastic differential equations with time delay.

Theorem 2.2. *For any initial data* $x(t) = \{(\xi_1(t), \xi_2(t), \xi_3(t)) : -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R_+^3)$, *the positive solution of system* (1.3) *has the property that*

$$I(t) \leqslant x(t) \leqslant \Phi(t),$$

i.e.,

$$I_i(t) \leqslant x_i(t) \leqslant \Phi_i(t), \qquad i = 1, 2, 3,$$

where

$$\Phi(t) = (\Phi_1(t), \Phi_2(t), \Phi_3(t))^\intercal, \qquad I(t) = (I_1(t), I_2(t), I_3(t))^\intercal,$$

are solutions of systems (2.3) *and* (2.4) *stochastic differential equations with time delay.*

Proof. Let $z_1(t) = \frac{1}{x_1(t)}$. Then, by Itô's formula, we have

$$dz_1(t) = d(\frac{1}{x_1(t)}) = -[(\frac{a_1}{x_1(t)} - \frac{b_{12}x_2(t-\tau)}{x_1(t)} - b_{11})dt + \frac{\sigma_1}{x_1(t)}dB_1(t)] + \frac{\sigma_1^2}{x_1(t)}dt$$

$$= [(\sigma_1^2 - a_1)z_1(t) + b_{11} + \frac{b_{12}x_2(t-\tau)}{x_1(t)}]dt - \sigma_1 z_1(t)dB_1(t)$$

$$= [(\sigma_1^2 - a_1)dt - \sigma_1 dB_1(t)]z_1(t) + (b_{11} + \frac{b_{12}x_2(t-\tau)}{x_1(t)})dt.$$

That is,

$$dz_1(t) = [(\sigma_1^2 - a_1)dt - \sigma_1 dB_1(t)]z_1(t) + (b_{11} + \frac{b_{12}x_2(t-\tau)}{x_1(t)})dt.$$

Then

$$z_1(t) = e^{\int_0^t (\frac{\sigma_1^2}{2} - a_1)ds - \sigma_1 dB_1(s)}[\frac{1}{x_1(0)} + \int_0^t (b_{11} + \frac{b_{12}x_2(t-\tau)}{x_1(t)})e^{\int_0^s (a_1 - \frac{\sigma_1^2}{2})d\tau + \sigma_1 dB_1(\tau)}ds]$$

$$= e^{(\frac{\sigma_1^2}{2} - a_1)t - \sigma_1 B_1(t)}[\frac{1}{x_1(0)} + \int_0^t (b_{11} + \frac{b_{12}x_2(t-\tau)}{x_1(t)})e^{(a_1 - \frac{\sigma_1^2}{2})s + \sigma_1 B_1(s)}ds]$$

$$\geqslant e^{(\frac{\sigma_1^2}{2} - a_1)t - \sigma_1 B_1(t)}[\frac{1}{x_1(0)} + \int_0^t b_{11}e^{(a_1 - \frac{\sigma_1^2}{2})s + \sigma_1 B_1(s)}ds]$$

$$= \Phi_1^{-1}(t).$$

By Lemma 2.1, we obtain that $\Phi_1(t)$ is the solution of the following equation

$$d\Phi_1(t) = \Phi_1(t)(a_1 - b_{11}\Phi_1(t))dt + \sigma_1\Phi_1(t)dB_1(t).$$

Hence, we have

$$x_1(t) \leqslant \Phi_1(t), \quad \text{a.s..}$$

On the other hand, let $z_2(t) = \frac{1}{x_2(t)}$. Then, by Itô's formula, we could derive that

$$
\begin{aligned}
dz_2(t) = d(\frac{1}{x_2(t)}) &= -[(-\frac{a_2}{x_2(t)} + \frac{b_{21}x_1(t-\tau)}{x_2(t)} - b_{22} - \frac{b_{23}x_3(t-\tau)}{x_2(t)})dt - \frac{\sigma_2}{x_2(t)}dB_2(t)] + \frac{\sigma_2^2}{x_2(t)}dt \\
&= [(\sigma_2^2 + a_2)z_2(t) + b_{22} - b_{21}x_1(t-\tau)z_2(t) - b_{23}x_3(t-\tau)z_2(t)]dt + \sigma_2 z_2(t)dB_2(t) \\
&= [(\sigma_2^2 + a_2 - b_{21}x_1(t-\tau) - b_{23}x_3(t-\tau))dt + \sigma_2 dB_2(t)]z_2(t) + b_{22}dt,
\end{aligned}
$$

then

$$
\begin{aligned}
z_2(t) &= \frac{1}{x_2(0)}e^{(\frac{\sigma_2^2}{2}+a_2)t+\sigma_2 B_2(s)-b_{21}\int_0^t x_1(s-\tau)ds+b_{23}\int_0^t x_3(s-\tau)ds} \\
&\quad + b_{22}\int_0^t e^{(a_2+\frac{\sigma_2^2}{2})(t-s)+\sigma_2(B_2(t)-B_2(s))-b_{21}\int_s^t x_1(\mu-\tau)d\mu+b_{23}\int_s^t x_3(\mu-\tau)d\mu}ds \\
&\geqslant \frac{1}{x_2(0)}e^{(\frac{\sigma_2^2}{2}+a_2)t+\sigma_2 B_2(s)-b_{21}\int_0^t x_1(s-\tau)ds} \\
&\quad + b_{22}\int_0^t e^{(a_2+\frac{\sigma_2^2}{2})(t-s)+\sigma_2(B_2(t)-B_2(s))-b_{21}\int_s^t \Phi_1(\mu-\tau)d\mu}ds \\
&= \Phi_2^{-1}(t).
\end{aligned}
$$

