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Norm inequalities of operators and commutators on generalized weighted morrey spaces

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Abstract

We prove that, if a class of operators, which includes singular integral operator with rough kernel, Bochner-Riesz operator and Marcinkiewicz integral operator, are bounded on weighted Lebesgue spaces and satisfy some local pointwise control, then these operators and associated commutators, formed by a BMO function and these operators, are also bounded on generalized weighted Morrey spaces.

Keywords: Singular integral with rough kernel, Bochner-Riesz operator, Marcinkiewicz integral, commutator, weighted Morrey space.
2010 MSC: 42B35, 47B38, 30H35.

1. Introduction and results

The classical Morrey space was introduced by Morrey [10] in 1938. It plays an important role in the theory of partial differential equations. Morrey space is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p}\|f\|_{L^p(B(x,r))} < \infty. \tag{1.1}$$

Note that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Let $\Phi(r), r > 0$ be a growth function, that is, a positive increasing function in $(0, \infty)$, which satisfies doubling condition

$$\Phi(2r) \leqslant D\Phi(r), \quad \forall\, r > 0,$$

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where $D = D(\Phi) \geqslant 1$ is a doubling constant independent of r. In [9] Mizuhara gave a generalization Morrey space $L^{p,\Phi}(\mathbb{R}^n)$ considering $\Phi(r)$ instead of r^λ in (1.1).

Komori and Shirai [8] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega,\mathbb{R}^n)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega,\mathbb{R}^n)$.

Let $1 \leqslant p < \infty$, $0 < \kappa < 1$ and ω be a weight function. Then the space $L^{p,\kappa}(\omega,\mathbb{R}^n)$ is defined by

$$L^{p,\kappa}(\omega,\mathbb{R}^n) = \{f \in L^p_{loc}(\omega) : \|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\omega(B(x,r))^\kappa}\int_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}}.$$

Let $1 \leqslant p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and ω be a non-negative measurable function on $\mathbb{R}^n$. We denote by $M^p_\varphi(\omega,\mathbb{R}^n)$ the generalized weighted Morrey space, the space of all functions $f \in L^p_{loc}(\omega)$ with finite norm

$$\|f\|_{M^p_\varphi(w,\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)}\left(\frac{1}{w(B(x,r))}\|f\|^p_{L^p(\omega,B(x,r))}\right)^{1/p},$$

where

$$\|f\|_{L^p(\omega,B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy\right)^{1/p}.$$

If $\omega = 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 \leqslant \lambda \leqslant n$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space. If $\varphi(x,r) = \omega(B(x,r))^{\frac{\kappa-1}{p}}$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\kappa}(\omega,\mathbb{R}^n)$ is the weighted Morrey space.

In this paper, we prove that, if a class of operators are bounded on weighted Lebesgue space and satisfy some local pointwise control, then these operators and associated commutators, formed by a BMO function and these operators, are also bounded on generalized weighted Morrey space. Our main results can be formulated as follows.

Theorem 1.1. *Let $1 \leqslant s' \leqslant p < \infty, \omega \in A_{p/s'}$ and T be a sublinear operator which satisfies*

$$\sup_{x\in B(x_0,l)} \left|T\left(f\chi_{(B(x_0,2l)^c)}\right)(x)\right| \leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|B(x_0,2^{j+1}l)|}\int_{B(x_0,2^{j+1}l)} |f(z)|^{s'} dz\right)^{1/s'}, \tag{1.2}$$

for any $x_0 \in \mathbb{R}^n$ and $l > 0$.

(i) *Suppose (φ_1,φ_2) satisfies the condition*

$$\int_l^\infty \frac{\operatorname{ess\,inf}_{r<t<\infty} \varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r} \leqslant c_0\varphi_2(x_0,l), \tag{1.3}$$

where c_0 does not depend on x and r. If T is bounded on $L^p(\omega,\mathbb{R}^n)$ for $p > 1$, then T is also bounded from $M^p_{\varphi_1}(\omega,\mathbb{R}^n)$ to $M^p_{\varphi_2}(\omega,\mathbb{R}^n)$ and

