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Strong convergence of implicit and explicit iterations for a class of variational inequalities in Banach spaces

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Abstract

In this paper, we introduce and analyze implicit and explicit iteration methods for solving a variational inequality problem over the set of common fixed points of an infinite family of nonexpansive mappings on a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Strong convergence results are given. Our results improve and extend the corresponding results in the literature.

Keywords: Nonexpansive mapping, fixed point, variational inequality, global convergence.
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1. Introduction

Let X be a real Banach space with its topological dual X^*, and C be a nonempty closed convex subset of X. Let $T : C \to X$ be a nonlinear mapping. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T. A mapping $T : C \to X$ is called L-Lipschitz continuous if there exists a constant $L \geqslant 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

T is called nonexpansive provided $L = 1$ and T is called contractive provided $L \in [0, 1)$.

The normalized dual mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) := \{\varphi \in X^* : \langle x, \varphi\rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot\rangle$ denotes the dual pairing between X and X^*, see e.g., [11] for further details.

Let $U := \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The space X is said to have a Gâteaux differentiable norm, if the limit

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t}, \tag{1.1}$$

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exists for each $x, y \in U$. The space X is said to have a uniformly Gâteaux differentiable norm, if the limit (1.1) is attained uniformly for $x \in U$. The space X is said to be strictly convex if and only if for $x, y \in U$ with $x \neq y$, we have $\|(1-\lambda)x + \lambda y\| < 1$, for all $\lambda \in (0,1)$. It is well-known in [11] that if X is smooth, then the normalized duality mapping is single-valued; and if the norm of X is uniformly Gâteaux differentiable, then the normalized duality mapping is norm to weak* uniformly continuous on every bounded subset of X. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Recall that the so-called classical variational inequality (VI) in Hilbert spaces is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C.$$

This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; see e.g., [13, 18–21, 33–38] and the references therein. A popular algorithm for solving this problem is extragradient method introduced by Korpelevich [22]. This method has been improved by several researchers; see e.g., [9, 12, 14, 27] and the references therein.

In case of Banach space setting, the VI is to find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geqslant 0, \quad \forall x \in C. \tag{1.2}$$

It is known ([1]) that (1.2) in a smooth Banach space is equivalent to a fixed-point equation, containing a sunny nonexpansive retraction from any point of the space onto the feasible set, which is usually assumed to be closed and convex. The sunny nonexpansive retraction is not easy to compute, due to the complexity of the feasible set. To overcome this drawback in a Hilbert space, where the retraction is a metric projection, in [32], Yamada assumed that the feasible set is the set of common fixed points of a finite family of nonexpansive mappings and introduced an explicit hybrid steepest-descent method. In this case, the variational inequality defined on such feasible set is also called a hierarchical variational inequality (HVI). Yamada's method is subsequently extended and modified to solve more complex problems, containing finite or infinite nonexpansive mappings (see, e.g., [3, 6, 40] and references therein). In [40], based on the Yamada result, Zeng and Yao introduced an implicit method that converges weakly to a solution of a variational inequality, involving a Lipschitz continuous and strongly monotone mapping in a Hilbert space H, where the feasible set is that of common fixed points of a finite family of nonexpansive mappings on H. In [7], Ceng et al. extended this result from nonexpansive mappings to Lipschitz pseudocontractive mappings and strictly pseudocontractive mappings on H. Recently, in [4], Buong and Anh proposed a strongly convergent implicit method, which is a modification of Yamada's result.

In the case where the feasible set is that of common fixed points of an infinite family of nonexpansive mappings on H, based on the W-mapping (see [29]) and Moudafi's viscosity approximation method (see [23]), in [16, 17], Kikkawa and Takahashi studied an implicit iteration scheme that converges strongly to a solution of the stated problem. Recently, by using the W-mappings, Ceng and Yao [8] introduced relaxed viscosity approximation methods for finding a common fixed point of an infinite family of nonexpansive mapping in a Banach space. It was proven in [8] that the sequences generated by the proposed methods converge strongly to a common fixed point, which solves some variational inequalities. In the meantime, in [30], Wang et al. proposed a new implicit iteration method, which converges strongly to a common fixed point for solving some variational inequalities in a Banach space with a weakly continuous duality mapping. Very recently, motivated by the ideas above and based on a V-mapping, which is simpler than the W-mapping, Buong and Phuong [3] introduced two new implicit iterative algorithms, which converge strongly in Banach spaces without weakly continuous duality mapping. These methods are two different combinations of the steepest-descent method with the V-mapping, a composition, and a convex combination.

Our purpose in this paper is to solve a HVI for an infinite family of nonexpansive mappings on a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. We introduce implicit and explicit iterative algorithms for finding a solution of the problem, and derive the strong convergence of the proposed algorithms to a unique solution of the problem, by using V-mappings instead of W-ones. Our results improve and extend the corresponding results announced by some others, e.g., Ceng and Yao [8] and Buong and Phuong [3].

2. Preliminaries and algorithms

Let X be a real Banach space with the dual space X^* and $\langle\cdot,\cdot\rangle$ be the dual pairing between X and X^*. For simplicity, the norms of X and X^* are denoted by the symbol $\|\cdot\|$. A mapping $J : X \to 2^{X^*}$, satisfying the condition

$$J(x) = \{\varphi \in X^* : \langle x, \varphi\rangle = \|\varphi\|^2 \quad \text{and} \quad \|\varphi\| = \|x\|\}, \quad \forall x \in X,$$

is called the normalized duality mapping of X. We know that $J(tx) = tJ(x)$ for all $t > 0$ and $x \in X$, and $J(-x) = -J(x)$. Throughout this paper, we denote the single-valued normalized duality mapping by j and denote the fixed point set of a mapping T by Fix(T). In addition, we shall use the notations: "$\rightharpoonup$", "$\rightharpoonup^*$", and "$\to$" stand for the weak convergence, weak* convergence, and strong convergence, respectively.

Let $U := \{x \in X : \|x\| = 1\}$ denote the unit sphere of x. Recall that a Banach space X is said to be strictly convex if $(\|x + y\|)/2 < 1$ for each $x, y \in U$ with $x \neq y$. If for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|\frac{x+y}{2}\| > 1 - \delta$ implies $\|x - y\| < \varepsilon$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth as well as X is smooth if and only if X^* is strictly convex.

Definition 2.1. A mapping F with domain D(F) and range R(F) in X is called

(a) accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant 0,$$

where J is the normalized duality mapping;

(b) δ-strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant \delta\|x - y\|^2 \quad \text{for some} \quad \delta \in (0, 1);$$

(c) α-inverse-strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant \alpha\|Fx - Fy\|^2 \quad \text{for some} \quad \alpha \in (0, 1);$$

(d) ζ-strictly pseudocontractive [2] if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \leqslant \|x - y\|^2 - \zeta\|x - y - (Fx - Fy)\|^2 \quad \text{for some} \quad \zeta \in (0, 1).$$

It is easy to see that the last inequality can be rewritten as (see [39])

$$\langle (I - F)x - (I - F)y, j(x - y)\rangle \geqslant \zeta\|(I - F)x - (I - F)y\|^2,$$

where I denotes the identity mapping of X. Clearly, if F is ζ-strictly pseudocontractive with $\zeta = 0$, then it is said to be pseudocontractive. It is not hard to find that every nonexpansive mapping is pseudocontractive.

Let C be a nonempty closed convex subset of a smooth Banach space X and $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C. Then we set $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. In 2013, Buong and Phuong [3] considered the following HVI with $C = X$: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{2.1}$$

In the case where $X = H$, a Hilbert space, we have $J = I$, and hence problem (2.1) reduces to the HVI: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), x - x^*\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{2.2}$$

Assume that $\mathcal{F} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ is the set of common fixed points of a family of N nonexpansive mappings T_i on H, and F is an L-Lipschitz continuous and η-strongly monotone mapping, i.e., $\|Fx - Fy\| \leqslant L\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2$ for all $x, y \in H$. Zeng and Yao [40] introduced the following implicit iteration: for an arbitrarily initial point $x_0 \in H$, the sequence $\{x_k\}_{k=1}^{\infty}$ is generated as follows:

$$x_k = \beta_k x_{k-1} + (1 - \beta_k)[T_{[k]}x_k - \lambda_k \mu F(T_{[k]}x_k)], \quad \forall k \geqslant 1, \tag{2.3}$$

where $T_{[n]} = T_{n \bmod N}$, for integer $n \geqslant 1$, with the mod function taking values in the set $\{1, 2, ..., N\}$. They proved the following result.

