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L_p-dual geominimal surface areas for the general L_p-intersection bodies

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Abstract

For $0 < p < 1$, Haberl and Ludwig defined the notions of symmetric and asymmetric L_p-intersection bodies. Recently, Wang and Li introduced the general L_p-intersection bodies. In this paper, we give the L_p-dual geominimal surface area forms for the extremum values and Brunn-Minkowski type inequality of general L_p-intersection bodies. Further, combining with the L_p-dual geominimal surface areas, we consider Busemann-Petty type problem for general L_p-intersection bodies.

Keywords: General L_p-intersection body, L_p-dual geominimal surface area, extremum value, Brunn-Minkowski inequality, Busemann-Petty problem.
2010 MSC: 52A20, 52A40, 52A39.

1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let $\mathcal{S}^n_o$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

The notion of intersection body was introduced by Lutwak (see [14]): For $K \in \mathcal{S}^n_o$, the intersection body, IK, of K is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u, i.e., for all $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp),$$

where V_{n-1} denotes $(n-1)$-dimensional volume.

In 2006, Haberl and Ludwing ([5]) introduced the L_p-intersection body as follows: For $K \in \mathcal{S}^n_o$, $0 < p < 1$, the L_p-intersection body, I_pK, of K is the origin-symmetric star body, whose radial function is defined by

$$\rho^p_{I_pK}(u) = \frac{1}{2}\int_K |u \cdot x|^{-p} dx = \frac{1}{2(n-p)}\int_{S^{n-1}} |u \cdot v|^{-p} \rho^{n-p}_K(v) dS(v), \tag{1.1}$$

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for all $u \in S^{n-1}$. For the convenient of this paper, here we add a coefficient 1/2 in (1.1).

Meanwhile, they ([5]) gave the following notion of asymmetric L_p-intersection body. For $K \in \mathcal{S}_o^n$, $0 < p < 1$, the asymmetric L_p-intersection body, $I_p^+ K$, of K is defined by

$$\rho_{I_p^+ K}^p(u) = \int_{K \cap u^+} |u \cdot x|^{-p} dx, \tag{1.2}$$

for all $u \in \mathcal{S}^{n-1}$, where $u^+ = \{x : u \cdot x \geqslant 0, x \in \mathbb{R}^n\}$ and $u \cdot x$ denotes the standard inner product of u and x. From (1.2), it follows that for all $u \in \mathcal{S}^{n-1}$,

$$\rho_{I_p^+ K}^p(u) = \frac{1}{n-p} \int_{S^{n-1} \bigcap u^+} |u \cdot v|^{-p} \rho_K^{n-p}(v) dS(v). \tag{1.3}$$

Further, the authors ([5]) also defined that

$$I_p^- K = I_p^+(-K).$$

This together with (1.3) yields that

$$\rho_{I_p^- K}^p(u) = \rho_{I_p^+(-K)}^p(u) = \frac{1}{n-p} \int_{S^{n-1} \bigcap u^+} |u \cdot v|^{-p} \rho_{-K}^{n-p}(v) dS(v). \tag{1.4}$$

Recently, Wang and Li ([26, 27]) introduced the notion of general L_p-intersection body with a parameter τ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$, the general L_p-intersection body, $I_p^\tau K \in \mathcal{S}_o^n$, of K is defined by

$$\rho_{I_p^\tau K}^p(u) = f_1(\tau) \rho_{I_p^+ K}^p(u) + f_2(\tau) \rho_{I_p^- K}^p(u), \tag{1.5}$$

for all $u \in \mathcal{S}^{n-1}$, where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \quad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.6}$$

Obviously, (1.6) deduces

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau), \tag{1.7}$$

$$f_1(\tau) + f_2(\tau) = 1. \tag{1.8}$$

In the meantime, they ([26, 27]) also showed that if $\tau = 0$, then $I_p^0 K = I_p K$ and

$$\rho_{I_p K}^p(u) = \frac{1}{2} \rho_{I_p^+ K}^p(u) + \frac{1}{2} \rho_{I_p^- K}^p(u), \tag{1.9}$$

for all $u \in \mathcal{S}^{n-1}$.

From (1.4), (1.5) and (1.7), we easily obtain for $\tau \in [-1, 1]$ (see [27]),

$$I_p^{-\tau} K = I_p^\tau(-K) = -I_p^\tau K. \tag{1.10}$$

For the general L_p-intersection bodies, Wang and Li in [27] obtained the following extremum values of volume and a Brunn-Minkowski type inequality with respect to L_q $(q > 0)$ radial combinations of star bodies, respectively.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$, then*

$$V(I_p K) \leqslant V(I_p^\tau K) \leqslant V(I_p^\pm K).$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

Theorem 1.2. *If* $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $0 < q < n-p$, *then for* $\tau \in [-1, 1]$,

$$V(I_p^\tau(K\tilde{+}_q L))^{\frac{pq}{n(n-p)}} \leqslant V(I_p^\tau K)^{\frac{pq}{n(n-p)}} + V(I_p^\tau L)^{\frac{pq}{n(n-p)}},$$

with equality if and only if K *and* L *are dilates.*

Here $K\tilde{+}_q L$ denotes the L_q $(q > 0)$ radial combination of star bodies K and L.