$$\|Tf\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

(ii) *Suppose (φ_1,φ_2) satisfies the condition*

$$\int_l^\infty \left(1+\ln\frac{r}{l}\right)\frac{\operatorname{ess\,inf}_{r<t<\infty} \varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r} \leqslant c_0\varphi_2(x_0,l), \tag{1.4}$$

where c_0 does not depend on x_0 and l. If $b \in BMO(\mathbb{R}^n)$, and $[b, T]$ is bounded on $L^p(\omega, \mathbb{R}^n)$ for $1 < p < \infty$, then $[b, T]$ is also bounded from $M^p_{\varphi_1}(\omega, \mathbb{R}^n)$ to $M^p_{\varphi_2}(\omega, \mathbb{R}^n)$ and

$$\|[b, T]f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

Remark 1.2. Let $\varphi_1(x, t) = (\Phi(t)t^{-n})^{\frac{1}{p}}$, let $\varphi_2(x, t) = (\Phi(t))^{\frac{1}{p}} t^{\frac{n}{q}}$, and let $\omega = 1$. If $1 \leqslant D(\Phi) \leqslant 2^n$, it is easy to prove (φ_1, φ_2) satisfies the conditions (1.3) and (1.4).

Remark 1.3. Let $\varphi_1(x_0, t) = \varphi_2(x_0, t) = \omega(B(x_0, t))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$, and $w \in A_\infty(\mathbb{R}^n)$, then (φ_1, φ_2) satisfies the conditions (1.3) and (1.4).

Then we have the following corollaries.

Corollary 1.4. *Let $1 \leqslant s' \leqslant p < \infty$, $\omega \in A_{p/s'}$, $1 \leqslant D(\Phi) \leqslant 2^n$, and let T be a sublinear operator which satisfies (1.4) for any $x_0 \in \mathbb{R}^n$ and $l > 0$. If T is bounded on $L^p(\omega, \mathbb{R}^n)$ for $p > 1$, then T is bounded on $L^{p,\Phi}(\mathbb{R}^n)$. If $b \in BMO(\mathbb{R}^n)$, and $[b, T]$ is bounded on $L^p(\omega, \mathbb{R}^n)$ for $p > 1$, then $[b, T]$ is also bounded on $L^{p,\Phi}(\mathbb{R}^n)$.*

Corollary 1.5. *Let $1 \leqslant s' \leqslant p < \infty$, $\omega \in A_{p/s'}$, $0 < \alpha < n$, and let T be a sublinear operator which satisfies (1.4) for any $x_0 \in \mathbb{R}^n$ and $l > 0$. If T is bounded on $L^p(\omega, \mathbb{R}^n)$ for $p > 1$, then T is bounded on $L^{p,\kappa}(\omega, \mathbb{R}^n)$. If $b \in BMO(\mathbb{R}^n)$, and $[b, T]$ is bounded on $L^p(\omega, \mathbb{R}^n)$ for $p > 1$, then $[b, T]$ is also bounded on $L^{p,\kappa}(\omega, \mathbb{R}^n)$.*

This paper is organized as follows. Section 2 is devoted to prove some preliminary results. In Section 3, we prove our main result and in Section 4 we give some applications to our main theorem.

2. Some preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B and let $\lambda B = B(x_0, \lambda r_B)$. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure by $\omega(E) = \int_E \omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $X \subsetneq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L^p(\omega, X)$ the space of all functions f satisfying

$$\|f\|_{L^p(\omega,X)} = \left(\int_X |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

A weight ω is said to belong to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} \leqslant C, \tag{2.1}$$

where p' is the dual of p such that $1/p + 1/p' = 1$. The class $A_1(\mathbb{R}^n)$ is defined by

$$\frac{1}{|B|}\int_B w(y)dy \leqslant C \cdot \operatorname*{ess\,inf}_{x\in B} w(x), \quad \text{for every ball } B \subset \mathbb{R}^n.$$

By (2.1), we have

$$\left(\omega^{-\frac{p'}{p}}(B)\right)^{\frac{1}{p'}} = \|\omega^{-\frac{1}{p}}\|_{L^{p'}(B)} \leqslant C|B|(\omega(B))^{-\frac{1}{p}}, \tag{2.2}$$

for $1 < p < \infty$.

Suppose $\omega \in A_p(\mathbb{R}^n), 1 < p < \infty$, by the definition of $A_p(\mathbb{R}^n)$, we know that $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.