Theorem 2.2 ([40, Theorem 2.1]). *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that, for some positive constants L and η, F is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive mappings on H such that $\mathcal{F} : \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, $x_0 \in H$, $\{\lambda_k\}_{k=1}^{\infty} \subset [0, 1)$, and $\{\beta_k\}_{k=1}^{\infty} \subset (0, 1)$ satisfying the condition $\sum_{k=1}^{\infty} \lambda_k < \infty$, and let $a \leqslant \beta_k \leqslant b, k \geqslant 1$ for some $a, b \in (0, 1)$. Then the sequence $\{x_k\}_{k=0}^{\infty}$, defined by (2.3), converges weakly to $x^* \in \mathcal{F}$, solving (2.2).*

It is well-known that if $\sum_{k=1}^{\infty} \lambda_k < \infty$, then $\lambda_k \to 0$ as $k \to \infty$, and the inversion is not right. Recently, in order to obtain the strong convergence and decrease the strictness of the condition on λ_k, Buong and Anh [4] proposed the following implicit iteration method:

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t \cdots T_1^t, \quad t \in (0, 1), \tag{2.4}$$

where $\{T_i^t\}_{i=0}^{N}$ are defined by

$$T_i^t x := (1 - \beta_t^i)x + \beta_t^i T_i x, \quad i = 1, \ldots, N, \quad T_0^t y := (I - \lambda_t \mu F)y, \quad x, y \in H, \tag{2.5}$$

and proved that the net $\{x_t\}$, defined by (2.4)-(2.5), converges strongly to an element x^* in (2.2) under the conditions on μ, β_t^i that are similar to Theorem 2.2, and $\lambda_t \to 0$ as $t \to 0^+$. When $N = 1$, X is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and T is a continuous pseudocontractive mapping, Ceng et al. [5] proved the following result.

Theorem 2.3 ([5, Proposition 4.3]). *Let F be a δ-strongly accretive and ζ-strictly pseudocontractive mapping with $\delta + \zeta > 1$ and let T be a continuous and pseudocontractive mapping on X, which is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, such that $\mathcal{F} := \mathrm{Fix}(T) \neq \emptyset$. For each $t \in (0, 1)$, choose a number $\mu_t \in (0, 1)$ arbitrarily and let $\{z_t\}$ be defined by*

$$z_t = t(I - \mu_t F)z_t + (1 - t)Tz_t. \tag{2.6}$$

Then, as $t \to 0^+$, $\{z_t\}$ converges strongly to $x^* \in \mathcal{F}$, solving (2.1).

Let C be a nonempty closed convex subset of a real Banach space X, $\{T_k\}_{k=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, and $\{\rho_k\}_{k=1}^{\infty}$ be a sequence of nonnegative numbers in [0, 1]. The following self-mapping W_n on C was considered and studied in [24, 25]:

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \rho_k T_k U_{k,k+1} + (1 - \rho_k)I, \\ U_{k,k-1} = \rho_{k-1} T_{k-1} U_{k,k} + (1 - \rho_{k-1})I, \\ \quad \vdots \\ U_{k,2} = \rho_2 T_2 U_{k,3} + (1 - \rho_2)I, \\ W_k = U_{k,1} = \rho_1 T_1 U_{k,2} + (1 - \rho_1)I. \end{cases} \tag{2.7}$$

Such a mapping W_k is called the W-mapping generated by $T_k, T_{k-1}, \ldots, T_1$ and $\rho_k, \rho_{k-1}, \ldots, \rho_1$; when $X = H$, see Takahashi [29] for more details. To find a common fixed point of an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive self-mappings on a nonempty, closed, and convex subset C in H, Kikkawa, and Takahashi [16] proved strong convergence of a sequence $\{x_k\}_{k=1}^{\infty}$, defined by the following implicit iterative scheme: $x_k = \gamma_k f(x_k) + (1-\gamma_k)W_k x_k$ with $0 < \rho_1 \leqslant 1$ and $0 < \rho_i \leqslant b < 1$, for $i = 2, 3, \dots$, where f is a contractive self-mapping on C. Subsequently, in [17], when C is a nonempty, closed, and convex subset of a uniformly convex Banach space X with a uniformly Gâteaux differentiable norm, they considered the following strongly convergent implicit method:

$$S_k x = (1-\frac{1}{k})Wx + \frac{1}{k}f(x), \quad \text{and} \quad Wx = \lim_{k\to\infty} W_k x = \lim_{k\to\infty} U_{k,1}x. \tag{2.8}$$

It is remarkable that the method (2.8) contains the limit mapping W, and hence it is quite difficult to realize.

Let X be a real reflexive and strictly convex Banach space X with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X. Let $f : C \to C$ be a contractive mapping, and let $\{T_k\}_{k=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that the common fixed point set $\mathcal{F} := \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \neq \emptyset$. Let W_k be the W-mapping defined by (2.7) where $\{\rho_k\}_{k=1}^{\infty} \subset (0, b]$ for some $b \in (0,1)$. Recently, in order to find a common fixed point of an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive mappings, Ceng and Yao [8] introduced a relaxed implicit viscosity approximation method.

Algorithm 2.4 ([8, (2)]). Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence in $(0,1)$ such that $\lim_{k\to\infty}\alpha_k = 0$ and $\{\beta_k\}_{k=1}^{\infty}$ be a sequence in $[0,1]$ such that $\lim_{k\to\infty}\beta_k/\alpha_k = 0$. The sequence $\{z_k\}_{k=1}^{\infty}$ is generated in the implicit manner

$$x_k = \alpha_k f((1-\beta_k)x_k + \beta_k W_k x_k) + (1-\alpha_k)W_k((1-\beta_k)x_k + \beta_k W_k x_k), \quad \forall k \geqslant 1, \tag{2.9}$$

where $f : C \to C$ is a contractive mapping with a contractive constant $\alpha \in (0,1)$.

It was proven in [8] that the net $\{x_k\}_{k=1}^{\infty}$ converges in norm, as $k \to \infty$, to the unique solution $x^* \in \mathcal{F}$ to the following VI:

$$\langle (I-f)(x^*), j(x-x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{2.10}$$

In the meantime, the authors [8] also proposed another relaxed explicit viscosity approximation method.

Algorithm 2.5 ([8, (14)]). Let $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ be two sequence in $(0,1)$ with $\alpha_k + \beta_k \leqslant 1$, for all $k \geqslant 1$, and $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence in $[0,1]$. Assume that $\lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=1}^{\infty}\alpha_k = \infty$, $0 < \liminf_{k\to\infty}\beta_k \leqslant \limsup_{k\to\infty}\beta_k < 1$, $\lim_{k\to\infty}|\gamma_{k+1}-\gamma_k| = 0$, and $\limsup_{k\to\infty}\gamma_k < 1$. For arbitrarily given $x_1 \in C$, let the sequence $\{x_k\}_{k=1}^{\infty}$ be generated in the explicit manner

$$\begin{cases} y_k = (1-\gamma_k)x_k + \gamma_k W_k x_k, \\ x_{k+1} = (1-\alpha_k-\beta_k)x_k + \alpha_k f(y_k) + \beta_k W_k y_k, \quad \forall k \geqslant 1, \end{cases}$$

where $f : C \to C$ is a contractive mapping with a contractive constant $\alpha \in (\frac{1}{2}, 1)$.

It was also proven in [8] that the sequence $\{x_k\}_{k=1}^{\infty}$ converges in norm, as $k \to \infty$, to the unique solution $x^* \in \mathcal{F}$ to the VI (2.10) provided $\lim_{k\to\infty}\gamma_k = 0$ and $\beta_k \equiv \beta$ for some fixed $\beta \in (0,1)$.