Further, Wang and Li in [26] researched the Busemann-Petty type problem for general L_p-intersection bodies, they respectively gave an affirmative and a negative form as follows:

Theorem 1.3. *Let* $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $\tau \in [-1, 1]$. *If* K *is a general* L_p*-intersection body, then*

$$I_p^\tau K \subseteq I_p^\tau L,$$

implies

$$V(K) \leqslant V(L).$$

The equality holds only if $K = L$.

Theorem 1.4. *Let* $K \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $\tau \in (-1, 1)$. *If* K *is not origin-symmetric, then there exists* $L \in \mathcal{S}_o^n$, *such that*

$$I_p^\tau K \subset I_p^\tau L.$$

But

$$V(K) > V(L).$$

The general L_p-intersection bodies belong to a new and rapidly evolving asymmetric L_p-Brunn-Minkowski theory that has its own origin in the work of Ludwig, Haberl and Schuster (see [4–7, 12, 13]). For the further researches of asymmetric L_p-Brunn-Minkowski theory, also see [1, 8, 11, 17–20, 22, 23, 25–32, 34, 35, 38].

Associated with L_p-mixed volumes, Lutwak ([16]) introduced the notion of L_p-geominimal surface area. For $K \in \mathcal{K}_o^n$ and $p \geqslant 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}.$$

Here $V_p(M, N)$ denotes L_p-mixed volume of $M, N \in \mathcal{K}_o^n$ (see [15, 16]). Obviously, if $p = 1$, $G_p(K)$ is just the geominimal surface area $G(K)$ which was given by Petty (see [21]). Some affine isoperimetric inequalities related to the L_p-geominimal surface areas can be found in [36, 37, 39–41].

Together with the L_p-dual mixed volumes, Wan and Wang ([24]) gave the notion of L_p-dual geominimal surface area. For $K \in \mathcal{S}_o^n$ and $p > 0$, the L_p-dual geominimal surface area, $\widetilde{G}_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} \widetilde{G}_p(K) = \sup\{n\widetilde{V}_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\}. \tag{1.11}$$

Here $\widetilde{V}_p(M, N)$ denotes L_p-dual mixed volume of $M, N \in \mathcal{S}_o^n$. For the studies of L_p-dual geominimal surface areas, also see [2, 10, 33].

In this paper, associated with the L_p-dual geominimal surface area, we continuously study general L_p-intersection bodies. First, corresponding to Theorem 1.1, we give L_p-dual geominimal surface area forms for the extremum values of general L_p-intersection bodies.

Theorem 1.5. *If* $K \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $\tau \in [-1, 1]$, *then*

$$\widetilde{G}_p(I_p K) \leqslant \widetilde{G}_p(I_p^\tau K) \leqslant \widetilde{G}_p(I_p^\pm K). \tag{1.12}$$

If K *is not origin-symmetric, equality holds in the left inequality if and only if* $\tau = 0$ *and equality holds in the right inequality if and only if* $\tau = \pm 1$.

Next, the L_p-dual geominimal surface area version of Theorem 1.2 is established as follows:

Theorem 1.6. *If* $K, L \in \mathcal{S}_o^n$, $n \geqslant 2$, $0 < p < 1$, $0 < q < n-p$ *and* $\tau \in [-1,1]$, *then*

$$\widetilde{G}_p(I_p^\tau(K \tilde{+}_q L))^{\frac{pq}{(n-p)^2}} \leqslant \widetilde{G}_p(I_p^\tau K)^{\frac{pq}{(n-p)^2}} + \widetilde{G}_p(I_p^\tau L)^{\frac{pq}{(n-p)^2}}, \tag{1.13}$$

with equality if and only if K *and* L *are dilates.*

Further, similar to Theorems 1.3–1.4, we give an affirmative and a negative form of the L_p-dual geominimal surface area for the Busemann-Petty type problems of general L_p-intersection bodies. Let $\mathcal{Z}_p^n$ denote the set of general L_p-intersection bodies. If $M \in \mathcal{Z}_p^n$ in (1.11), then we write $\widetilde{G}_p^\circ(K)$ by

$$\omega_n^{\frac{p}{n}} \widetilde{G}_p^\circ(K) = \sup\{n\widetilde{V}_p(K, M)V(M^*)^{\frac{p}{n}} : M \in \mathcal{Z}_p^n\}. \tag{1.14}$$

Here, associated with (1.14), we obtain an affirmative form of the Busemann-Petty type problem for general L_p-intersection bodies.