The classical $A_p(\mathbb{R}^n)$ weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [11].

Following [7], a locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B|b(x)-b_B|dx=\|b\|_*<\infty,$$

where

$$b_B=\frac{1}{|B|}\int_B b(y)dy.$$

Lemma 2.1 ([6]). *Suppose $\omega\in A_\infty(\mathbb{R}^n)$ and $b\in BMO(\mathbb{R}^n)$. Then for any $1\leqslant p<\infty$ and $r_1,r_2>0$, we have*

$$\left(\frac{1}{\omega(B(x_0,r_1))}\int_{B(x_0,r_1)}|b(x)-b_{B(x_0,r_2)}|^p\omega(x)dx\right)^{1/p}\leqslant C\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*.$$

In order to prove Theorem 1.1, we need to prove the following lemmas.

Lemma 2.2. *Suppose that $1\leqslant s'\leqslant p<\infty$, $p>1$, and $\omega\in A_{p/s'}(\mathbb{R}^n)$. If T is bounded on $L^p(\omega,\mathbb{R}^n)$ and satisfies (1.2), then for any $l>0$, there is a constant C independent of f such that*

$$\|T(f)\|_{L^p(\omega,B(x_0,l))}\leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}. \tag{2.3}$$

Proof. We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2l)}(y)$, $\chi_{B(x_0,2l)}$ denotes the characteristic function of $B(x_0,2l)$. Then

$$\|T(f)\|_{L^p(\omega,B(x_0,l))}\leqslant\|T(f_1)\|_{L^p(\omega,B(x_0,l))}+\|T(f_2)\|_{L^p(\omega,B(x_0,l))}.$$

Since $f_1\in L^p(\omega,\mathbb{R}^n)$, from the boundedness of T on $L^p(\omega,\mathbb{R}^n)(p>1)$ it follows that

$$\begin{aligned}\|T(f_1)\|_{L^p(\omega,B(x_0,l))}&\leqslant\|T(f_1)\|_{L^p(\omega,\mathbb{R}^n)}\\&\leqslant C\|f_1\|_{L^p(\omega,\mathbb{R}^n)}\\&=C\|f\|_{L^p(\omega,B(x_0,2l))}.\end{aligned}$$

By Hölder's inequality,

$$|B(x_0,l)|\leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\|\omega^{-\frac{1}{p}}\|_{L^{p'}(B(x_0,l))}.$$

Then, for any $p>1$,

$$\begin{aligned}\|f\|_{L^p(\omega,B(x_0,2l))}&\leqslant C|B(x_0,l)|\|f\|_{L^p(\omega,B(x_0,2l))}\int_{2l}^{\infty}\frac{dr}{r^{n+1}}\\&\leqslant C|B(x_0,l)|\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\frac{dr}{r^{n+1}}\\&\leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\|\omega^{-\frac{1}{p}}\|_{L^{p'}(B(x_0,l))}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\frac{dr}{r^{n+1}}\\&\leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r)}\|\omega^{-\frac{1}{p}}\|_{L^{p'}(B(x_0,r))}\frac{dr}{r^{n+1}}.\end{aligned}$$

Then, by (2.2) we get

$$\|T(f_1)\|_{L^p(\omega,B(x_0,l))}\leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}. \tag{2.4}$$

When $1 \leqslant s' < p < \infty$, set $\nu = p/s' > 1$. Since T satisfies (1.2), it follows from Hölder's inequality that

$$\begin{aligned}\sup_{x\in B(x_0,l)} |T(f_2)(x)| &\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|B(x_0,2^{j+1}l)|}\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}}\|f\|_{L^p(\omega,B(x_0,2^{j+1}l))}\|\omega^{-\frac{1}{p}}\|_{L^{s'\nu'}(B(x_0,2^{j+1}l))}\\ &\leqslant C\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}(2^{j+1}l)^{-(1+\frac{n}{s'})}\|f\|_{L^p(\omega,B(x_0,r))}\|\omega^{-\frac{1}{p}}\|_{L^{s'\nu'}(B(x_0,r))}dr\\ &\leqslant C\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\|\omega^{-\frac{1}{p}}\|_{L^{s'\nu'}(B(x_0,r))}\frac{dr}{r^{1+n/s'}}.\end{aligned}$$