Therefore

$$
x_2(t) \leqslant \Phi_2(t), \quad a.s..
$$

By Lemma 2.1, we obtain that $\Phi_2(t)$ is the solution of the following equation

$$
d\Phi_2(t) = \Phi_2(t)\left(-a_2 + b_{21}\Phi_1(t-\tau) - b_{22}\Phi_2(t)\right)dt - \sigma_2\Phi_2(t)dB_2(t).
$$

At last, let $z_3(t) = \frac{1}{x_3(t)}$. Then, by Itô's formula, we could derive that

$$
\begin{aligned}
dz_3(t) = d(\frac{1}{x_3(t)}) &= -[(-\frac{a_3}{x_3(t)} + \frac{b_{32}x_2(t-\tau)}{x_3(t)} - b_{33})dt - \frac{\sigma_3}{x_3(t)}dB_3(t)] + \frac{\sigma_3^2}{x_3(t)}dt \\
&= [(\sigma_3^2 + a_3)z_2(t) + b_{33} - b_{32}x_2(t-\tau)z_3(t)]dt + \sigma_3 z_3(t)dB_3(t) \\
&= [(\sigma_3^2 + a_3 - b_{32}x_2(t-\tau))dt + \sigma_2 dB_3(t)]z_2(t) + b_{33}dt,
\end{aligned}
$$

then

$$
\begin{aligned}
z_3(t) &= \frac{1}{x_3(0)}e^{(\frac{\sigma_3^2}{2}+a_3)t+\sigma_3 B_3(s)-b_{32}\int_0^t x_2(s-\tau)ds} \\
&\quad + b_{33}\int_0^t e^{(a_3+\frac{\sigma_3^2}{2})(t-s)+\sigma_3(B_3(t)-B_3(s))-b_{32}\int_s^t x_2(\mu-\tau)d\mu}ds \\
&\geqslant \frac{1}{x_3(0)}e^{(\frac{\sigma_3^2}{2}+a_3)t+\sigma_3 B_3(s)-b_{32}\int_0^t \Phi_2(s-\tau)ds} \\
&\quad + b_{33}\int_0^t e^{(a_3+\frac{\sigma_3^2}{2})(t-s)+\sigma_3(B_3(t)-B_3(s))-b_{32}\int_s^t \Phi_2(\mu-\tau)d\mu}ds \\
&= \Phi_3^{-1}(t),
\end{aligned}
$$

then, it is easy to see that $\Phi_3(t)$ is the solution of the following equation

$$
d\Phi_3(t) = \Phi_3(t)\left(-a_3 + b_{32}\Phi_2(t-\tau) - b_{33}\Phi_3(t)\right)dt - \sigma_3\Phi_3(t)dB_3(t),
$$

and

$$x_3(t) \leqslant \Phi_3(t), \quad \text{a.s..}$$

In the same method, we could derive that

$$x_i(t) \geqslant I_i(t) \quad \text{a.s., } i = 1, 2, 3,$$

where $I(t) = (I_1(t), I_2(t), I_3(t))^\intercal$ is the solution of system (2.4). $\qquad\square$

Remark 2.3. From Lemma 2.1, we know

$$\frac{1}{\Phi_1(t)} = \frac{1}{x_1(0)} e^{(\frac{\sigma_1^2}{2} - a_1)t - \sigma_1 B_1(t)} + b_{11} \int_0^t e^{(\frac{\sigma_1^2}{2} - a_1)(t-s) - \sigma_1(B_1(t) - B_1(s))} ds,$$

$$\frac{1}{\Phi_2(t)} = \frac{1}{x_2(0)} e^{(\frac{\sigma_2^2}{2} + a_2)t + \sigma_2 B_2(t) - b_{21} \int_0^t \Phi_1(s-\tau) ds} + b_{22} \int_0^t e^{(\frac{\sigma_2^2}{2} + a_2)(t-s) + \sigma_2(B_2(t) - B_2(s)) - b_{21} \int_s^t \Phi_1(\mu-\tau) d\mu} ds,$$

$$\frac{1}{\Phi_3(t)} = \frac{1}{x_3(0)} e^{(\frac{\sigma_3^2}{2} + a_3)t + \sigma_3 B_3(t) - b_{32} \int_0^t \Phi_2(s-\tau) ds} + b_{33} \int_0^t e^{(\frac{\sigma_3^2}{2} + a_3)(t-s) + \sigma_3(B_3(t) - B_3(s)) - b_{32} \int_s^t \Phi_2(\mu-\tau) d\mu} ds;$$

and

$$\frac{1}{I_1(t)} = \frac{1}{x_1(0)} e^{(\frac{\sigma_1^2}{2} - a_1)t - \sigma_1 B_1(t) + b_{12} \int_0^t \Phi_2(s-\tau) ds} + b_{11} \int_0^t e^{(\frac{\sigma_1^2}{2} - a_1)(t-s) - \sigma_1(B_1(t) - B_1(s)) + b_{12} \int_0^t \Phi_2(\mu-\tau) d\mu} ds,$$