Theorem 1.7. *If* $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $\tau \in [-1,1]$, *then*

$$I_p^\tau K \subseteq I_p^\tau L,$$

implies

$$\widetilde{G}_p^\circ(K) \leqslant \widetilde{G}_p^\circ(L).$$

The equality holds only if $K = L$.

Finally, according to (1.11), we give a negative form of the Busemann-Petty type problem for general L_p-intersection bodies as follows:

Theorem 1.8. *Let* $K \in \mathcal{S}_o^n$, $0 < p < 1$ *and* $\tau \in (-1,1)$. *If* K *is not origin-symmetric, then there exists* $L \in \mathcal{S}_o^n$, *such that*

$$I_p^\tau K \subset I_p^\tau L.$$

But

$$\widetilde{G}_p(K) > \widetilde{G}_p(L).$$

2. L_p-dual mixed volumes and general L_p-dual Blaschke bodies

In order to complete the proofs of Theorems 1.5–1.8, we will require the following two notions.

2.1. L_p-*dual mixed volumes*

If K is a compact star shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [3])

$$\rho(K, u) = \max\{\lambda \geqslant 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}.$$

For $K, L \in S_o^n$, $p > 0$ and $\lambda, \mu \geqslant 0$ (not both zero), the L_p-radial combination, $\lambda \cdot K \tilde{+}_p \mu \cdot L \in S_o^n$, of K and L is defined by (see [4])

$$\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p, \tag{2.1}$$

where $\lambda \cdot K$ denotes the L_p-radial scalar multiplication, and we easily know $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

Associated with the L_p-radial combinations of star bodies, Haberl ([4]) introduced the notion of L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_p(K, L)$, of K and L is defined by

$$\frac{n}{p}\widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \tilde{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

From above definition, the integral representation of the L_p-dual mixed volume can be given by (see [26])

$$\widetilde{V}_p(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p}(u)\rho_L^p(u)dS(u), \tag{2.2}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1} .

2.2. General L_p-dual Blaschke bodies

The notion of dual Blaschke combination was given by Lutwak (see [14]). For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geqslant 0$ (not both zero), $n \geqslant 2$, the dual Blaschke combination, $\lambda \circ K \oplus \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by

$$\rho(\lambda \circ K \oplus \mu \circ L, \cdot)^{n-1} = \lambda\rho(K, \cdot)^{n-1} + \mu\rho(L, \cdot)^{n-1},$$

where the operation "$\oplus$" is called dual Blaschke addition and $\lambda \circ K$ denotes dual Blaschke scalar multiplication.

In 2015, Wang and Wang ([29]) introduced the notion of L_p-dual Blaschke combination as follows: For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geqslant 0$ (not both zero), $n > p > 0$, the L_p-dual Blaschke combination, $\lambda \circ K \oplus_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by

$$\rho(\lambda \circ K \oplus_p \mu \circ L, \cdot)^{n-p} = \lambda\rho(K, \cdot)^{n-p} + \mu\rho(L, \cdot)^{n-p}, \tag{2.3}$$

where the operation "$\oplus_p$" is called L_p-dual Blaschke addition and $\lambda \circ K = \lambda^{\frac{1}{n-p}}$.

Let $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (2.3), then the L_p-dual Blaschke body, $\overline{\nabla}_p K$, of $K \in \mathcal{S}_o^n$ is given by

$$\overline{\nabla}_p K = \frac{1}{2} \circ K \oplus_p \frac{1}{2} \circ (-K). \tag{2.4}$$

According to (2.3), Wang and Li in [26] (also see [29]) defined general L_p-dual Blaschke bodies as follows: For $K \in \mathcal{S}_o^n$, $n > p > 0$ and $\tau \in [-1, 1]$, the general L_p-dual Blaschke body, $\overline{\nabla}_p^\tau K$, of K is defined by

$$\rho(\overline{\nabla}_p^\tau K, \cdot)^{n-p} = f_1(\tau)\rho(K, \cdot)^{n-p} + f_2(\tau)\rho(-K, \cdot)^{n-p}, \tag{2.5}$$

where $f_1(\tau)$, $f_2(\tau)$ satisfy (1.6).