Note that $\omega \in A_\nu$, by (2.2) we get

$$\|\omega^{-\frac{1}{p}}\|_{L^{s'\nu'}(B(x_0,r))} \leqslant Cr^{\frac{n}{s'}}\omega(B(x_0,r))^{-\frac{1}{p}}.$$

Then

$$\sup_{x\in B(x_0,l)}|T(f_2)(x)| \leqslant C\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}. \tag{2.5}$$

When $s' = p$, then $\omega \in A_1$. Then for any $p > 1$,

$$\begin{aligned}\sup_{x\in B(x_0,l)}|T(f_2)(x)| &\leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{p}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^p dy\right)^{\frac{1}{p}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{p}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^p w(x)dy\right)^{\frac{1}{p}}\left(\operatorname*{ess\,inf}_{x\in B(x_0,2^{j+1}l)} w(x)\right)^{-\frac{1}{p}}\\ &\leqslant C\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}\|f\|_{L^p(\omega,B(x_0,2^{j+1}l))}\omega(B(x_0,2^{j+1}l))^{-\frac{1}{p}}\frac{dr}{r}\\ &\leqslant C\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}\\ &\leqslant C\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.\end{aligned} \tag{2.6}$$

By (2.5) and (2.6) we get

$$\|T(f_2)\|_{L^p(\omega,B(x_0,l))} \leqslant C\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r)}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r^{n+1}}. \tag{2.7}$$

Combining (2.4) and (2.7), we complete the proof of Lemma 2.2. □

Lemma 2.3. *Suppose that* $1 \leqslant s' \leqslant p < \infty$, $p > 1$, *and* $\omega \in A_{p/s'}(\mathbb{R}^n)$. *If* T *satisfies* (1.2) *and* $[b,T]$ *is bounded on* $L^p(\omega,\mathbb{R}^n)$, *then for any* $l > 0$, *there is a constant* C *independent of* f *such that*

$$\|[b,T](f)\|_{L^p(\omega,B(x_0,l))} \leqslant C\|b\|_*\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}. \tag{2.8}$$

Proof. We represent f as

$$f(y) = f_1(y) + f_2(y), \quad f_1(y) = f(y)\chi_{B(x_0,2l)}(y).$$

Then

$$\|[b,T](f)\|_{L^p(\omega,B(x_0,l))} \leqslant \|[b,T](f_1)\|_{L^p(\omega,B(x_0,l))} + \|[b,T](f_2)\|_{L^p(\omega,B(x_0,l))}.$$

Since $[b,T]$ is bounded on $L^p(\omega,\mathbb{R}^n)$, as the proof of (2.4) we get

$$\begin{aligned}\|[b,T](f_1)\|_{L^p(\omega,B(x_0,l))} &\leqslant C\|b\|_*\|f\|_{L^p(\omega,B(x_0,2l))}\\ &\leqslant C\|b\|_*\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.\end{aligned}$$

We now turn to deal with the term $\|[b,T](f_2)\|_{L^p(\omega,B(x_0,l))}$. For any given $x\in B(x_0,l)$, we have

$$\begin{aligned}|[b,T](f_2)(x)| &\leqslant C|b(x)-b_{B(x_0,l)}||T(f_2)(x)|+C|T((b-b_{B(x_0,l)})f_2)(x)|\\ &= I_1+I_2.\end{aligned}$$

Since T satisfies (1.2), by (2.5) and (2.6),

$$I_1\leqslant C|b(y)-b_{B(x_0,l)}|\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.$$

Applying Lemma 2.1 we get

$$\|I_1\|_{L^p(\omega,B(x_0,l))}\leqslant C\|b\|_*\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.$$

On the other hand, it follows from (1.2) that

$$I_2\leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}l)}|(b(y)-b_{B(x_0,l)})f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Set $\nu=p/s'$. From $\omega\in A_\nu$ we know $\omega^{1-\nu'}\in A_{\nu'}$. By Hölder's inequality

$$\left(\int_{B(x_0,2^{j+1}s)}|b(y)-b_{B(x_0,l)}|^{q'}|f(y)|^{q'}dy\right)^{\frac{1}{q'}}\leqslant C\|f\|_{L^p(\omega,B(x_0,2^{j+1}l))}\left\|b(\cdot)-b_{B(x_0,l)}\right\|_{L^{s'\nu'}(\omega^{1-\nu'},B(x_0,2^{j+1}l))}.$$