$$\frac{1}{I_2(t)} = \frac{1}{x_2(0)} e^{(\frac{\sigma_2^2}{2} + a_2)t + \sigma_2 B_2(t) - b_{21} \int_0^t I_1(s-\tau) ds + b_{23} \int_0^t \Phi_3(s-\tau) ds}$$

$$+ b_{22} \int_0^t e^{(\frac{\sigma_2^2}{2} + a_2)(t-s) + \sigma_2(B_2(t) - B_2(s)) - b_{21} \int_s^t I_1(\mu-\tau) d\mu + b_{23} \int_0^t \Phi_3(\mu-\tau) d\mu} ds,$$

$$\frac{1}{I_3(t)} = \frac{1}{x_3(0)} e^{(\frac{\sigma_3^2}{2} + a_3)t + \sigma_3 B_3(t) - b_{32} \int_0^t I_2(s-\tau) ds} + b_{33} \int_0^t e^{(\frac{\sigma_3^2}{2} + a_3)(t-s) + \sigma_3(B_3(t) - B_3(s)) - b_{32} \int_s^t I_2(\mu-\tau) d\mu} ds.$$

From the representations of $\Phi_i(t)$ and $I_i(t)$, (i=1,2,3), Theorem 2.2 tells us the species will not reach zero in finite time.

From now on, we denote the unique global positive solution of system (1.3) with the given initial data $\xi = \{\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t)) : -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R_3^+)$ by $x(t, \xi)$. In the same way, we define the solutions of systems (2.3) and (2.4) by $\Phi(t, \xi), I(t, \xi)$.

3. Persistent in time average

There is no equilibrium of system (1.3). Hence we can not show the permanence of the system by proving the stability of the positive equilibrium as the deterministic system. In this section we first show that this system is persistent in mean. Before we give the result, we should do some preparation work.

Chen et al. in [2] proposed the definition of persistence in mean for the deterministic system. Here, we also use this definition for the stochastic system.

Definition 3.1. System (1.3) is said to be persistent in mean, if

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_3(s) ds > 0, \quad \text{a.s..}$$

Before giving the result, we do some preparation work.

Lemma 3.2 ([27]). *Let* $f \in C\left([0, +\infty) \times \Omega, (0, +\infty)\right)$, $F \in C\left([0, +\infty) \times \Omega, R\right)$. *If there exist positive constants* λ_0, λ, *such that*

$$\log f(t) \geqslant \lambda t - \lambda_0 \int_0^t f(s)\,ds + F(t), \quad t \geqslant 0 \quad a.s.,$$

and $\lim\limits_{t \to \infty} \dfrac{F(t)}{t} = 0$ *a.s., then*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t f(s)\,ds \geqslant \frac{\lambda}{\lambda_0}, \quad a.s..$$

From Lemma 3.2, it is easy to see that we could get Lemmas 3.3 and 3.4 with the same method.

Lemma 3.3. *Let* $f \in C\left([0, +\infty) \times \Omega, (0, +\infty)\right)$, $F \in C\left([0, +\infty) \times \Omega, R\right)$. *If there exist positive constants* λ_0, λ, *such that*

$$\log f(t) \leqslant \lambda t - \lambda_0 \int_0^t f(s)\,ds + F(t), \quad t \geqslant 0, \quad a.s.,$$

and $\lim\limits_{t \to \infty} \dfrac{F(t)}{t} = 0$, *a.s., then*

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t f(s)\,ds \leqslant \frac{\lambda}{\lambda_0}, \, a.s..$$

Lemma 3.4. *Let* $f \in C\left([0, +\infty) \times \Omega, (0, +\infty)\right)$, $F \in C\left([0, +\infty) \times \Omega, R\right)$. *If there exist positive constants* λ_0, λ, *such that*

$$\log f(t) = \lambda t - \lambda_0 \int_0^t f(s)\,ds + F(t), \quad t \geqslant 0, \quad a.s.,$$

and $\lim\limits_{t \to \infty} \dfrac{F(t)}{t} = 0$, *a.s., then*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(s)\,ds = \frac{\lambda}{\lambda_0}, \quad a.s..$$

Assumption 3.5.

$$r_1 - \frac{b_{11}}{b_{21}} r_2 - \frac{b_{11} b_{22} + b_{12} b_{21}}{b_{21} b_{32}} r_3 > 0, \quad r_1 = a_1 - \frac{\sigma_1^2}{2} > 0, \quad r_i = a_i + \frac{\sigma_i^2}{2} \quad i = 2, 3.$$

Lemma 3.6. *If Assumption 3.5 is satisfied, then the solution* $\Phi(t, \xi)$ *of system* (2.2) *has the following property:*

$$\lim_{t \to \infty} \frac{\log \Phi_i(t)}{t} = 0, \qquad \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(s)\,ds = M_i, \quad a.s.,$$

where

$$M_1 = \frac{r_1}{b_{11}}, \quad M_2 = \frac{r_1 b_{21} - r_2 b_{11}}{b_{11}}, \quad M_3 = \frac{r_1 b_{21} b_{32} - r_2 b_{11} b_{32} - r_3 b_{11} b_{22}}{b_{11} b_{22} b_{33}}.$$