In [3], motivated by methods (2.4) and (2.6), by introducing a mapping V_k, defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1} \cdots T^k, \quad T^i = (1-\alpha_i)I + \alpha_i T_i, \quad i = 1, 2, \dots, k, \tag{2.11}$$

where

$$\alpha_i \in (0,1) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i < \infty, \tag{2.12}$$

Buong and Phuong considered two implicit methods. In both methods, the iteration sequence $\{x_k\}_{k=1}^{\infty}$ is defined, respectively, by

$$x_k = V_k(I - \lambda_k F)x_k, \quad \forall k \geqslant 1, \tag{2.13}$$

and

$$x_k = \gamma_k(I - \lambda_k F)x_k + (I - \gamma_k)V_k x_k, \quad \forall k \geqslant 1, \tag{2.14}$$

where λ_k and γ_k are the positive parameters, satisfying some additional conditions. The authors [3] proved the strong convergence theorems for the methods (2.13) and (2.14).

We will make use of the following well-known results.

Lemma 2.6 ([15]). *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that C is a nonempty closed convex subset of X, that $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and that $f : C \to C$ is a fixed contractive mapping. Let $\{x_t\}$ be defined by $x_t = tf(x_t) + (1-t)Tx_t$. Then as $t \to 0$, $\{x_t\}$ converges strongly to a unique solution $x^* \in \mathrm{Fix}(T)$ to the following VI:*

$$\langle (I - f)(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T).$$

Lemma 2.7 ([10]). *Let X be a real Banach space. Then for all $x, y \in X$*

(i) $\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle$ *for all* $j(x+y) \in J(x+y)$;
(ii) $\|x + y\|^2 \geqslant \|x\|^2 + 2\langle y, j(x)\rangle$ *for all* $j(x) \in J(x)$.

Let LIM be a continuous linear functional on l^∞ and $s = (a_1, a_2, \ldots) \in l^\infty$. We write $\mathrm{LIM}_k a_k$ instead of $\mathrm{LIM}(s)$. LIM is called a Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}_k 1 = 1$ and $\mathrm{LIM}_k a_{k+1} = \mathrm{LIM}_k a_k$ for all $(a_1, a_2, \ldots) \in l^\infty$. If LIM is a Banach limit, then there hold the following:

(i) for all $k \geqslant 1$, $a_k \leqslant c_k$ implies $\mathrm{LIM}_k a_k \leqslant \mathrm{LIM}_k c_k$;
(ii) $\mathrm{LIM}_k a_{k+m} = \mathrm{LIM}_k a_k$ for any fixed positive integer m;
(iii) $\liminf_{k\to\infty} a_k \leqslant \mathrm{LIM}_k a_k \leqslant \limsup_{k\to\infty} a_k$ for all $(a_1, a_2, \ldots) \in l^\infty$.

Lemma 2.8 ([41]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_k a_k \leqslant a$ for all Banach limit LIM. If $\limsup_{k\to\infty}(a_{k+m} - a_k) \leqslant 0$, then $\limsup_{k\to\infty} a_k \leqslant a$.*

In particular, if $m = 1$ in Lemma 2.8, then we immediately obtain the following corollary.

Corollary 2.9 ([26]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_k a_k \leqslant a$ for all Banach limit LIM. If $\limsup_{k\to\infty}(a_{k+1} - a_k) \leqslant 0$, then $\limsup_{k\to\infty} a_k \leqslant a$.*

Lemma 2.10 ([5]). *Let X be a real smooth Banach space and $F : C \to X$ be a mapping.*

(a) *If F is ζ-strictly pseudocontractive, then F is Lipschitz continuous with constant $1 + \frac{1}{\zeta}$.*
(b) *If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$.*
(c) *If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$, then for any fixed number $\lambda \in (0, 1)$, $I - \lambda F$ is contractive with constant $1 - \lambda(1 - \sqrt{\frac{1-\delta}{\zeta}}) \in (0, 1)$.*

3. Iterative algorithms and convergence criteria

In this section, we study iterative methods for computing approximate solutions of a HVI for an infinite family of nonexpansive mappings. We introduce implicit and explicit iterative algorithms for solving such a problem, and derive the strong convergence theorems for the sequences generated by the proposed algorithms.

The following lemmas will be used to prove our main results in the sequel.

Lemma 3.1 ([3, Lemma 3.1]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^k$, $k \geqslant 1$ be k nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^k \mathrm{Fix}(T_i) \neq \emptyset$. Let a, b and α_i, $i = 1, 2, \ldots, k$, be real numbers such that $0 < a \leqslant \alpha_i \leqslant b < 1$, and let V_k be a mapping defined by (2.9) for all $k \geqslant 1$. Then, $\mathrm{Fix}(V_k) = \mathcal{F}$.*

Lemma 3.2 ([3, Lemma 3.2]). *Let C be a nonempty closed convex subset of a Banach space X and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let V_k be a mapping defined by (2.9), and let α_i satisfy (2.10). Then, for each $x \in C$ and $i \geqslant 1$, $\lim_{k\to\infty} V_k^i x$ exists.*

Remark 3.3.

(i) We can define the mappings

$$V_\infty^i x := \lim_{k\to\infty} V_k^i x \quad \text{and} \quad Vx := V_\infty^1 x = \lim_{k\to\infty} V_k x, \quad \forall x \in C.$$

(ii) It can be readily seen from the proof of Lemma 3.2 that if D is a nonempty and bounded subset of C, then the following holds:

$$\lim_{k\to\infty} \sup_{x\in D} \|V_k^i x - V_\infty^i x\| = 0, \quad \forall i \geqslant 1.$$

In particular, whenever $i = 1$, we have

$$\lim_{k\to\infty} \sup_{x\in D} \|V_k x - Vx\| = 0.$$

Lemma 3.4 ([3, Lemma 3.3]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let α_i satisfy the first condition in (2.12). Then, $\mathrm{Fix}(V) = \mathcal{F}$.*

Lemma 3.5 ([28]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_k\}$ be a sequence in $[0, 1]$ such that*

$$0 < \liminf_{k\to\infty} \alpha_k \leqslant \limsup_{k\to\infty} \alpha_k < 1.$$

Suppose that $x_{k+1} = \alpha_k x_k + (1 - \alpha_k) z_k$, for all $k \geqslant 1$, and

$$\limsup_{k\to\infty} (\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|) \leqslant 0.$$

Then $\lim_{k\to\infty} \|z_k - x_k\| = 0$.

Lemma 3.6 ([31]). *Assume that $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leqslant (1 - \gamma_k) a_k + \gamma_k \delta_k, \quad \forall k \geqslant 1,$$

where $\{\gamma_k\}$ is a sequence in $[0, 1]$ and $\{\delta_k\}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{k=1}^{\infty} \gamma_k = \infty$;
(ii) $\limsup_{k\to\infty} \delta_k \leqslant 0$ *or* $\sum_{k=1}^{\infty} |\gamma_k \delta_k| < \infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

Now, we are in a position to prove the following main results.

Theorem 3.7. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, let F be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$, and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings on X such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (2.11) and (2.12). Let $\{\lambda_k\}_{k=1}^{\infty} \subset (0, 1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0, 1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset [0, 1]$ such that $\lim_{k\to\infty} \gamma_k = 0$, $\lim_{k\to\infty} \beta_k = 0$ and $\limsup_{k\to\infty} \beta_k/(\lambda_k\gamma_k) < \infty$. Let $\{x_k\}_{k=1}^{\infty}$ be defined by*

$$x_k = \gamma_k (I - \lambda_k F)((1 - \beta_k) x_k + \beta_k V_k x_k) + (1 - \gamma_k) V_k ((1 - \beta_k) x_k + \beta_k V_k x_k), \quad \forall k \geqslant 1. \tag{3.1}$$

Then $\{x_k\}_{k=1}^{\infty}$ converges strongly to a unique solution $x^* \in \mathcal{F}$ to the following VI:

$$\langle F(x^*), j(x - x^*) \rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{3.2}$$