Consequently,

$$\begin{aligned}I_2 &\leqslant \sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}(2^{j+1}l)^{-(1+\frac{n}{s'})}\|f\|_{L^p(\omega,B(x_0,2^{j+1}l))}\left\|b(\cdot)-b_{B(x_0,l)}\right\|_{L^{s\nu'}(\omega^{1-\nu'},B(x_0,2^{j+1}l))}dr\\ &\leqslant C\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\left\|b(\cdot)-b_{B(x_0,l)}\right\|_{L^{s'\nu'}(\omega^{1-\nu'},r))}\frac{dr}{r^{1+n/s'}}.\end{aligned}$$

Since $\omega^{-\frac{\nu'}{s'}}=\omega^{1-\nu'}\in A_{\nu'}$, we get

$$(\omega^{1-\nu'}B(x_0,r))^{\frac{1}{s'\nu'}}\leqslant Cr^{\frac{n}{\nu}}((B(x_0,r))^{-\frac{1}{p}}.$$

By Lemma 2.1 and the fact that $\omega\in A_\nu$, we obtain

$$\begin{aligned}\left(\int_{B(x_0,2^{j+1}l)}|b(y)-b_{B(x_0,l)}|^{s'\nu'}\omega^{1-\nu'}(y)dy\right)^{\frac{1}{s'\nu'}} &\leqslant C\|b\|_*\left(1+\ln\frac{r}{l}\right)(\omega^{1-\nu'}(B(x_0,r)))^{\frac{1}{s\nu'}}\\ &\leqslant C\|b\|_*r^{\frac{n}{s'}}\left(1+\ln\frac{r}{l}\right)\omega(B(x_0,r))^{-\frac{1}{p}}.\end{aligned}$$

Then

$$I_2\leqslant C\|b\|_*\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.$$

Therefore

$$\|I_2\|_{L^p(\omega,B(x_0,l))}\leqslant C\|b\|_*\omega(B(x_0,l))^{\frac{1}{p}}\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}.$$

□

3. Proof of Theorem 1.1

Proof. For $f \in M^p_{\varphi_1}(w, \mathbb{R}^n)$, from the fact $\|f\|_{L^p(\omega,B(x_0,r))}$ is a non-decreasing function of r, and

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)},$$

holds for any real-valued nonnegative function f and measurable on E ([14, p.143]), we get

$$\begin{aligned}
\frac{\|f\|_{L^p(\omega,B(x_0,r))}}{\operatorname*{ess\,inf}_{0<r<t<\infty} \varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}} &\leqslant \operatorname*{ess\,sup}_{0<r<t<\infty} \frac{\|f\|_{L^p(\omega,B(x_0,r))}}{\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}\\
&\leqslant \operatorname*{ess\,sup}_{t>0,x_0\in\mathbb{R}^n} \frac{\|f\|_{L^p(\omega,B(x_0,t))}}{\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}\\
&\leqslant \|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}.
\end{aligned}$$

Since $p > 1$, and (φ_1, φ_2) satisfies (1.3), we have

$$\begin{aligned}
&\int_l^\infty \|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}\\
&\leqslant \int_l^\infty \frac{\|f\|_{L^p(\omega,B(x_0,r))}}{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}} \frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r}\\
&\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}\int_l^\infty \frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r}\\
&\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}\varphi_2(x_0,l).
\end{aligned}$$

Then by (2.3) we get

$$\begin{aligned}
\|T(f)\|_{M^p_{\varphi_2}(w,\mathbb{R}^n)} &\leqslant C \sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\left(\frac{1}{w(B(x_0,l))}\int_{B(x_0,l)}|T(f)(y)|^p w(y)dy\right)^{1/p}\\
&\leqslant C \sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\int_l^\infty \|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}\\
&\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}.
\end{aligned}$$