Proof. From the results in [16] and if Assumption 3.5 is satisfied, we know

$$\lim_{t \to \infty} \frac{\log \Phi_1(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi_1(s)\,ds = \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}} = \frac{r_1}{b_{11}} = M_1, \quad a.s., \tag{3.1}$$

besides, according to Itô's formula, the second population of system (2.2) is changed into

$$d \log \Phi_2(t) = (-r_2 + b_{21} \Phi_1(t - \tau) - b_{22} \Phi_2(t))\,dt - \sigma_2 dB_2(t).$$

It then follows

$$\log \Phi_2(t) = \log \Phi_2(0) - r_2 t + b_{21} \int_0^t \Phi_1(s - \tau)\,ds - b_{22} \int_0^t \Phi_2(s)\,ds - \sigma_2 B_2(t). \tag{3.2}$$

Notice that

$$\int_0^t \Phi_1(s-\tau)ds = \int_{-\tau}^{t-\tau} \Phi_1(s)ds = \int_{-\tau}^0 \xi_1(s)ds + \int_0^t \Phi_1(s)ds - \int_{t-\tau}^t \Phi_1(s)ds, \tag{3.3}$$

and from the second equation of (3.1), we get $\lim_{t\to\infty} \frac{1}{t}\int_{t-\tau}^t \Phi_1(s)ds = 0$, dividing the equation (3.3) both sides by t, and taking $t \to \infty$, yields

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi_1(s-\tau)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi_1(s)ds = M_1,$$

so

$$\lim_{t\to 0} \frac{\log \Phi_2(0) - r_2 t + b_{21}\int_0^t \Phi_1(s-\tau)ds - \sigma_2 B_2(t)}{t} = -r_2 + b_{21}\frac{r_1}{b_{11}}.$$

With Lemma 3.4 and Assumption 3.5 we could get

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi_2(s)ds = \frac{-r_2 + b_{21}\frac{r_1}{b_{11}}}{b_{22}} = \frac{r_1 b_{21} - r_2 b_{11}}{b_{11}b_{22}} = M_2 > 0.$$

Let (3.2) divide t, and $t \to \infty$, together with (3.1) and (3.3), consequently

$$\lim_{t\to\infty} \frac{\log \Phi_2(t)}{t} = 0.$$

Similarly, according to Ito's formula, the third population of system (2.2) is changed into

$$d\log \Phi_3(t) = (-r_3 + b_{32}\Phi_2(t-\tau) - b_{33}\Phi_3(t))dt - \sigma_3 dB_3(t),$$

it then follows

$$\log \Phi_3(t) = \log \Phi_3(0) - r_3 t + b_{32}\int_0^t \Phi_2(s-\tau)ds - b_{33}\int_0^t \Phi_3(s)ds - \sigma_3 B_3(t),$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi_3(s)ds = \frac{-r_3 + b_{32}\frac{r_1 b_{21}-r_2 b_{11}}{b_{11}b_{22}}}{b_{33}} = M_3 > 0, \quad \lim_{t\to\infty} \frac{\log \Phi_3(t)}{t} = 0.$$

$\square$

From this, together with Theorem 2.2 and Lemma 3.6, the following result is obviously true.

Theorem 3.7. *If Assumption 3.5 is satisfied, then the solution* $x(t,\xi)$ *of system* (1.3) *has the following property:*

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \leqslant 0, \quad i = 1,2,3.$$

By above all, we could get the following result.

Theorem 3.8. *If Assumption 3.5 is satisfied, then the solution* $x(t,\xi)$ *of system* (1.3) *has the following property*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_3(s)ds \geqslant \tilde{x}_3^*, \quad a.s.,$$

where $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*)$ *is the only nonnegative solution of the following equation,*

$$\begin{cases} r_1 - b_{11}x_1 - b_{12}x_2 = 0, \\ -r_2 + b_{21}x_1 - b_{22}x_2 - b_{23}x_3 = 0, \\ -r_3 + b_{32}x_2 - b_{33}x_3 = 0. \end{cases}$$

Proof. From system (1.3),

$$\frac{\log x_1(t) - \log x_1(0)}{t} = r_1 - b_{11}\frac{1}{t}\int_0^t x_1(s)\,ds - b_{12}\frac{1}{t}\int_0^t x_2(s-\tau)\,ds + \frac{\sigma_1 B_1(t)}{t}$$

$$= r_1 - b_{11}\frac{1}{t}\int_0^t x_1(s)\,ds - b_{12}\frac{1}{t}\left(\int_{-\tau}^0 \xi_2(s)\,ds - \int_{t-\tau}^t x_2(s)\,ds\right)$$

$$- b_{12}\frac{1}{t}\int_0^t x_2(s)\,ds + \frac{\sigma_1 B_1(t)}{t},$$

similarly,

$$\frac{\log x_2(t) - \log x_2(0)}{t} = -r_2 + b_{21}\frac{1}{t}\int_0^t x_1(s)\,ds + b_{21}\left(\frac{1}{t}\int_{-\tau}^0 \xi_1(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_1(s)\,ds\right) - b_{22}\frac{1}{t}\int_0^t x_2(s)\,ds$$

$$- b_{23}\frac{1}{t}\int_0^t x_3(s)\,ds - b_{23}\left(\frac{1}{t}\int_{-\tau}^0 \xi_3(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_3(s)\,ds\right) - \frac{\sigma_2 B_2(t)}{t},$$