Associated with the definition of L_p-dual Blaschke combination, it easily follows that

$$\overline{\nabla}_p^\tau K = f_1(\tau) \circ K \oplus_p f_2(\tau) \circ (-K). \tag{2.6}$$

Besides, by (1.6), (2.4) and (2.6), we may get that if $\tau = 0$, then $\overline{\nabla}_p^0 K = \overline{\nabla}_p K$, if $\tau = \pm 1$, then

$$\overline{\nabla}_p^{+1} K = K, \quad \overline{\nabla}_p^{-1} K = -K. \tag{2.7}$$

3. Proofs of Theorems 1.5–1.6

In this section, we shall complete the proofs of Theorems 1.5–1.6. The proof of Theorem 1.5 requires the following lemmas.

Lemma 3.1. *If $K, L \in \mathcal{S}_o^n$, $0 < p < \frac{n}{2}$ and $\lambda, \mu \geqslant 0$ (not both zero), then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(\lambda \cdot K \tilde{+}_p \mu \cdot L, Q)^{\frac{p}{n-p}} \leqslant \lambda \widetilde{V}_p(K, Q)^{\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q)^{\frac{p}{n-p}}, \tag{3.1}$$

with equality if and only if K and L are dilates.

Proof. Since $0 < p < \frac{n}{2}$, thus $\frac{n-p}{p} > 1$. Hence by (2.1), (2.2) and the Minkowski integral inequality (see [9]), we have for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{V}_p(\lambda \cdot K \tilde{+}_p \mu \cdot L, Q)^{\frac{p}{n-p}} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, u)^{n-p} \rho(Q, u)^p dS(u)\right]^{\frac{p}{n-p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, u)^p \rho(Q, u)^{\frac{p^2}{n-p}}\right)^{\frac{n-p}{p}} dS(u)\right]^{\frac{p}{n-p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left((\lambda\rho(K, u)^p + \mu\rho(L, u)^p)\rho(Q, u)^{\frac{p^2}{n-p}}\right)^{\frac{n-p}{p}} dS(u)\right]^{\frac{p}{n-p}}
\end{aligned}$$

$$\leqslant \lambda\left[\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-p}\rho(Q,u)^p dS(u)\right]^{\frac{p}{n-p}} + \mu\left[\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-p}\rho(Q,u)^p dS(u)\right]^{\frac{p}{n-p}}$$
$$\leqslant \lambda\widetilde{V}_p(K,Q)^{\frac{p}{n-p}} + \mu\widetilde{V}_p(L,Q)^{\frac{p}{n-p}}.$$

According to the equality condition of Minkowski integral inequality, we see that equality holds in (3.1) if and only if K and L are dilates. □

Lemma 3.2. *If $K, L \in \mathcal{S}_o^n$, $0 < p < \frac{n}{2}$ and $\lambda, \mu \geqslant 0$ (not both zero), then*

$$\widetilde{G}_p(\lambda\cdot K\tilde{+}_p\mu\cdot L)^{\frac{p}{n-p}} \leqslant \lambda\widetilde{G}_p(K)^{\frac{p}{n-p}} + \mu\widetilde{G}_p(L)^{\frac{p}{n-p}}, \tag{3.2}$$

with equality if and only if K and L are dilates.

Proof. From definition (1.11) and inequality (3.1), and notice $\frac{p}{n-p} > 0$, we have

$$\begin{aligned}
[\omega_n^{\frac{p}{n}}\widetilde{G}_p(\lambda\cdot K\tilde{+}_p\mu\cdot L)]^{\frac{p}{n-p}} &= \sup\{[n\widetilde{V}_p(\lambda\cdot K\tilde{+}_p\mu\cdot L, Q)V(Q^*)^{\frac{p}{n}}]^{\frac{p}{n-p}} : Q\in\mathcal{K}_{os}^n\}\\
&= \sup\{[n\widetilde{V}_p(\lambda\cdot K\tilde{+}_p\mu\cdot L, Q)]^{\frac{p}{n-p}}[V(Q^*)^{\frac{p}{n}}]^{\frac{p}{n-p}} : Q\in\mathcal{K}_{os}^n\}\\
&\leqslant \sup\{[\lambda(n\widetilde{V}_p(K,Q))^{\frac{p}{n-p}} + \mu(n\widetilde{V}_p(L,Q))^{\frac{p}{n-p}}][V(Q^*)^{\frac{p}{n}}]^{\frac{p}{n-p}} : Q\in\mathcal{K}_{os}^n\}\\
&\leqslant \sup\{\lambda(n\widetilde{V}_p(K,Q))^{\frac{p}{n-p}}[V(Q^*)^{\frac{p}{n}}]^{\frac{p}{n-p}} : Q\in\mathcal{K}_{os}^n\}\\
&\quad + \sup\{\mu(n\widetilde{V}_p(L,Q))^{\frac{p}{n-p}}[V(Q^*)^{\frac{p}{n}}]^{\frac{p}{n-p}} : Q\in\mathcal{K}_{os}^n\}\\
&= \lambda[\omega_n^{\frac{p}{n}}\widetilde{G}_p(K)]^{\frac{p}{n-p}} + \mu[\omega_n^{\frac{p}{n}}\widetilde{G}_p(L)]^{\frac{p}{n-p}}.
\end{aligned}$$