When $f \in M^p_{\varphi_1}(w, \mathbb{R}^n)$ and (φ_1, φ_2) satisfies (1.4), then for $p > 1$, we have

$$\begin{aligned}
&\int_l^\infty \left(1+\ln\frac{r}{l}\right)\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}\\
&\leqslant \int_l^\infty \left(1+\ln\frac{r}{l}\right)\frac{\|f\|_{L^p(\omega,B(x_0,r))}}{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}} \frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r}\\
&\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}\int_l^\infty \left(1+\ln\frac{r}{l}\right)\frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x_0,t)\omega(B(x_0,t))^{\frac{1}{p}}}{\omega(B(x_0,r))^{\frac{1}{p}}}\frac{dr}{r}\\
&\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}\varphi_2(x_0,l).
\end{aligned}$$

By (2.8) we get

$$\begin{aligned}\|[b,T](f)\|_{M^p_{\varphi_2}(w,\mathbb{R}^n)} &\leqslant C \sup_{x_0\in\mathbb{R}^n, l>0} \frac{1}{\varphi_2(x_0,l)}\left(\frac{1}{w(B(x_0,l))}\int_{B(x_0,l)} |[b,T_\Omega](f)(y)|^p w(y)dy\right)^{1/p}\\ &\leqslant C \sup_{x_0\in\mathbb{R}^n, l>0} \frac{1}{\varphi_2(x_0,l)}\int_s^\infty \left(1+\ln\frac{r}{l}\right)\|f\|_{L^p(\omega,B(x_0,r))}\omega(B(x_0,r))^{-\frac{1}{p}}\frac{dr}{r}\\ &\leqslant C\|f\|_{M^p_{\varphi_1}(w,\mathbb{R}^n)}.\end{aligned}$$

□

4. Some applications

In this section, we shall apply Theorem 1.1 to several particular operators such as singular integral operators with rough kernel, Bochner-Riesz operators and Marcinkiewicz integral operators.

4.1. Singular integral operators with rough kernels

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n (n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(S^{n-1})$ with $1 < s < \infty$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y)dy.$$

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and T_Ω is defined by

$$[b, T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x).$$

Following [5], there is a constant C independent of f such that

$$\|T_\Omega f\|_{L^p(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\omega,\mathbb{R}^n)},$$

for every $s' \leqslant p < \infty$ and $\omega \in A_{p/s'}$. By the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez et al. (see [1]), we see that

$$\|[b, T_\Omega]f\|_{L^p(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{L^p(\omega,\mathbb{R}^n)},$$

holds for all $b \in BMO, s' \leqslant p < \infty$ and $\omega \in A_{p/s'}$.

Theorem 4.1. *Suppose $\Omega \in L^s(S^{n-1})$ with $1 < s < \infty$. Let $s' \leqslant p < \infty, \omega \in A_{p/s'}$ and $b \in BMO(\mathbb{R}^n)$. If (φ_1, φ_2) satisfies the condition (1.3), then there is a constant $C > 0$ independent of f such that*

$$\|T_\Omega f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

If (φ_1, φ_2) satisfies the condition (1.4), then there is a constant $C > 0$ independent of f such that

$$\|[b, T_\Omega]f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

Proof. We only need to prove T_Ω satisfies (1.2). By Hölder's inequality,

$$\begin{aligned}\sup_{x\in B(x_0,l)}\left|T_\Omega\left(f\chi_{(B(x_0,2l)^c}\right)(x)\right| &\leqslant \sup_{x\in B(x_0,l)}\left|\int_{B(x_0,2l)^c}\frac{\Omega((x-y)')}{|x-y|^n}f(y)dy\right|\\ &\leqslant \sup_{x\in B(x_0,l)}\sum_{j=1}^{\infty}\left(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}|\Omega((x-y)')|^s dy\right)^{\frac{1}{s}}\\ &\quad\times\left(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}\frac{|f(y)|^{s'}}{|x-y|^{ns'}}dy\right)^{\frac{1}{s'}}.\end{aligned}$$

When $x\in B(x_0,l)$ and $y\in B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)$, by a direct calculation, we can see that $2^{j-1}l\leqslant|y-x|<2^{j+1}l$. Hence

$$\left(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}|\Omega((x-y)')|^s dy\right)^{\frac{1}{s}}\leqslant C\|\Omega\|_{L^s(S^{n-1})}|B(x_0,2^{j+1}l)|^{\frac{1}{s}}. \tag{4.1}$$