and

$$\frac{\log x_3(t) - \log x_3(0)}{t} = -r_3 + b_{32}\frac{1}{t}\int_0^t x_2(s)\,ds + b_{32}\left(\frac{1}{t}\int_{-\tau}^0 \xi_2(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_2(s)\,ds\right)$$

$$- b_{33}\frac{1}{t}\int_0^t x_3(s)\,ds - \frac{\sigma_3 B_3(t)}{t}.$$

Hence

$$\frac{c_1(\log x_1(t) - \log x_1(0)) + c_2(\log x_2(t) - \log x_2(0)) + c_3(\log x_3(t) - \log x_3(0))}{t}$$

$$= (r_1 c_1 - r_2 c_2 - r_3 c_3) + (-b_{11} c_1 + b_{21} c_2)\frac{1}{t}\int_0^t x_1(s)\,ds$$

$$+ (-b_{12} c_1 - b_{22} c_2 + b_{32} c_3)\frac{1}{t}\int_0^t x_2(s)\,ds - (b_{23} c_2 + b_{33} c_3)\frac{1}{t}\int_0^t x_3(s)\,ds$$

$$- c_1 b_{12}\frac{1}{t}\left(\int_{-\tau}^0 \xi_2(s)\,ds - \int_{t-\tau}^t x_2(s)\,ds\right) + c_2 b_{21}\left(\frac{1}{t}\int_{-\tau}^0 \xi_1(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_1(s)\,ds\right) \qquad (3.4)$$

$$- c_2 b_{23}\left(\frac{1}{t}\int_{-\tau}^0 \xi_3(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_3(s)\,ds\right) + c_3 b_{32}\left(\frac{1}{t}\int_{-\tau}^0 \xi_2(s)\,ds - \frac{1}{t}\int_{t-\tau}^t x_2(s)\,ds\right)$$

$$+ \frac{c_1 \sigma_1 B_1(t)}{t} - \frac{c_2 \sigma_2 B_2(t)}{t} - \frac{c_3 \sigma_3 B_3(t)}{t}.$$

From Theorem 2.2, we get

$$x_i(t) \leqslant \Phi_i(t), \qquad (i = 1, 2, 3),$$

then

$$\lim_{t \to \infty} \frac{1}{t}\int_{t-\tau}^t x_i(s)\,ds = 0. \qquad (3.5)$$

Let $c_1 = b_{21}$, $c_2 = b_{11}$, $c_3 = \dfrac{b_{11} b_{22} + b_{12} b_{21}}{b_{32}}$, together with Assumption 3.5, we know

$$r_1 c_1 - r_2 c_2 - r_3 c_3 > 0.$$

According to Theorem 3.7 and equation (3.5), together with $\lim\limits_{t\to\infty} \frac{B_i(t)}{t} = 0$, $(i = 1, 2, 3)$, we could get

$$\limsup_{t\to\infty} \frac{c_1(\log x_1(t) - \log x_1(0)) + c_2(\log x_2(t) - \log x_2(0)) + c_3(\log x_3(t) - \log x_3(0))}{t}$$

$$= (r_1 c_1 - r_2 c_2 - r_3 c_3) - (c_2 b_{23} + c_3 b_{33}) \liminf_{t\to\infty} \frac{1}{t} \int_0^t x_3(s) ds \leqslant 0.$$

Such that,

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_3(s) ds \geqslant \frac{r_1 c_1 - r_2 c_2 - r_3 c_3}{c_2 b_{23} + c_3 b_{33}} = \tilde{x}_3^*,$$

where $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*)$ is the only nonnegative solution of the following equation when Assumption 3.5 is satisfied,

$$\begin{cases} r_1 - b_{11} x_1 - b_{12} x_2 = 0, \\ -r_2 + b_{21} x_1 - b_{22} x_2 - b_{23} x_3 = 0, \\ -r_3 + b_{32} x_2 - b_{33} x_3 = 0. \end{cases}$$

$\square$

4. Non-persistence

In the previous section, we showed the solution $x(t, \xi)$ of system (1.3) is stable in time average, and in this section, we discuss the dynamics of system (1.3) when the white noise is getting larger. We show the situation when the population of system (1.3) will be non-persistent of the white noise is large, which does not happen in the deterministic system in three cases.

Definition 4.1. System (1.3) is said to be non-persistent, if there are positive constants c_i, $(i = 1, 2, 3)$ such that

$$\lim_{t\to\infty} \prod_{i=1}^3 x_i^{c_i}(t) = 0, \quad \text{a.s..}$$

Now we present conditions for all species or some species of (1.3) to be extinct. Consider the following cases.

Case (i): $r_1 < 0$.