Thus

$$\widetilde{G}_p(\lambda\cdot K\tilde{+}_p\mu\cdot L)^{\frac{p}{n-p}} \leqslant \lambda\widetilde{G}_p(K)^{\frac{p}{n-p}} + \mu\widetilde{G}_p(L)^{\frac{p}{n-p}}.$$

According to the equality condition of (3.1), we see that equality holds in (3.2) if and only if K and L are dilates. □

Lemma 3.3 ([27]). *If $K \in \mathcal{S}_o^n$ and $0 < p < 1$, then $I_p^+K = I_p^-K$ if and only if K is origin-symmetric.*

Lemma 3.4 ([27]). *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$ and $\tau \neq 0$, then $I_p^\tau K = I_p^{-\tau}K$ if and only if K is origin-symmetric.*

Proof of Theorem 1.5. Since $K \in \mathcal{S}_o^n$, $0 < p < 1$, by (1.5) and (3.2), we have

$$\begin{aligned}
\widetilde{G}_p(I_p^\tau K)^{\frac{p}{n-p}} &= \widetilde{G}_p(f_1(\tau)\cdot I_p^+K\tilde{+}_p f_2(\tau)\cdot I_p^-K)^{\frac{p}{n-p}}\\
&\leqslant f_1(\tau)\widetilde{G}_p(I_p^+K)^{\frac{p}{n-p}} + f_2(\tau)\widetilde{G}_p(I_p^-K)^{\frac{p}{n-p}}.
\end{aligned} \tag{3.3}$$

Since $I_p^+K = -I_p^-K$ and notice that $Q \in \mathcal{K}_{os}^n$ implies $\rho(Q,u) = \rho(-Q,u) = \rho(Q,-u)$ for all $u \in S^{n-1}$, thus by (2.2) we get that

$$\widetilde{V}_p(I_p^-K, Q) = \widetilde{V}_p(-I_p^+K, Q) = \widetilde{V}_p(I_p^+K, Q).$$

Therefore, from definition (1.11), it follows that

$$\widetilde{G}_p(I_p^+K) = \widetilde{G}_p(I_p^-K). \tag{3.4}$$

Combining with (3.3), (3.4), and (1.8), we can get

$$\widetilde{G}_p(I_p^\tau K)^{\frac{p}{n-p}} \leqslant \widetilde{G}_p(I_p^\pm K)^{\frac{p}{n-p}},$$

i.e.,

$$\widetilde{G}_p(I_p^\tau K) \leqslant \widetilde{G}_p(I_p^\pm K). \tag{3.5}$$

According to the equality condition of inequality (3.2), we know that equality holds in (3.5) if and only if I_p^+K and I_p^-K are dilates. Since $I_p^+K = -I_p^-K$, this means $I_p^+K = I_p^-K$. Hence from Lemma 3.3, we see that if K is not origin-symmetric, then equality holds in (3.5) if and only if $\tau = \pm 1$.

Now, we prove the left inequality of (1.12). By (2.1), (1.5), (1.7), and (1.8), we have

$$\begin{aligned}\rho(I_p^\tau K,\cdot)^p + \rho(I_p^{-\tau}K,\cdot)^p &= f_1(\tau)\rho(I_p^+K,\cdot)^p + f_2(\tau)\rho(I_p^-K,\cdot)^p + f_1(-\tau)\rho(I_p^+K,\cdot)^p + f_2(-\tau)\rho(I_p^-K,\cdot)^p\\ &= f_1(\tau)\rho(I_p^+K,\cdot)^p + f_2(\tau)\rho(I_p^-K,\cdot)^p + f_2(\tau)\rho(I_p^+K,\cdot)^p + f_1(\tau)\rho(I_p^-K,\cdot)^p\\ &= \rho(I_p^+K,\cdot)^p + \rho(I_p^-K,\cdot)^p.\end{aligned} \tag{3.6}$$

Therefore, (3.6) can be written as

$$\frac{1}{2}\rho(I_p^\tau K,\cdot)^p + \frac{1}{2}\rho(I_p^{-\tau}K,\cdot)^p = \frac{1}{2}\rho(I_p^+K,\cdot)^p + \frac{1}{2}\rho(I_p^-K,\cdot)^p.$$