We also note that if $x\in B(x_0,l), y\in B(x_0,2l)^c$, then $|y-x|\sim|y-x_0|$. Consequently

$$\left(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}\frac{|f(y)|^{s'}}{|x-y|^{ns'}}dy\right)^{\frac{1}{s'}}\leqslant\frac{1}{|B(x_0,2^{j+1}l)|}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}. \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\sup_{x\in B(x_0,l)}\left|T_\Omega\left(f\chi_{(B(x_0,2l)^c}\right)(x)\right|\leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

□

4.2. Bochner-Riesz operators

Bochner-Riesz operators were first introduced by Bochner [2] in connection with summation of multiple Fourier series and played an important role in harmonic analysis. The Bochner-Riesz operators of order $\delta>0$ in $\mathbb{R}^n(n\geqslant 2)$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$(T_R^\delta f)^\wedge(\xi)=\left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta\hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of f. These operators can be expressed as convolution operators by the formula

$$T_R^\delta f(x)=(f*\phi_{1/R})(x),$$

where $\phi_{1/R}(x)=R^n f(Rx)$, and for all $\delta\geqslant(n-1)/2$,

$$|\phi(x)|\leqslant\frac{C}{(1+|x|)^{\frac{n+1}{2}+\delta}}. \tag{4.3}$$

The associated maximal Bochner-Riesz operator is defined by

$$T_*^\delta(f)(x)=\sup_{R>0}|T_R^\delta f(x)|.$$

When $\delta>(n-1)/2$, it is well-known that ([13])

$$T_*^\delta(f)(x)\leqslant CM(f)(x).$$

By [12], if $\delta = (n-1)/2$ and $\omega \in A_p$, then there exists a constant $C > 0$ such that

$$\|T_*^\delta(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}, \qquad \text{for } 1 < p < \infty.$$

Then, by the boundedness of maximal function $M(f)$ on $L^p(\omega)$, we know that if $\omega \in A_p$ $(1 < p < \infty)$, then for all $\delta \geqslant (n-1)/2$,

$$\|T_*^\delta(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}$$

holds.

Let b be a locally integrable function on $\mathbb{R}^n$, for any given $R > 0$, the commutator of b and T_R^δ is defined as follows

$$[b, T_R^\delta]f(x) = b(x)T_R^\delta f(x) - T_R^\delta(Tf)(x).$$

Note that $T_R^\delta f(x) \leqslant T_*^\delta(f)(x)$, then, if $\omega \in A_p(1 < p < \infty)$, the equality

$$\|T_R^\delta(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}$$

holds for all $\delta \geqslant (n-1)/2$. Therefore, by the boundedness criterion for the commutators of linear operators, we see that if $b \in BMO$, then $[b, T_R^\delta]$ is also bounded on $L^p(\omega)$ for all $1 < p < \infty$ and $\omega \in A_p$.

Theorem 4.2. *Suppose $\delta \geqslant (n-1)/2$ and $1 < p < \infty$. Let $b \in BMO$, and let $\omega \in A_p$. If (φ_1, φ_2) satisfies the condition (1.3), then there is a constant $C > 0$ independent of f such that*

$$\|T_*^\delta f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

If (φ_1, φ_2) satisfies the condition (1.4), then there is a constant $C > 0$ independent of f such that

$$\|[b, T_R^\delta]f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

Proof. Note that when $\delta \geqslant (n-1)/2$, then by the estimate (4.3), we have

$$|\phi(x)| \leqslant \frac{C}{|x|^n}.$$

We also observe that when $x \in B(x_0, l), y \in (B(x_0, 2l))^c$, then $|x-y| \sim |x-x_0|$. Hence

$$\begin{aligned}
\sup_{x\in B(x_0,l)} \left|T_R^\delta \left(f\chi_{(B(x_0,2l))^c}\right)(x)\right| &\leqslant \sup_{x\in B(x_0,l)} T_*^\delta \left(f\chi_{(B(x_0,2l))^c}\right)(x) \\
&= C \sup_{x\in B(x_0,l)} \sup_{R>0} \left|\left(f\chi_{(B(x_0,2l))^c}\right) * \phi_{1/R}(x)\right| \\
&\leqslant C \sup_{x\in B(x_0,l)} \sup_{R>0} \int_{(B(x_0,2l))^c} \frac{R^n}{(R|x-y|)^n}|f(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{|B(x_0, 2^{j+1}l)|} \int_{B(x_0,2^{j+1}l)} |f(y)|dy.
\end{aligned}$$