According to Itô's formula, the first population of system (2.3) is changed into

$$d \log \Phi_1(t) \leqslant (r_1 - b_{11} \Phi_1(t)) dt - \sigma_1 dB_1(t).$$

If $r_1 < 0$, we could get

$$\limsup_{t\to\infty} \frac{\log \Phi_1(t)}{t} = r_1 < 0 \quad \text{a.s.,}$$

from the stochastic comparison theorem, we have

$$\limsup_{t\to\infty} \frac{\log x_1(t)}{t} < 0 \quad \text{a.s.,}$$

hence

$$\lim_{t\to\infty} x_1(t) = 0, \quad \text{a.s..}$$

From the second population of system (2.3) and equation (3.3), we have

$$\frac{\log \Phi_2(t) - \log \Phi_2(0)}{t} \leqslant -r_2 + b_{21} \frac{1}{t} \int_0^t \Phi_1(s - \tau) ds - \frac{\sigma_2 dB_2(t)}{t} \quad \text{a.s.,} \tag{4.1}$$

hence

$$\limsup_{t \to \infty} \frac{\log \Phi_2(t)}{t} \leqslant -r_2 + b_{21} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \Phi_1(s - \tau) ds$$

$$= -r_2 + b_{21} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \Phi_1(s) ds = -r_2 \leqslant 0, \quad \text{a.s..}$$

Similarly,

$$\limsup_{t \to \infty} \frac{\log \Phi_3(t)}{t} = -r_3 \leqslant 0, \quad \text{a.s.,}$$

and

$$\lim_{t \to \infty} x_i(t) = 0, \quad \text{a.s. } i = 2, 3.$$

Case (ii): $r_1 > 0$, $r_1 - \frac{b_{11}}{b_{21}} r_2 < 0$.

It is clear that from the equations (4.1) and (3.1), we get

$$\limsup_{t \to \infty} \frac{\log \Phi_2(t)}{t} \leqslant -r_2 + b_{21} \frac{r_1}{b_{11}} < 0, \quad \text{a.s..}$$

Similarly

$$\limsup_{t \to \infty} \frac{\log \Phi_3(t)}{t} \leqslant -r_3 + b_{32} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \Phi_2(s) ds = -r_3 < 0, \quad \text{a.s.,}$$

thus,

$$\lim_{t \to \infty} x_i(t) = 0, \quad \text{a.s., } i = 2, 3.$$

By above all, and from the conclusion in [14], we could easily know that the distribution of $x_1(t)$ converges weekly to the probability measure with density

$$f^*(\zeta) = C_0 \zeta^{2r_1/\sigma_1^2 - 1} e^{-2b_{11}\zeta/\sigma_1^2},$$

where $C_0 = (2b_{11}/\sigma_1^2)^{2r_1/\sigma_1^2}/\Gamma(2r_1/\sigma_1^2)$, and

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t x_1(s) ds = \frac{r_1}{b_{11}}, \quad \text{a.s..}$$

Case (iii): $r_1 - \frac{b_{11}}{b_{21}} r_2 - \frac{b_{11}b_{22} + b_{12}b_{21}}{b_{21}b_{32}} r_3 < 0$.

It is clear that from (3.4), letting $c_1 = b_{21}$, $c_2 = b_{11}$, $c_3 = \frac{b_{11}b_{22} + b_{12}b_{21}}{b_{32}}$, with $\lim_{t \to \infty} \frac{B_i(t)}{t} = 0, i = 1, 2, 3$, we get

$$\limsup_{t \to \infty} \frac{c_1(\log x_1(t) - \log x_1(0)) + c_2(\log x_2(t) - \log x_2(0)) + c_3(\log x_3(t) - \log x_3(0))}{t}$$

$$= (r_1 c_1 - r_2 c_2 - r_3 c_3) - (c_2 b_{23} + c_3 b_{33}) \liminf_{t \to \infty} \frac{1}{t} \int_0^t x_3(s) ds$$

$$\leqslant r_1 c_1 - r_2 c_2 - r_3 c_3,$$

moreover,

$$\limsup_{t\to\infty} \frac{\log x_1^{c_1}(t)x_2^{c_2}(t)x_3^{c_3}(t)}{t} \leqslant r_1 c_1 - r_2 c_2 - r_3 c_3 < 0,$$

then

$$\lim_{t\to\infty} x_1^{c_1}(t)x_2^{c_2}(t)x_3^{c_3}(t) = 0, \quad a.s..$$

Therefore, by the above arguments, we get the following conclusion.

Theorem 4.2. *Let $x(t, \xi)$ be the solution of system (1.3), the following conclusions are founded.*

(1) *If $r_1 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0, \quad a.s., \quad i = 1, 2, 3.$$

(2) *If $r_1 > 0$, $r_1 - \frac{b_{11}}{b_{21}}r_2 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0, \quad a.s., \quad i = 2, 3,$$

and the distribution of $x_1(t)$ converges weekly to the probability measure with density

$$f^*(\zeta) = C_0 \zeta^{2r_1/\sigma_1^2 - 1} e^{-2b_{11}\zeta/\sigma_1^2},$$

where $C_0 = (2b_{11}/\sigma_1^2)^{2r_1/\sigma_1^2}/\Gamma(2r_1/\sigma_1^2)$, and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = \frac{r_1}{b_{11}}, \quad a.s..$$

(3) *If $r_1 - \dfrac{b_{11}}{b_{21}}r_2 - \dfrac{b_{11}b_{22} + b_{12}b_{21}}{b_{21}b_{32}}r_3 < 0$, then*

$$\lim_{t\to\infty} x_1^{c_1}(t)x_2^{c_2}(t)x_3^{c_3}(t) = 0, \quad a.s.,$$

where $c_1 = b_{21}$, $c_2 = b_{11}$, $c_3 = \dfrac{b_{11}b_{22} + b_{12}b_{21}}{b_{32}}$.