Proof. Consider the mapping $U_k x = \gamma_k(I-\lambda_k F)((1-\beta_k)x+\beta_k V_k x)+(1-\gamma_k)V_k((1-\beta_k)x+\beta_k V_k x)$, for all $k \geqslant 1$ and $x \in X$. Then, from Lemma 2.10 (c), it follows that for each $x, y \in X$,

$$\begin{aligned}\|U_k x - U_k y\| &= \|\gamma_k(I-\lambda_k F)((1-\beta_k)x+\beta_k V_k x)+(1-\gamma_k)V_k((1-\beta_k)x+\beta_k V_k x)\\ &\quad -[\gamma_k(I-\lambda_k F)((1-\beta_k)y+\beta_k V_k y)+(1-\gamma_k)V_k((1-\beta_k)y+\beta_k V_k y)]\|\\ &= \|\gamma_k[(I-\lambda_k F)((1-\beta_k)x+\beta_k V_k x)-(I-\lambda_k F)((1-\beta_k)y+\beta_k V_k y)]\\ &\quad +(1-\gamma_k)[V_k((1-\beta_k)x+\beta_k V_k x)-V_k((1-\beta_k)y+\beta_k V_k y)]\\ &\leqslant \gamma_k(1-\lambda_k\tau)\|((1-\beta_k)x+\beta_k V_k x)-((1-\beta_k)y+\beta_k V_k y)\|\\ &\quad +(1-\gamma_k)\|((1-\beta_k)x+\beta_k V_k x)-((1-\beta_k)y+\beta_k V_k y)\|\\ &= (I-\gamma_k\lambda_k\tau)\|((1-\beta_k)x+\beta_k V_k x)-((1-\beta_k)y+\beta_k V_k y)\|\\ &= (I-\gamma_k\lambda_k\tau)\|(1-\beta_k)(x-y)+\beta_k(V_k x-V_k y)\|\\ &\leqslant (I-\gamma_k\lambda_k\tau)[(1-\beta_k)\|x-y\|+\beta_k\|V_k x-V_k y\|]\\ &\leqslant (I-\gamma_k\lambda_k\tau)\|x-y\|,\end{aligned}$$

where $\tau = 1-\sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$ (due to $\delta+\zeta>1$). Since $\gamma_k\lambda_k\tau \in (0,1)$, U_k is a contraction of X into itself. By Banach's contraction principle, there exists a unique element $x_k \in X$, satisfying (3.1).

Next, we divide the rest of the proof into several steps.

Step 1. We show that $\{x_k\}_{k=1}^{\infty}$ is bounded. Indeed, take an arbitrarily given $p \in \mathcal{F}$. Then, by Lemma 3.1, we have $V_k p = p$, and hence $\|V_k x_k - p\| \leqslant \|x_k - p\|$. So, by Lemma 2.10 (c) we get

$$\begin{aligned}\|x_k-p\|^2 &\leqslant \gamma_k\|(I-\lambda_k F)((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &\quad +(1-\gamma_k)\|V_k((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &= \gamma_k\|(I-\lambda_k F)((1-\beta_k)x_k+\beta_k V_k x_k)-(I-\lambda_k F)p-\lambda_k F(p)\|^2\\ &\quad +(1-\gamma_k)\|V_k((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &\leqslant \gamma_k[\|(I-\lambda_k F)((1-\beta_k)x_k+\beta_k V_k x_k)-(I-\lambda_k F)p\|+\lambda_k\|F(p)\|]^2\\ &\quad +(1-\gamma_k)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &\leqslant \gamma_k[(1-\lambda_k\tau)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|+\lambda_k\|F(p)\|]^2\\ &\quad +(1-\gamma_k)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &= \gamma_k[(1-\lambda_k\tau)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|+\lambda_k\tau\cdot\tau^{-1}\|F(p)\|]^2\\ &\quad +(1-\gamma_k)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &\leqslant \gamma_k[(1-\lambda_k\tau)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2+\lambda_k\tau^{-1}\|F(p)\|^2]\\ &\quad +(1-\gamma_k)\|((1-\beta_k)x_k+\beta_k V_k x_k)-p\|^2\\ &\leqslant \gamma_k[(1-\lambda_k\tau)\|x_k-p\|^2+\lambda_k\tau^{-1}\|F(p)\|^2]+(1-\gamma_k)\|x_k-p\|^2\\ &= (1-\gamma_k\lambda_k\tau)\|x_k-p\|^2+\gamma_k\lambda_k\tau^{-1}\|F(p)\|^2.\end{aligned}$$

Therefore, $\|x_k-p\| \leqslant \|F(p)\|/\tau$, which implies the boundedness of $\{x_k\}_{k=1}^{\infty}$. So, the sequences $\{V_k x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$, $\{V_k y_k\}_{k=1}^{\infty}$, and $\{F(y_k)\}_{k=1}^{\infty}$, where $y_k = (1-\beta_k)x_k+\beta_k V_k x_k$, are also bounded. Since $\gamma_k \to 0$ as $k \to \infty$, and the following relation holds

$$\|x_k - V_k y_k\| = \gamma_k\|(I-\lambda_k F)y_k - V_k y_k\| \leqslant \gamma_k(\|y_k - V_k y_k\|+\lambda_k\|F(y_k)\|) \leqslant \gamma_k(\|y_k\|+\|V_k y_k\|+\|F(y_k)\|),$$

we obtain from the boundedness of $\{y_k\}_{k=1}^{\infty}$, $\{V_k y_k\}_{k=1}^{\infty}$ and $\{F(y_k)\}_{k=1}^{\infty}$ that $\|x_k - V_k y_k\| \to 0$ as $k \to \infty$. Note that $\|y_k - x_k\| = \beta_k\|V_k x_k - x_k\|$ and

$$\begin{aligned}\|x_k - V_k x_k\| \leqslant \|x_k - V_k y_k\| + \|V_k y_k - V_k x_k\| &\leqslant \|x_k - V_k y_k\| + \|y_k - x_k\|\\ &= \|x_k - V_k y_k\| + \beta_k\|V_k x_k - x_k\|.\end{aligned}$$

So, from $\|x_k - V_k y_k\| \to 0$ and $\beta_k \to 0$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_k - V_k x_k\| = 0. \tag{3.3}$$

Step 2. We show that $\mathrm{LIM}_k\|x_k - Vz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2$ for any Banach limit LIM, where for each $n \geqslant 1$, z_n is a unique element in X such that $z_n = \frac{1}{n}(I-F)z_n + (1-\frac{1}{n})Vz_n$.

Indeed, in terms of Lemma 2.10 (b) we know that $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$. Utilizing Lemmas 2.6 and 3.4, we conclude that $\{z_n\}$ converges strongly to a unique solution $x^* \in \mathrm{Fix}(V) = \mathcal{F}$ to the following VI:

$$\langle (I-(I-F))x^*, j(x-x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{3.4}$$

Since the VI (3.4) is equivalent to the VI (3.2), we know that $\{z_n\}$ converges strongly to a unique solution $x^* \in \mathcal{F}$ to the VI (3.2). Moreover, since V_k is a nonexpansive mapping for each $k \geqslant 1$, V is a nonexpansive mapping on X. Note that $x_k = \gamma_k(I-\lambda_k F)y_k + (1-\gamma_k)V_k y_k$, where $y_k = (1-\beta_k)x_k + \beta_k V_k x_k$. Also, observe that for each $k, n \geqslant 1$

$$\begin{aligned}
\|x_k - Vz_n\| &= \|\gamma_k[(I-\lambda_k F)y_k - Vz_n] + (1-\gamma_k)(V_k y_k - Vz_n)\| \\
&\leqslant \gamma_k\|(I-\lambda_k F)y_k - Vz_n\| + (1-\gamma_k)\|V_k y_k - V_k z_n\| + (1-\gamma_k)\|V_k z_n - Vz_n\| \\
&\leqslant \gamma_k(\|y_k - Vz_n\| + \lambda_k\|F(y_k)\|) + \|y_k - z_n\| + \|V_k z_n - Vz_n\| \\
&\leqslant \gamma_k(\|y_k - Vz_n\| + \|F(y_k)\|) + (1-\beta_k)\|x_k - z_n\| + \beta_k\|V_k x_k - z_n\| + \|V_k z_n - Vz_n\| \\
&\leqslant \gamma_k(\|y_k - Vz_n\| + \|F(y_k)\|) + \beta_k\|V_k x_k - z_n\| + \|x_k - z_n\| + \|V_k z_n - Vz_n\|.
\end{aligned} \tag{3.5}$$

Furthermore, from Remark 3.3 (ii), we deduce that if D is a nonempty and bounded subset of X, then, for $\varepsilon > 0$, there exists $k_0 > i$ such that for all $k > k_0$

$$\sup_{x\in D} \|V_k^i x - V_\infty^i x\| \leqslant \varepsilon. \tag{3.6}$$

Taking $D = \{z_n : n \geqslant 1\}$, $\{x_k : k \geqslant 1\}$, respectively, and setting $i = 1$, from (3.6) we have

$$\|V_k z_n - Vz_n\| \leqslant \sup_{x\in D}\|V_k x - Vx\| \leqslant \varepsilon \quad \text{and} \quad \|V_k x_k - Vx_k\| \leqslant \sup_{x\in D}\|V_k x - Vx\| \leqslant \varepsilon,$$

which immediately imply that

$$\lim_{k\to\infty} \|V_k x_k - Vx_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|V_k z_n - Vz_n\| = 0, \quad \forall n \geqslant 1. \tag{3.7}$$