This means that T_R^δ and T_*^δ satisfy (1.2). □

4.3. Marcinkiewicz integral operators

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n (n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(S^{n-1})$ with $1 < s \leqslant \infty$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The Marcinkiewicz integral of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We will also consider the commutator generated by Marcinkiewcz integral μ_Ω and b is defined as follows

$$[b,\mu_\Omega](f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Suppose $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leqslant \infty$ and $b \in BMO$. Then by [3], for every $s' < p < \infty$ and $\omega \in A_{p/s'}$, there is a constant C independent of f such that

$$\|\mu_\Omega(f)\|_{L^p(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\omega,\mathbb{R}^n)}.$$

By [4], for every $s' < p < \infty$ and $\omega \in A_{p/s'}$, there is a constant C independent of f such that

$$\|[b,\mu_\Omega](f)\|_{L^p(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{L^p(\omega,\mathbb{R}^n)}.$$

Theorem 4.3. *Suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leqslant \infty$. Let $s' < p < \infty, \omega \in A_{p/s'}$ and $b \in BMO$. If (φ_1, φ_2) satisfies the condition (1.3), then there is a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

If (φ_1, φ_2) satisfies the condition (1.4), then there is a constant $C > 0$ independent of f such that

$$\|[b,\mu_\Omega]f\|_{M^p_{\varphi_2}(\omega,\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega,\mathbb{R}^n)}.$$

Proof. Observe that when $x \in B(x_0, l)$ and $y \in B(x_0, 2^{j+1}l) \setminus B(x_0, 2^j l)(j \geqslant 1)$, then

$$t \geqslant |x-y| \geqslant |y-x_0| - |x-x_0| \geqslant 2^{j-1}l.$$

Then, by Minkowski's inequality we have

$$\begin{aligned}
\mu_\Omega(f\chi_{(B(x_0,2l))^c})(x) &= \left(\int_0^\infty \left|\int_{B(x_0,2l)^c\cap\{y:|x-y|\leqslant t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \left|\sum_{j=1}^\infty \int_{(B(x_0,2^{j+1}l)\setminus B(x_0,2^j l))^c\cap\{y:|x-y|\leqslant t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \qquad (4.4)\\
&\leqslant C\sum_{j=1}^\infty \left(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^j l)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy\right)\left(\int_{2^{j-1}l}^\infty \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leqslant C\sum_{j=1}^\infty (2^{j+1}l)^{-1} \int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^j l)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy.
\end{aligned}$$

When $\Omega \in L^\infty(\mathbb{S}^{n-1})$, then

$$\sup_{x\in B(x_0,l)} \mu_\Omega(f\chi_{(B(x_0,2l))^c})(x) \leqslant C\sum_{j=1}^\infty (2^{j+1}l)^{-n} \int_{B(x_0,2^{j+1}l)} |f(y)|dy. \qquad (4.5)$$

When $\Omega \in L^s(\mathbb{S}^{n-1}), 1 < s < \infty$, then by Hölder's inequality,

$$\begin{aligned}&\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^j l)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy \\ &\leqslant C\Big(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^j l)} |\Omega((x-y)')|^s dy\Big)^{1/s}\Big(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^j l)} \frac{|f(y)|^{s'}}{|x-y|^{(n-1)s'}}dy\Big)^{1/s'}. \end{aligned} \tag{4.6}$$

It follows from (4.1), (4.4) and (4.6) that

$$\sup_{x\in B(x_0,l)} \mu_\Omega(f\chi_{(B(x_0,2l))^c})(x) \leqslant C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}}\Big(\int_{B(x_0,2^{j+1}l)} |f(y)|^{s'}dy\Big)^{1/s'}. \tag{4.7}$$

Combining (4.5) with (4.7), the proof of Theorem 4.3 is completed. □

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