That is to say, the large white noise will lead to the population system non-persistent.

5. Numerical simulation

In this section, we give out the numerical experiment to support our results. Consider the equation

$$\begin{cases} dx_1(t) = x_1(t)\left(a_1 - b_{11}x_1(t) - b_{12}x_2(t-\tau)\right)dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left(-a_2 + b_{21}x_1(t-\tau) - b_{22}x_2(t) - b_{23}x_3(t-\tau)\right)dt - \sigma_2 x_2(t)dB_2(t), \\ dx_3(t) = x_3(t)\left(-a_3 + b_{32}x_2(t-\tau) - b_{33}x_3(t)\right)dt - \sigma_3 x_3(t)dB_3(t). \end{cases}$$

By the method in [11], we have the difference equation

$$\begin{cases} x_{1,k+1} = x_{1,k} + x_{1,k}[(a_1 - b_{11}x_{1,k} - b_{12}x_{2,k-m})\Delta t + \sigma_1\epsilon_{1,k}\sqrt{\Delta t} + \frac{\sigma_1^2}{2}(\epsilon_{1,k}^2\Delta t - \Delta t)], \\[2mm] x_{2,k+1} = x_{2,k} + x_{2,k}[(-a_2 + b_{21}x_{1,k-m} - b_{22}x_{2,k} - b_{23}x_{3,k-m})\Delta t - \sigma_2\epsilon_{2,k}\sqrt{\Delta t} + \frac{\sigma_2^2}{2}(\epsilon_{2,k}^2\Delta t - \Delta t)], \\[2mm] x_{3,k+1} = x_{3,k} + x_{3,k}[(-a_3 + b_{32}x_{2,k-m} - b_{33}x_{3,k})\Delta t - \sigma_3\epsilon_{3,k}\sqrt{\Delta t} + \frac{\sigma_3^2}{2}(\epsilon_{3,k}^2\Delta t - \Delta t)], \end{cases}$$

where $\epsilon_{1,k}, \epsilon_{2,k}$ and $\epsilon_{3,k}, i = 1,2,3$ are the Gaussian random variables $N(0,1)$, $r_1 = a_1 - \frac{\sigma_1^2}{2} > 0$, $r_i = a_i + \frac{\sigma_i^2}{2}$, $(i = 2,3)$, and m represents the integer part $\tau / \triangle t - 1$. Choose $(x_1(0), x_2(0), x_3(0)) \in R_+^3$, $t \in [-\tau, 0]$, and suitable parameters, by Matlab, we get Figs. 1, 2, and 3.
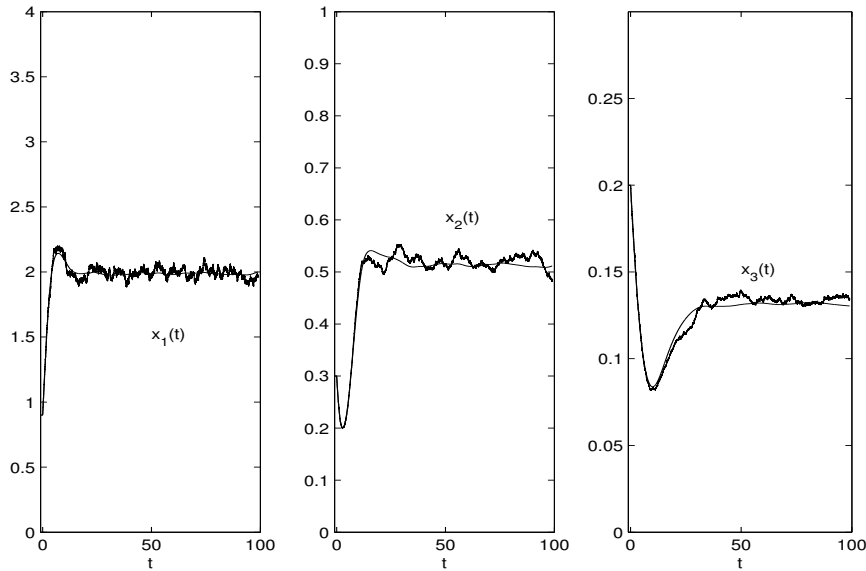