This together with (1.9) yields

$$\rho(I_pK,\cdot)^p = \frac{1}{2}\rho(I_p^\tau K,\cdot)^p + \frac{1}{2}\rho(I_p^{-\tau}K,\cdot)^p,$$

so by (2.1) we have

$$I_pK = \frac{1}{2}\cdot I_p^\tau K \tilde{+}_p \frac{1}{2}\cdot I_p^{-\tau}K.$$

Thus from inequality (3.2), we obtain

$$\begin{aligned}\widetilde{G}_p(I_pK)^{\frac{p}{n-p}} &= \widetilde{G}_p\left(\frac{1}{2}\cdot I_p^\tau K \tilde{+}_p \frac{1}{2}\cdot I_p^{-\tau}K\right)^{\frac{p}{n-p}}\\ &\leqslant \frac{1}{2}\widetilde{G}_p(I_p^\tau K)^{\frac{p}{n-p}} + \frac{1}{2}\widetilde{G}_p(I_p^{-\tau}K)^{\frac{p}{n-p}}.\end{aligned} \tag{3.7}$$

Due to $I_p^{-\tau}K = -I_p^\tau K$ by (1.10), similar to the proof of (3.4), we have

$$\widetilde{G}_p(I_p^\tau K) = \widetilde{G}_p(-I_p^\tau K). \tag{3.8}$$

From (3.7) and (3.8), we deduce

$$\widetilde{G}_p(I_pK) \leqslant \widetilde{G}_p(I_p^\tau K). \tag{3.9}$$

Using $I_p^\tau K = -I_p^{-\tau}K$ and the equality condition of inequality (3.2), we know that equality holds in (3.9) if and only if $I_p^\tau K = I_p^{-\tau}K$. By Lemma 3.4, we see that if K is not origin-symmetric, then equality holds in (3.9) if and only if $\tau = 0$. □

In order to prove Theorem 1.6, the following lemmas are essential.

Lemma 3.5 ([27]). *If $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $0 < q < n - p$ and $\tau \in [-1, 1]$, then for all $u \in S^{n-1}$,*

$$\rho_{I_p^\tau(K\tilde{+}_qL)}^{\frac{pq}{n-p}}(u) \leqslant \rho_{I_p^\tau K}^{\frac{pq}{n-p}}(u) + \rho_{I_p^\tau L}^{\frac{pq}{n-p}}(u), \tag{3.10}$$

with equality if and only if K and L are dilates.

Lemma 3.6. *If $K, L \in \mathcal{S}_o^n$, $n \geqslant 2$, $0 < p < 1$, $0 < q < n - p$ and $\tau \in [-1, 1]$, then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(I_p^\tau(K\tilde{+}_qL), Q)^{\frac{pq}{(n-p)^2}} \leqslant \widetilde{V}_p(I_p^\tau K, Q)^{\frac{pq}{(n-p)^2}} + \widetilde{V}_p(I_p^\tau L, Q)^{\frac{pq}{(n-p)^2}}, \tag{3.11}$$

with equality if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates.

Proof. Since $n \geqslant 2$, $0 < p < 1$ and $0 < q < n-p$, thus $\frac{(n-p)^2}{pq} > 1$. Hence by (2.2), (3.10) and the Minkowski integral inequality (see [9]), we have that for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{V}_p(I_p^\tau(K\tilde{+}_q L), Q)^{\frac{pq}{(n-p)^2}} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau(K\tilde{+}_q L), u)^{n-p}\rho(Q,u)^p dS(u)\right]^{\frac{pq}{(n-p)^2}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho(I_p^\tau(K\tilde{+}_q L), u)^{\frac{pq}{n-p}}\rho(Q,u)^{\frac{p^2q}{(n-p)^2}}\right)^{\frac{(n-p)^2}{pq}} dS(u)\right]^{\frac{pq}{(n-p)^2}} \\
&\leqslant \left[\frac{1}{n}\int_{S^{n-1}} \left((\rho(I_p^\tau K, u)^{\frac{pq}{n-p}} + \rho(I_p^\tau L, u)^{\frac{pq}{n-p}})\rho(Q,u)^{\frac{p^2q}{(n-p)^2}}\right)^{\frac{(n-p)^2}{pq}} dS(u)\right]^{\frac{pq}{(n-p)^2}} \\
&\leqslant \left[\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau K, u)^{n-p}\rho(Q,u)^p dS(u)\right]^{\frac{pq}{(n-p)^2}} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau L, u)^{n-p}\rho(Q,u)^p dS(u)\right]^{\frac{pq}{(n-p)^2}} \\
&= \widetilde{V}_p(I_p^\tau K, Q)^{\frac{pq}{(n-p)^2}} + \widetilde{V}_p(I_p^\tau L, Q)^{\frac{pq}{(n-p)^2}}.
\end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds in (3.11) if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates. □