Since $\gamma_k \to 0$ and $\beta_k \to 0$ as $k \to \infty$, from (3.5) and (3.7) we obtain

$$\mathrm{LIM}_k\|x_k - Vz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2. \tag{3.8}$$

Step 3. We show that $\mathrm{LIM}_k\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0$. Indeed, since $z_n = \frac{1}{n}(I-F)z_n + (1-\frac{1}{n})Vz_n$, we have

$$x_k - z_n = \frac{1}{n}(x_k - (I-F)z_n) + (1-\frac{1}{n})(x_k - Vz_n),$$

that is,

$$(1-\frac{1}{n})(x_k - Vz_n) = x_k - z_n - \frac{1}{n}(x_k - (I-F)z_n). \tag{3.9}$$

From Lemma 2.7 (ii) and (3.9) it follows that

$$\begin{aligned}
(1-\frac{1}{n})^2\|x_k - Vz_n\|^2 &\geqslant \|x_k - z_n\|^2 - \frac{2}{n}\langle x_k - z_n + z_n - (I-F)z_n, j(x_k - z_n)\rangle \\
&= (1-\frac{2}{n})\|x_k - z_n\|^2 + \frac{2}{n}\langle F(z_n), j(z_n - x_k)\rangle.
\end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$(1-\frac{1}{n})^2\mathrm{LIM}_k\|x_k-z_n\|^2 \geqslant (1-\frac{2}{n})\mathrm{LIM}_k\|x_k-z_n\|^2+\frac{2}{n}\mathrm{LIM}_k\langle F(z_n),j(z_n-x_k)\rangle,$$

and hence

$$\frac{1}{n^2}\mathrm{LIM}_k\|x_k-z_n\|^2 \geqslant \frac{2}{n}\mathrm{LIM}_k\langle F(z_n),j(z_n-x_k)\rangle.$$

This implies that $\frac{1}{2n}\mathrm{LIM}_k\|x_k-z_n\|^2 \geqslant \mathrm{LIM}_k\langle F(z_n),j(z_n-x_k)\rangle$. Since $z_n \to x^* \in \mathcal{F}$ as $n \to \infty$, by the uniform Gâteaux differentiability of the norm of X we have

$$\mathrm{LIM}_k\langle F(x^*),j(x^*-x_k)\rangle \leqslant 0. \tag{3.11}$$

Step 4. We show that $\mathrm{LIM}_k\|x_k-x^*\|^2=0$. Indeed, since $x_k=\gamma_k(I-\lambda_kF)y_k+(1-\gamma_k)V_ky_k$, where $y_k=(1-\beta_k)x_k+\beta_kV_kx_k$, we have

$$\begin{aligned}x_k-(I-\lambda_kF)x_k &= \gamma_k[(I-\lambda_kF)y_k-(I-\lambda_kF)x_k]+(1-\gamma_k)[V_ky_k-(I-\lambda_kF)x_k]\\ &= \gamma_k[(I-\lambda_kF)y_k-(I-\lambda_kF)x_k]+(1-\gamma_k)[V_ky_k-V_kx_k+V_kx_k-x_k+x_k-(I-\lambda_kF)x_k],\end{aligned}$$

which hence implies that

$$\lambda_kF(x_k)=x_k-(I-\lambda_kF)x_k=(I-\lambda_kF)y_k-(I-\lambda_kF)x_k+\frac{1-\gamma_k}{\gamma_k}(V_ky_k-V_kx_k)-\frac{1-\gamma_k}{\gamma_k}(I-V_k)x_k.$$

Consequently, for $x^*\in\mathcal{F}$ we conclude that

$$\begin{aligned}\lambda_k\langle F(x_k),j(x_k-x^*)\rangle &= \langle(I-\lambda_kF)y_k-(I-\lambda_kF)x_k,j(x_k-x^*)\rangle+\frac{1-\gamma_k}{\gamma_k}\langle(V_ky_k-V_kx_k),j(x_k-x^*)\rangle\\ &\quad-\frac{1-\gamma_k}{\gamma_k}\langle(I-V_k)x_k-(I-V_k)x^*,j(x_k-x^*)\rangle\\ &\leqslant \langle(I-\lambda_kF)y_k-(I-\lambda_kF)x_k,j(x_k-x^*)\rangle+\frac{1-\gamma_k}{\gamma_k}\langle(V_ky_k-V_kx_k),j(x_k-x^*)\rangle\\ &\leqslant \|(I-\lambda_kF)y_k-(I-\lambda_kF)x_k\|\|x_k-x^*\|+\frac{1-\gamma_k}{\gamma_k}\|V_ky_k-V_kx_k\|\|x_k-x^*\|\\ &\leqslant (1-\lambda_k\tau)\|y_k-x_k\|\|x_k-x^*\|+\frac{1-\gamma_k}{\gamma_k}\|y_k-x_k\|\|x_k-x^*\|\\ &= (\frac{1}{\gamma_k}-\lambda_k\tau)\|y_k-x_k\|\|x_k-x^*\|\\ &\leqslant \frac{\beta_k}{\gamma_k}\|V_kx_k-x_k\|\|x_k-x^*\|,\end{aligned}$$

which immediately leads to

$$\langle F(x_k),j(x_k-x^*)\rangle \leqslant \frac{\beta_k}{\lambda_k\gamma_k}\|V_kx_k-x_k\|\|x_k-x^*\|. \tag{3.12}$$

On the other hand, utilizing Lemma 2.10 (b) we get

$$\begin{aligned}\langle F(x_k),j(x_k-x^*)\rangle &= \langle(I-(I-F))x_k,j(x_k-x^*)\rangle\\ &= \|x_k-x^*\|^2+\langle(I-(I-F))x^*,j(x_k-x^*)\rangle+\langle(I-F)x^*-(I-F)x_k,j(x_k-x^*)\rangle\\ &\geqslant (1-\sqrt{\frac{1-\delta}{\zeta}})\|x_k-x^*\|^2+\langle F(x^*),j(x_k-x^*)\rangle\\ &= \tau\|x_k-x^*\|^2+\langle F(x^*),j(x_k-x^*)\rangle.\end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\|x_k - x^*\|^2 \leqslant \frac{1}{\tau}[\langle F(x^*), j(x^* - x_k)\rangle + \frac{\beta_k}{\lambda_k\gamma_k}\|V_k x_k - x_k\|\|x_k - x^*\|].$$

This together with (3.3), (3.11), and $\limsup_{k\to\infty} \frac{\beta_k}{\lambda_k\gamma_k} < \infty$, implies that $\mathrm{LIM}_k\|x_k - x^*\|^2 \leqslant 0$, that is,

$$\mathrm{LIM}_k\|x_k - x^*\|^2 = 0.$$

Step 5. We show that $\lim_{k\to\infty}\|x_k - x^*\|^2 = 0$. Indeed, from $\mathrm{LIM}_k\|x_k - x^*\|^2 = 0$, it follows that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges strongly to $x^* \in \mathcal{F}$. Noting that $\|x_k - Vx_k\| \leqslant \|x_k - V_k x_k\| + \|V_k x_k - Vx_k\|$, we deduce from (3.3) and (3.7) that

$$\lim_{k\to\infty}\|x_k - Vx_k\| = 0.$$

Now assume that there exists another subsequence $\{x_{m_i}\}$ of $\{x_k\}$ such that $x_{m_i} \to \hat{x} \in \mathrm{Fix}(V) = \mathcal{F}$ (because $\|x_k - Vx_k\| \to 0$ as $k \to \infty$). Then we have that $\|F(x_{m_i}) - F(\hat{x})\| \to 0$ as $i \to \infty$. We claim that $\hat{x}$ is a solution in $\mathcal{F}$ to the VI (3.2). Indeed, since for any $p \in \mathcal{F}$ the sequences $\{x_{m_i} - p\}$ and $\{F(x_{m_i})\}$ are bounded and j is norm to weak* uniformly continuous on bounded subsets of X, we obtain that as $i \to \infty$

$$|\langle F(x_{m_i}), j(x_{m_i} - p)\rangle - \langle F(\hat{x}), j(\hat{x} - p)\rangle| \leqslant \|F(x_{m_i}) - F(\hat{x})\|\|x_{m_i} - p\| + |\langle F(\hat{x}), j(x_{m_i} - p) - j(\hat{x} - p)\rangle| \to 0.$$