Figure 1: The solutions of system (1.2) and system (1.3) with $(x_1(0), x_2(0), x_3(0)) = (0.9, 0.3, 0.2)$, $t \in [-\tau, 0]$, $a_1 = 0.7$, $a_2 = 0.3$, $a_3 = 0.1$, $b_{11} = 0.3$, $b_{12} = 0.2$, $b_{21} = 0.3$, $b_{22} = 0.5$, $b_{23} = 0.3$, $b_{32} = 0.4$, $b_{33} = 0.8$. The light lines represent the solution of system (1.2), while the dark lines represent the solution of system (1.3) with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $\sigma_3 = 0.01$.
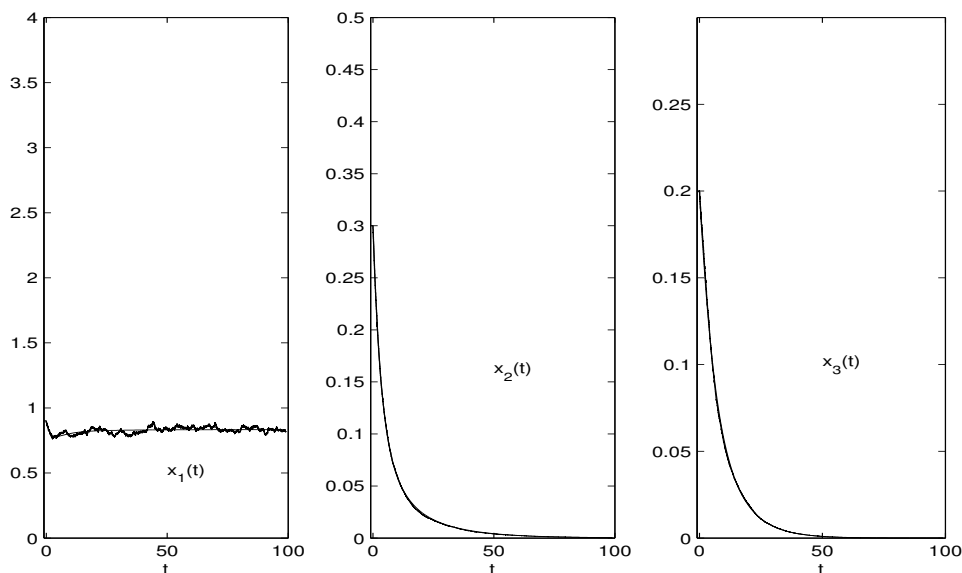

Figure 2: Two of the species will die out in probability. The solutions of system (1.2) and system (1.3) with $(x_1(0), x_2(0), x_3(0)) = (0.9, 0.3, 0.2)$, $t \in [-\tau, 0]$, $a_1 = 0.5$, $a_2 = 0.3$, $a_3 = 0.1$, $b_{11} = 0.6$, $b_{12} = 0.2$, $b_{21} = 0.3$, $b_{22} = 0.5$, $b_{23} = 0.3$, $b_{32} = 0.4$, $b_{33} = 0.8$. The light lines represent the solution of system (1.2), while the dark lines represent the solution of system (1.3) with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $\sigma_3 = 0.01$.
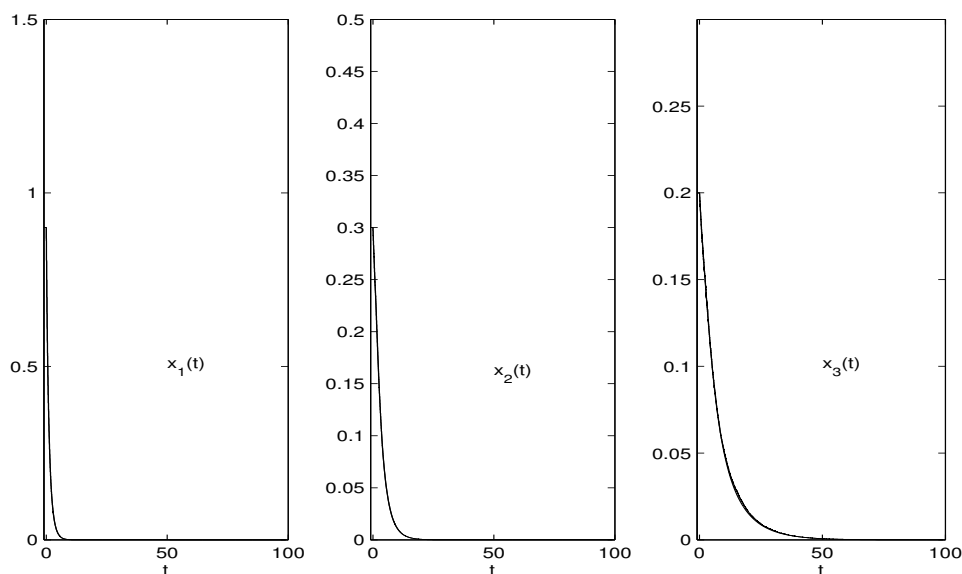
Figure 3: One of the species or both species will die out in probability. The solutions of system (1.2) and system (1.3) with $(x_1(0), x_2(0), x_3(0)) = (0.9, 0.3, 0.2)$, $t \in [-\tau, 0]$, $a_1 = -0.7$ $a_2 = 0.3$, $a_3 = 0.1$, $b_{11} = 0.3$, $b_{12} = 0.2$, $b_{21} = 0.3$, $b_{22} = 0.5$, $b_{23} = 0.3$, $b_{32} = 0.4$, $b_{33} = 0.8$. The light lines represent the solution of system (1.2), while the dark lines represent the solution of system (1.3) with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $\sigma_3 = 0.01$.

In Fig. 1, when the noise is small, choose parameters satisfying the condition of Theorem 3.7, then the solution of system (1.2) will persist in time average.

In Fig. 2, we observe Case (iii) in Theorem 4.2 and choose parameters $r_1 > 0$, $r_1 - \frac{b_{11}}{b_{21}} r_2 < 0$. As Theorem 4.2 indicated that two predators will die out in probability, then the prey solution of system (1.2) will persist in time average.

In Fig. 3, we observe Case (i) in Theorem 4.2 and choose parameters $r_1 < 0$. As Theorem 4.2 indicated that not only predators but also prey will die out in probability when the noise of the prey is large, and it does not happen in the deterministic system, these simulated results are consistent with our theorems.

Acknowledgment

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