Proof of Theorem 1.6. Since $\frac{pq}{(n-p)^2} > 0$, thus by definition (1.1) and inequality (3.11) we obtain

$$\begin{aligned}
\left[\omega_n^{\frac{p}{n}}\widetilde{G}_p(I_p^\tau(K\tilde{+}_q L))\right]^{\frac{pq}{(n-p)^2}} &= \sup\left\{\left[n\widetilde{V}_p(I_p^\tau(K\tilde{+}_q L), Q)V(Q^*)^{\frac{p}{n}}\right]^{\frac{pq}{(n-p)^2}} : Q \in \mathcal{K}_{os}^n\right\} \\
&= \sup\left\{\left[n\widetilde{V}_p(I_p^\tau(K\tilde{+}_q L), Q)\right]^{\frac{pq}{(n-p)^2}} V(Q^*)^{\frac{p^2q}{n(n-p)^2}} : Q \in \mathcal{K}_{os}^n\right\} \\
&\leqslant \sup\left\{\left[(n\widetilde{V}_p(I_p^\tau K, Q))^{\frac{pq}{(n-p)^2}} + (n\widetilde{V}_p(I_p^\tau L, Q))^{\frac{pq}{(n-p)^2}}\right] V(Q^*)^{\frac{p^2q}{n(n-p)^2}} : Q \in \mathcal{K}_{os}^n\right\} \\
&\leqslant \sup\left\{\left[n\widetilde{V}_p(I_p^\tau K, Q)V(Q^*)^{\frac{p}{n}}\right]^{\frac{pq}{(n-p)^2}} : Q \in \mathcal{K}_{os}^n\right\} \\
&\quad + \sup\left\{\left[n\widetilde{V}_p(I_p^\tau L, Q)V(Q^*)^{\frac{p}{n}}\right]^{\frac{pq}{(n-p)^2}} : Q \in \mathcal{K}_{os}^n\right\} \\
&= \left[\omega_n^{\frac{p}{n}}\widetilde{G}_p(I_p^\tau K)\right]^{\frac{pq}{(n-p)^2}} + \left[\omega_n^{\frac{p}{n}}\widetilde{G}_p(I_p^\tau L)\right]^{\frac{pq}{(n-p)^2}},
\end{aligned}$$

i.e.,

$$\widetilde{G}_p(I_p^\tau(K\tilde{+}_q L))^{\frac{pq}{(n-p)^2}} \leqslant \widetilde{G}_p(I_p^\tau K)^{\frac{pq}{(n-p)^2}} + \widetilde{G}_p(I_p^\tau L)^{\frac{pq}{(n-p)^2}}.$$

This gives inequality (1.13).

By the equality condition of (3.11), we see that equality holds in (1.13) if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates. □

4. Busemann-Petty type problems

In this section, we give the proofs of Theorems 1.7–1.8.

Lemma 4.1 ([26]). *For* $K, L \in \mathcal{S}_o^n$ *and* $0 < p < 1$, *if for every* $\tau \in [-1, 1]$, $I_p^\tau K \subseteq I_p^\tau L$, *then for any* $M \in \mathcal{Z}_p^n$,

$$\widetilde{V}_p(K, M) \leqslant \widetilde{V}_p(L, M).$$

The equality holds only if $K = L$.

Proof of Theorem 1.7. From Lemma 4.1 and (1.14), we know that if $I_p^\tau K \subseteq I_p^\tau L$, then

$$\widetilde{G}_p^\circ(K) = \sup\{n\widetilde{V}_p(K,M)V(M^*)^{\frac{p}{n}} : M \in \mathcal{Z}_p^n\} \leqslant \sup\{n\widetilde{V}_p(L,M)V(M^*)^{\frac{p}{n}} : M \in \mathcal{Z}_p^n\} = \widetilde{G}_p^\circ(L).$$

According to the equality condition in Lemma 4.1, we know that equality holds in Theorem 1.7 only if $K = L$. □

Lemma 4.2. *If $K \in \mathcal{S}_o^n$, $0 < p < n$ and $\tau \in [-1,1]$, then*

$$\widetilde{G}_p(\overline{\nabla}_p^\tau K) \leqslant \widetilde{G}_p(K). \tag{4.1}$$

For $\tau \in (-1,1)$, equality holds if and only if K is origin-symmetric. For $\tau = \pm 1$, (4.1) becomes an equality.