In addition, repeating the same arguments as those of (3.24), we obtain that for any $p \in \mathcal{F}$

$$\langle F(x_k), j(x_k - p)\rangle \leqslant \frac{\beta_k}{\lambda_k\gamma_k}\|V_k x_k - x_k\|\|x_k - p\|,$$

which immediately yields

$$\langle F(\hat{x}), j(\hat{x} - p)\rangle = \lim_{i\to\infty}\langle F(x_{m_i}), j(x_{m_i} - p)\rangle \leqslant 0.$$

That is, $\hat{x} \in \mathcal{F}$ is a solution of the VI (3.2) and hence $\hat{x} = x^*$ by uniqueness. Therefore, each cluster point of $\{x_k\}$ equals x^*, and so $\{x_k\}$ converges strongly to x^*, which is the unique solution of the VI (3.2) in $\mathcal{F}$. This completes the proof. □

Whenever $\beta_k = 0$ for all $k \geqslant 1$, Theorem 3.7 reduces to Buong and Phuong's Theorem 3.2 in [3].

Corollary 3.8 ([3, Theorem 3.2])**.** *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, let F be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$ and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings on X such that $\mathcal{F} := \bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i) \neq \emptyset$. Then, the sequence $\{x_k\}_{k=1}^{\infty}$, defined by* (2.11), (2.12), *and* (2.14) *with $\{\gamma_k\}_{k=1}^{\infty} \subset (0,1)$ and $\{\lambda_k\} \subset (0,1]$ such that $\gamma_k \to 0$, as $k \to \infty$, converges strongly to a unique solution $x^* \in \mathcal{F}$ to the VI* (3.2)*.*

Proof. Putting $\beta_k \equiv 0$ in Theorem 3.7, we know that the iterative scheme (3.1) reduces to (2.14). In this case, $\limsup_{k\to\infty}\beta_k/(\lambda_k\gamma_k) = 0 < \infty$. Thus, utilizing Theorem 3.7, we obtain the desired result. □

Theorem 3.9. *Let $X, F, \{T_i\}_{i=1}^{\infty}$ and $\mathcal{F}$ be as in Theorem* 3.7*. Let $\{V_k\}_{k=1}^{\infty}$ be defined by* (2.11) *and* (2.12)*. For arbitrarily given $x_1 \in X$, let $\{x_k\}_{k=1}^{\infty}$ be defined by*

$$y_k = (1-\beta_k)x_k + \beta_k V_k x_k, \quad x_{k+1} = (1-\gamma_k-\delta_k)x_k + \gamma_k(I-\lambda_k F)y_k + \delta_k V_k y_k, \quad \forall k \geqslant 1, \tag{3.14}$$

where $\{\lambda_k\}_{k=1}^{\infty} \subset (0,1]$, $\{\beta_k\}_{k=1}^{\infty} \subset [0,1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0,1)$, and $\{\delta_k\}_{k=1}^{\infty} \subset (0,1)$ such that $\gamma_k + \delta_k \leqslant 1$, for all $k \geqslant 1$. Assume that:

(i) $\lim_{k\to\infty}\gamma_k/\lambda_k = 0$, $\sum_{k=1}^{\infty}\gamma_k\lambda_k = \infty$ *and* $0 < \liminf_{k\to\infty}\delta_k \leqslant \limsup_{k\to\infty}\delta_k < 1$;

(ii) $\lim_{k\to\infty} |\beta_{k+1} - \beta_k| = 0$ *and* $\limsup_{k\to\infty} \beta_k < 1$.

Then there hold the following:

(I) $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$;

(II) *the sequence* $\{x_k\}_{k=1}^{\infty}$ *converges strongly to a unique solution* $x^* \in \mathcal{F}$ *to the VI* (3.2) *provided* $\lim\limits_{k\to\infty} \beta_k/\lambda_k = 0$ *and* $\delta_k \equiv \sigma$ *for some fixed* $\sigma \in (0, 1)$.

Proof.

Step 1. The proof of conclusion (I). First, we claim that $\{x_k\}_{k=1}^{\infty}$ is bounded. Indeed, take an arbitrarily given $p \in \mathcal{F}$. Then, observe that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|(1-\gamma_k-\delta_k)x_k + \gamma_k(I-\lambda_k F)y_k + \delta_k V_k y_k - p\| \\
&\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + \gamma_k\|(I-\lambda_k F)y_k - p\| + \delta_k\|V_k y_k - p\| \\
&= (1-\gamma_k-\delta_k)\|x_k - p\| + \gamma_k\|(I-\lambda_k F)y_k - (I-\lambda_k F)p - \lambda_k F(p)\| + \delta_k\|V_k y_k - p\| \\
&\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + \gamma_k[(1-\lambda_k\tau)\|y_k - p\| + \lambda_k\|F(p)\|] + \delta_k\|y_k - p\| \\
&\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + \gamma_k \max\{\|y_k - p\|, \frac{\|F(p)\|}{\tau}\} + \delta_k\|y_k - p\| \\
&\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + (\gamma_k+\delta_k) \max\{\|y_k - p\|, \frac{\|F(p)\|}{\tau}\},
\end{aligned}$$

and

$$\begin{aligned}
\|y_k - p\| = \|(1-\beta_k)(x_k - p) + \beta_k(V_k x_k - p)\| &\leqslant (1-\beta_k)\|x_k - p\| + \beta_k\|V_k x_k - p\| \\
&\leqslant (1-\beta_k)\|x_k - p\| + \beta_k\|x_k - p\| = \|x_k - p\|.
\end{aligned}$$

Combining these two inequalities, we have

$$\begin{aligned}
\|x_{k+1} - p\| &\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + (\gamma_k+\delta_k) \max\{\|y_k - p\|, \frac{\|F(p)\|}{\tau}\} \\
&\leqslant (1-\gamma_k-\delta_k)\|x_k - p\| + (\gamma_k+\delta_k) \max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\} \\
&\leqslant \max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\}.
\end{aligned}$$

By induction,

$$\|x_k - p\| \leqslant \max\{\|x_1 - p\|, \frac{\|F(p)\|}{\tau}\}, \quad \forall k \geqslant 1.$$

Hence it follows that $\{x_k\}$ is bounded, and so are $\{V_k x_k\}$, $\{y_k\}$, $\{V_k y_k\}$, and $\{F(y_k)\}$.

Second, we claim that $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$. Indeed, define a sequence $\{w_n\}$ by