Proof. For $\tau \in (-1,1)$, by definition (1.11), (2.2) and (2.5) we get

$$\begin{aligned}\omega_n^{\frac{p}{n}}\widetilde{G}_p(\overline{\nabla}_p^\tau K) &= \sup\left\{n\widetilde{V}_p(\widehat{\nabla}_p^\tau K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}\\ &= \sup\left\{n\widetilde{V}_p\left(f_1(\tau)\circ K \oplus_p f_2(\tau)\circ(-K),Q\right)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}\\ &= \sup\left\{\int_{S^{n-1}}\left[\rho(f_1(\tau)\circ K\oplus_p f_2(\tau)\circ(-K),u)^{n-p}\rho(Q,u)^p\right]dS(u)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}\\ &= \sup\left\{\int_{S^{n-1}}[f_1(\tau)\rho(K,u)^{n-p}+f_2(\tau)\rho(-K,u)^{n-p}]\rho(Q,u)^p\,dS(u)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}\\ &= \sup\left\{nf_1(\tau)\widetilde{V}_p(K,Q)V(Q^*)^{\frac{p}{n}}+nf_2(\tau)\widetilde{V}_p(-K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}\\ &\leqslant f_1(\tau)\sup\left\{n\widetilde{V}_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}+f_2(\tau)\sup\left\{n\widetilde{V}_p(-K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\right\}.\end{aligned} \tag{4.2}$$

Notice $Q \in \mathcal{K}_{os}^n$, we easily get $\widetilde{V}_p(-K,Q) = \widetilde{V}_p(K,Q)$. This together with (4.2) yields

$$\widetilde{G}_p(\overline{\nabla}_p^\tau K) \leqslant \widetilde{G}_p(K). \tag{4.3}$$

Because of equality holds in (4.2) if and only if K and $-K$ are dilates, this gives $K = -K$, i.e., K is origin-symmetric. Hence, equality holds in (4.3) if and only if K is origin-symmetric.
For $\tau = \pm 1$, by (2.7) we see that (4.1) is an equality. □

Lemma 4.3 ([26]). *If $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1,1]$, then*

$$I_p^+(\overline{\nabla}_p^\tau K) = I_p^\tau K,$$

$$I_p^-(\overline{\nabla}_p^\tau K) = I_p^{-\tau} K.$$

Proof of Theorem 1.8. Since K is not origin-symmetric, thus by Lemma 4.2 we know for $\tau \in (-1,1)$,

$$\widetilde{G}_p(\overline{\nabla}_p^\tau K) < \widetilde{G}_p(K).$$

Choose $\varepsilon > 0$, such that $\widetilde{G}_p((1+\varepsilon)\overline{\nabla}_p^\tau K) < \widetilde{G}_p(K)$. Therefore, let $L = (1+\varepsilon)\overline{\nabla}_p^\tau K$, then $L \in \mathcal{S}_o^n$ ($L \in \mathcal{S}_{os}^n$ when $\tau = 0$) and

$$\widetilde{G}_p(K) > \widetilde{G}_p(L).$$

But from Lemma 4.3, we have for $\tau \in (-1,1)$,

$$\begin{aligned}\rho(I_p^+L,\cdot) = \rho(I_p^+(1+\varepsilon)\overline{\nabla}_p^\tau K,\cdot) &= (1+\varepsilon)^{\frac{n-p}{p}}\rho(I_p^+\overline{\nabla}_p^\tau K,\cdot)\\ &= (1+\varepsilon)^{\frac{n-p}{p}}\rho(I_p^\tau K,\cdot) > \rho(I_p^\tau K,\cdot).\end{aligned} \tag{4.4}$$

Similarly, from Lemma 4.3, we obtain for $\tau \in (-1, 1)$,

$$\rho(I_p^- L, \cdot) > \rho(I_p^{-\tau} K, \cdot). \tag{4.5}$$

Notice that $\tau \in (-1, 1)$ is equivalent to $-\tau \in (-1, 1)$, then by (4.5) we see for $\tau \in (-1, 1)$,

$$\rho(I_p^- L, \cdot) > \rho(I_p^{\tau} K, \cdot). \tag{4.6}$$

Because of $f_1(\tau)$, $f_2(\tau) > 0$ for $\tau \in (-1, 1)$, thus by (4.4) and (4.6) we obtain for $0 < p < 1$,

$$f_1(\tau)\rho(I_p^\tau K, \cdot)^p + f_2(\tau)\rho(I_p^\tau K, \cdot)^p < f_1(\tau)\rho(I_p^+ L, \cdot)^p + f_2(\tau)\rho(I_p^- L, \cdot)^p.$$

This together with (1.5) and (1.8), we have for $\tau \in (-1, 1)$,

$$\rho(I_p^\tau K, \cdot)^p < \rho(I_p^\tau L, \cdot)^p,$$

i.e.,

$$I_p^\tau K \subset I_p^\tau L.$$

□

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