$$x_{k+1} = \rho_k x_k + (1-\rho_k)w_k, \quad \forall k \geqslant 1,$$

where $\rho_k = 1 - \gamma_k - \delta_k$, for all $k \geqslant 1$. Then we have

$$\begin{aligned}
w_{k+1} - w_k &= \frac{x_{k+2} - \rho_{k+1}x_{k+1}}{1-\rho_{k+1}} - \frac{x_{k+1} - \rho_k x_k}{1-\rho_k} \\
&= \frac{\gamma_{k+1}(I-\lambda_{k+1}F)y_{k+1} + \delta_{k+1}V_{k+1}y_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k(I-\lambda_k F)y_k + \delta_k V_k y_k}{1-\rho_k} \\
&= \frac{\gamma_{k+1}(I-\lambda_{k+1}F)y_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k(I-\lambda_k F)y_k}{1-\rho_k} + \frac{\delta_{k+1}}{1-\rho_{k+1}}(V_{k+1}y_{k+1} - V_{k+1}y_k) \\
&\quad + V_{k+1}y_k - V_k y_k + \frac{\gamma_k}{1-\rho_k}V_k y_k - \frac{\gamma_{k+1}}{1-\rho_{k+1}}V_{k+1}y_k,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\|y_{k+1}-y_k\| &= \|(1-\beta_{k+1})x_{k+1}+\beta_{k+1}V_{k+1}x_{k+1}-(1-\beta_k)x_k-\beta_kV_kx_k\| \\ &\leqslant (1-\beta_{k+1})\|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|\|x_k\| \\ &\quad +\beta_{k+1}\|V_{k+1}x_{k+1}-V_kx_k\|+|\beta_{k+1}-\beta_k|\|V_kx_k\| \\ &\leqslant (1-\beta_{k+1})\|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|\|x_k\| \\ &\quad +\beta_{k+1}(\|V_{k+1}x_{k+1}-V_{k+1}x_k\|+\|V_{k+1}x_k-V_kx_k\|)+|\beta_{k+1}-\beta_k|\|V_kx_k\| \\ &\leqslant \|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|\|x_k\|+\beta_{k+1}\|V_{k+1}x_k-V_kx_k\|+|\beta_{k+1}-\beta_k|\|V_kx_k\| \\ &\leqslant \|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|\|x_k\|+\alpha_{k+1}\beta_{k+1}\|T_{k+1}x_k-x_k\|+|\beta_{k+1}-\beta_k|\|V_kx_k\| \\ &= \|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|(\|x_k\|+\|V_kx_k\|)+\alpha_{k+1}\beta_{k+1}\|T_{k+1}x_k-x_k\|.\end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$\begin{aligned}&\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\| \\ &\leqslant \frac{\gamma_{k+1}}{1-\rho_{k+1}}(\|(I-\lambda_{k+1}F)y_{k+1}\|+\|V_{k+1}y_k\|)+\frac{\gamma_k}{1-\rho_k}(\|(I-\lambda_kF)y_k\|+\|V_ky_k\|) \\ &\quad +\frac{\delta_{k+1}}{1-\rho_{k+1}}\|V_{k+1}y_{k+1}-V_{k+1}y_k\|+\|V_{k+1}y_k-V_ky_k\|-\|x_{k+1}-x_k\| \\ &\leqslant \frac{\gamma_{k+1}}{1-\rho_{k+1}}(\|(I-\lambda_{k+1}F)y_{k+1}\|+\|V_{k+1}y_k\|)+\frac{\gamma_k}{1-\rho_k}(\|(I-\lambda_kF)y_k\|+\|V_ky_k\|) \\ &\quad +\frac{\delta_{k+1}}{1-\rho_{k+1}}\{\|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|(\|x_k\|+\|V_kx_k\|) \\ &\quad +\alpha_{k+1}\beta_{k+1}\|T_{k+1}x_k-x_k\|\}+\alpha_{k+1}\|T_{k+1}y_k-y_k\|-\|x_{k+1}-x_k\| \\ &\leqslant \frac{\gamma_{k+1}}{1-\rho_{k+1}}(\|y_{k+1}\|+\|F(y_{k+1})\|+\|V_{k+1}y_k\|)+\frac{\gamma_k}{1-\rho_k}(\|y_k\|+\|F(y_k)\|+\|V_ky_k\|) \\ &\quad +\frac{\delta_{k+1}}{1-\rho_{k+1}}\{|\beta_{k+1}-\beta_k|(\|x_k\|+\|V_kx_k\|)+\alpha_{k+1}\|T_{k+1}x_k-x_k\|\}+\alpha_{k+1}\|T_{k+1}y_k-y_k\|.\end{aligned} \tag{3.17}$$

Thus, from (3.17), $\lim_{k\to\infty}\alpha_k=0$, and conditions (i), (ii), it follows that (noticing the boundedness of $\{x_k\}$ and $\{y_k\}$)

$$\limsup_{k\to\infty}(\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\|)\leqslant 0.$$

Since $\lim_{k\to\infty}\gamma_k=0$ and $0<\liminf_{k\to\infty}\delta_k\leqslant\limsup_{k\to\infty}\delta_k<1$, we have

$$0<\liminf_{k\to\infty}\rho_k\leqslant\limsup_{k\to\infty}\rho_k<1.$$

Thus by Lemma 3.5 we get $\lim_{k\to\infty}\|w_k-x_k\|=0$. Consequently,

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(1-\rho_k)\|w_k-x_k\|=0. \tag{3.18}$$

Step 2. The proof of conclusion (II).

Suppose that $\lim_{k\to\infty}\beta_k/\lambda_k=0$ and $\delta_k\equiv\sigma$ for some fixed $\sigma\in(0,1)$. In this case, conditions (i), (ii) are still satisfied. Let $\{z_n\}_{n=1}^{\infty}$ be defined by $z_n=\frac{1}{n}(I-F)z_n+(1-\frac{1}{n})Vz_n$. Then $\{z_n\}$ converges strongly to a unique solution $x^*\in\mathcal{F}$ to the VI (3.2). Observe that for each $k,n\geqslant 1$

$$\begin{aligned}\|x_{k+1}-Vz_n\| &= \|(1-\gamma_k-\sigma)(x_k-Vz_n)+\gamma_k((I-\lambda_kF)y_k-Vz_n)+\sigma(V_ky_k-Vz_n)\| \\ &\leqslant (1-\gamma_k-\sigma)\|x_k-Vz_n\|+\gamma_k\|(I-\lambda_kF)y_k-Vz_n\| \\ &\quad +\sigma(\|V_ky_k-V_kz_n\|+\|V_kz_n-Vz_n\|) \\ &\leqslant (1-\gamma_k-\sigma)\|x_k-Vz_n\|+\gamma_k\|(I-\lambda_kF)y_k-Vz_n\| \\ &\quad +\sigma(\|y_k-z_n\|+\|V_kz_n-Vz_n\|) \\ &\leqslant (1-\gamma_k-\sigma)\|x_k-Vz_n\|+\gamma_k\|(I-\lambda_kF)y_k-Vz_n\| \\ &\quad +\sigma(\|x_k-z_n\|+\|y_k-x_k\|+\|V_kz_n-Vz_n\|) \\ &= (1-\gamma_k-\sigma)\|x_k-Vz_n\|+\gamma_k\|(I-\lambda_kF)y_k-Vz_n\| \\ &\quad +\sigma(\|x_k-z_n\|+\beta_k\|V_kx_k-x_k\|+\|V_kz_n-Vz_n\|) \\ &\leqslant \epsilon_k+(1-\sigma)\|x_k-Vz_n\|+\sigma\|x_k-z_n\|,\end{aligned} \tag{3.19}$$

where $\epsilon_k = \gamma_k\|(I-\lambda_k F)y_k - Vz_n\| + \sigma(\beta_k\|V_k x_k - x_k\| + \|V_k z_n - Vz_n\|)$. Repeating the same arguments as those of (3.7) in the proof of Theorem 3.7, we obtain $\lim_{k\to\infty}\|V_k z_n - Vz_n\| = 0$. Since $\lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\beta_k = 0$, we know that $\epsilon_k \to 0$ as $k\to\infty$. From (3.19) we get

$$\begin{aligned}\|x_{k+1} - Vz_n\|^2 &\leqslant ((1-\sigma)\|x_k - Vz_n\| + \sigma\|x_k - z_n\|)^2 \\ &\quad + \epsilon_k[2((1-\sigma)\|x_k - Vz_n\| + \sigma\|x_k - z_n\|) + \epsilon_k] \\ &= (1-\sigma)^2\|x_k - Vz_n\|^2 + \sigma^2\|x_k - z_n\|^2 + 2\sigma(1-\sigma)\|x_k - Vz_n\|\|x_k - z_n\| + r_k \\ &\leqslant (1-\sigma)^2\|x_k - Vz_n\|^2 + \sigma^2\|x_k - z_n\|^2 + \sigma(1-\sigma)(\|x_k - Vz_n\|^2 + \|x_k - z_n\|^2) + r_k \\ &= (1-\sigma)\|x_k - Vz_n\|^2 + \sigma\|x_k - z_n\|^2 + r_k,\end{aligned} \tag{3.20}$$

where $r_k = \epsilon_k[2((1-\sigma)\|x_k - Vz_n\| + \sigma\|x_k - z_n\|) + \epsilon_k] \to 0$ as $k\to\infty$.

Repeating the same arguments as those of (3.8) in the proof of Theorem 3.7, we obtain $\mathrm{LIM}_k\|x_k - Vz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2$. For any Banach limit LIM, from (3.20) we derive

$$\mathrm{LIM}_k\|x_k - Vz_n\|^2 = \mathrm{LIM}_k\|x_{k+1} - Vz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2.$$

Observe that $x_k - z_n = \frac{1}{n}(x_k - (I-F)z_n) + (1-\frac{1}{n})(x_k - Vz_n)$. By the same arguments as those of (3.11) in the proof of Theorem 3.7, we can get

$$\mathrm{LIM}_k\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0. \tag{3.21}$$

On the other hand, from (3.18), it follows that

$$\lim_{k\to\infty}|\langle F(x^*), j(x^* - x_{k+1})\rangle - \langle F(x^*), j(x^* - x_k)\rangle| = 0,$$

which together with (3.21) and Lemma 2.8, yields

$$\limsup_{k\to\infty}\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0. \tag{3.22}$$

Finally we show that $x_k \to x^*$ as $k\to\infty$. From Lemma 2.7 (i) and (3.14) with $\delta_k = \sigma$, we have

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|(1-\gamma_k-\sigma)(x_k - x^*) + \gamma_k((I-\lambda_k F)y_k - x^*) + \sigma(V_k y_k - x^*)\|^2 \\ &\leqslant \|(1-\gamma_k-\sigma)(x_k - x^*) + \sigma(V_k y_k - x^*)\|^2 + 2\gamma_k\langle (I-\lambda_k F)y_k - x^*, j(x_{k+1} - x^*)\rangle \\ &\leqslant [(1-\gamma_k-\sigma)\|x_k - x^*\| + \sigma\|y_k - x^*\|]^2 + 2\gamma_k\langle (I-\lambda_k F)y_k - x^*, j(x_{k+1} - x^*)\rangle \\ &\leqslant (1-\gamma_k)^2\|x_k - x^*\|^2 + 2\gamma_k[\langle (I-\lambda_k F)y_k - (I-\lambda_k F)x_k, j(x_{k+1} - x^*)\rangle \\ &\quad + \langle (I-\lambda_k F)x_k - (I-\lambda_k F)x^*, j(x_{k+1} - x^*)\rangle + \langle (I-\lambda_k F)x^* - x^*, j(x_{k+1} - x^*)\rangle] \\ &\leqslant (1-\gamma_k)^2\|x_k - x^*\|^2 + 2\gamma_k[(1-\lambda_k\tau)\|y_k - x_k\|\|x_{k+1} - x^*\| \\ &\quad + (1-\lambda_k\tau)\|x_k - x^*\|\|x_{k+1} - x^*\| + \langle (I-\lambda_k F)x^* - x^*, j(x_{k+1} - x^*)\rangle] \\ &\leqslant (1-\gamma_k)^2\|x_k - x^*\|^2 + \gamma_k(1-\lambda_k\tau)[\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|^2] \\ &\quad + 2\gamma_k[(1-\lambda_k\tau)\beta_k\|V_k x_k - x_k\|\|x_{k+1} - x^*\| + \lambda_k\langle F(x^*), j(x^* - x_{k+1})\rangle],\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leqslant \frac{(1-\gamma_k)^2 + \gamma_k(1-\lambda_k\tau)}{1-\gamma_k(1-\lambda_k\tau)}\|x_k - x^*\|^2 \\ &\quad + \frac{2\gamma_k}{1-\gamma_k(1-\lambda_k\tau)}[(1-\lambda_k\tau)\beta_k\|V_k x_k - x_k\|\|x_{k+1} - x^*\| + \lambda_k\langle F(x^*), j(x^* - x_{k+1})\rangle].\end{aligned} \tag{3.23}$$

Observe that for all $k \geqslant 1$

$$\frac{(1-\gamma_k)^2 + \gamma_k(1-\lambda_k\tau)}{1-\gamma_k(1-\lambda_k\tau)} = \frac{1-(1-\lambda_k\tau)\gamma_k + \gamma_k - 2\gamma_k[1-(1-\lambda_k\tau)] + \gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)}$$

$$\begin{aligned}&= 1 - \frac{2\gamma_k[1-(1-\lambda_k\tau)]}{1-\gamma_k(1-\lambda_k\tau)} + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)}\\ &\leqslant 1 - 2\gamma_k[1-(1-\lambda_k\tau)] + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)}\\ &= 1 - 2\gamma_k\lambda_k\tau + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)}.\end{aligned}$$

Then it follows from (3.23) that

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\leqslant (1-2\gamma_k\lambda_k\tau)\|x_k-x^*\|^2 + \frac{2\gamma_k}{1-\gamma_k(1-\lambda_k\tau)}[\frac{\gamma_k}{2}\|x_k-x^*\|^2\\ &\quad + (1-\lambda_k\tau)\beta_k\|V_kx_k-x_k\|\|x_{k+1}-x^*\| + \lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle]\\ &= (1-2\gamma_k\lambda_k\tau)\|x_k-x^*\|^2 + 2\gamma_k\lambda_k\tau\cdot\frac{1}{\tau-\tau\gamma_k(1-\lambda_k\tau)}[\frac{\gamma_k}{2\lambda_k}\|x_k-x^*\|^2\\ &\quad + \frac{\beta_k}{\lambda_k}\cdot(1-\lambda_k\tau)\|V_kx_k-x_k\|\|x_{k+1}-x^*\| + \langle F(x^*), j(x^*-x_{k+1})\rangle].\end{aligned} \tag{3.24}$$

Since $\lim_{k\to\infty}\gamma_k/\lambda_k = \lim_{k\to\infty}\beta_k/\lambda_k = 0$, we deduce from (3.22) that

$$\limsup_{k\to\infty}\frac{\frac{\gamma_k}{2\lambda_k}\|x_k-x^*\|^2 + \frac{\beta_k}{\lambda_k}\cdot(1-\lambda_k\tau)\|V_kx_k-x_k\|\|x_{k+1}-x^*\| + \langle F(x^*), j(x^*-x_{k+1})\rangle}{\tau-\tau\gamma_k(1-\lambda_k\tau)} \leqslant 0.$$

Noticing $\sum_{k=1}^{\infty}\gamma_k\lambda_k = \infty$, we get $\sum_{k=1}^{\infty}2\gamma_k\lambda_k\tau = \infty$. Therefore, according to Lemma 3.6 we conclude from (3.24) that $\lim_{k\to\infty}\|x_k-x^*\| = 0$. This completes the proof. □

Whenever $\beta_k = 0$ for all $k \geqslant 1$, Theorem 3.9 reduces to the following.

Corollary 3.10. *Let* X, F, $\{T_i\}_{i=1}^{\infty}$, *and* $\mathcal{F}$ *be as in Theorem 3.7. Let* $\{V_k\}_{k=1}^{\infty}$ *be defined by* (2.11) *and* (2.12). *For arbitrarily given* $x_1 \in X$, *let* $\{x_k\}_{k=1}^{\infty}$ *be defined by*

$$x_{k+1} = (1-\gamma_k-\delta_k)x_k + \gamma_k(I-\lambda_kF)x_k + \delta_kV_kx_k, \quad \forall k \geqslant 1, \tag{3.25}$$

where $\{\lambda_k\}_{k=1}^{\infty} \subset (0,1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0,1)$, *and* $\{\delta_k\}_{k=1}^{\infty} \subset (0,1)$ *such that* $\gamma_k + \delta_k \leqslant 1$, *for all* $k \geqslant 1$. *Assume that* $\lim_{k\to\infty}\gamma_k/\lambda_k = 0$, $\sum_{k=1}^{\infty}\gamma_k\lambda_k = \infty$, *and* $0 < \liminf_{k\to\infty}\delta_k \leqslant \limsup_{k\to\infty}\delta_k < 1$. *Then there hold the following:*

(I) $\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0$;

(II) *the sequence* $\{x_k\}_{k=1}^{\infty}$ *converges strongly to a unique solution* $x^* \in \mathcal{F}$ *to the VI* (3.2) *provided* $\delta_k \equiv \sigma$ *for some fixed* $\sigma \in (0,1)$.

Proof. Putting $\beta_k \equiv 0$ in Theorem 3.9, we know that the iterative scheme (3.14) reduces to (3.25). In this case, we have that $\lim_{k\to\infty}|\beta_{k+1}-\beta_k| = 0$ and $\limsup_{k\to\infty}\beta_k < 1$. Thus the condition (ii) in Theorem 3.9 is satisfied. In the meantime, it is easy to see that $\lim_{k\to\infty}\beta_k/\lambda_k = 0$. Consequently, utilizing Theorem 3.9, we derive the desired result. □

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