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The viscosity approximation forward-backward splitting method for the implicit midpoint rule of quasi inclusion problems in Banach spaces

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Abstract

The purpose of this paper is to introduce a viscosity approximation forward-backward splitting method for the implicit midpoint rule of an accretive operators and m-accretive operators in Banach spaces. The strong convergence of this viscosity method is proved under certain assumptions imposed on the sequence of parameters. The results presented in the paper extend and improve some recent results announced in the current literature. Moreover, some applications to the minimization optimization problem and the linear inverse problem are presented.

Keywords: Viscosity approximation, Banach space, splitting method, forward-backward algorithm, the implicit midpoint rule.
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1. Introduction

Let X be a real Banach space. We study the following inclusion problem: find $x^* \in X$ such that

$$0 \in Ax^* + Bx^*, \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form.

One of the popular iterative methods used for solving problem (1.1) is the forward-backward splitting method [15, 21, 27, 37] which is defined by the following manner: $x_1 \in X$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \geqslant 1,$$

where $r > 0$. We see that each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of A and B. This method includes, in particular, the proximal point

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algorithm [7, 9, 19, 24, 33] and the gradient method [6, 18]. Lions-Mercier [21] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n, \quad n \geqslant 1,$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \quad n \geqslant 1,$$

where $J_r^T = (I + rT)^{-1}$ is the resolvent of T. The first one is often called Peaceman-Rachford algorithm [28] and the second one is called Douglas-Rachford algorithm [17]. We note that both algorithms can be weakly convergent in general [27].

In 2012, Takashashi et al. [36] proved some strong convergence theorems of the Halpern-type iteration in a Hilbert space H, which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_n}^B(x_n - r_n A x_n)), \quad \forall n \geqslant 1, \tag{1.2}$$

where $u \in H$ is a given point and A is an α-inverse strongly monotone mapping on H and B is a maximal monotone operator on H, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\}$,$\{\beta_n\} \subset (0, 1]$. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a solution of the inclusion problem (1.1).

Recently, López et al. [22] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \tag{1.3}$$

where $u \in X$, A is an α-inverse strongly accretive mapping on X and B is an m-accretive operator on X. $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}$, $\{b_n\}$ are error sequences in X. They proved that the sequence $\{x_n\}$ generated by (1.3) strongly converges to a solution of the inclusion problem (1.1) under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces). For more details, see [11, 14, 22, 29, 30, 36, 37, 39, 42].

In 2015, Cholamjiak [12] studied a generalized forward-backward method for solving the inclusion problem (1.1) for an accretive and m-accretive operators in Banach spaces.

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^B(x_n - r_n A x_n) + e_n, \quad n \geqslant 1.$$

They then proved its strong convergence under some mild conditions.

The viscosity approximation method for nonexpansive mapping in Hilbert spaces was introduced by Moudafi [26], following the ideas of Attouch [2]. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [41].

Let $T : X \to X$ be a nonexpansive mapping and $f : X \to X$ be a contraction. Explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n)Tx_n, \quad n \geqslant 0,$$

where I is the identity of X. It is well-known [26, 41] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of T.

The implicit midpoint rule is one of the powerful methods for solving ordinary differential equations, see [3, 4, 16, 34, 35, 38] and the references therein. For instance, consider the initial value problem for the differential equation $y'(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from R^d to R^d. The implicit midpoint rule is that which generates a sequence $\{y_n\}$ via the relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} + y_n}{2}).$$

The implicit midpoint rule has been extended [1] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the implicit procedure:

$$x_{n+1} = (1 - t_n)x_n + t_n T(\frac{x_{n+1} + x_n}{2}), \quad n \geqslant 0.$$

Motivated and inspired by the research going on in this direction. The purpose of this paper is to introduce a viscosity approximation forward-backward splitting method for the implicit midpoint rule of an accretive operators and m-accretive operators in the framework of Banach spaces. More precisely, we consider the following iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J^B_{r_n}(I - r_n A)(\frac{x_{n+1} + x_n}{2}) + e_n, \quad n \geqslant 1. \tag{1.4}$$

Under certain assumptions imposed on the sequence of parameters, the strong convergence of this viscosity method is proved. Finally, we discuss applications of algorithms (1.4) to the minimization optimization problem and the linear inverse problem.

2. preliminaries

In order to prove the main results of the paper, we need the following basic concepts, notations and lemmas.

We assume that X is a real Banach space with norm $\|\cdot\|$ and dual space X^*. Let T be a nonlinear mapping. We denote the fixed point set of T by Fix(T).

Let $\delta(\epsilon) : (0, 2] \to [0, 1]$ be modulus of convexity of X defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geqslant \epsilon\}.$$

A Banach space X is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

Let $\rho : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of X defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : x, y \in X, \|x\| = \|y\| = 1\}.$$

A Banach space X is said to be uniformly smooth if $\frac{\rho(t)}{t} \to 0$ as $t \to 0$. Let q be a fixed real number with $q > 1$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $b > 0$ such that $\rho(t) \leqslant bt^q$ for all $t > 0$. It is well-known that every q-uniformly smooth Banach space is uniformly smooth.

Let $J_q(q > 1)$ denote the generalized duality mapping from X into 2^{X^*} given by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. In particular, $J_2 := J$ is called the normalized duality mapping on X. It is also known (e.g., [40, p.1128]]) that

$$j_q(x) = \|x\|^{q-2}J(x), \quad x \neq 0.$$

Some properties of the duality mappings are collected as follows.

Lemma 2.1 ([13])**.** *Let* $1 < q < \infty$.

(i) *The Banach space* X *is smooth if and only if the duality mapping* J_q *is single-valued.*

(ii) *The Banach space* X *is uniformly smooth if and only if the duality mapping* J_q *is single-valued and norm-to-norm uniformly continuous on bounded subsets of* X.

Using the concept of sub-differentials, we know the following inequality:

Lemma 2.2 ([10, p. 33])**.** *Let* $q > 1$ *and* X *be a real normed space with the generalized duality mapping* J_q. *Then, for any* $x, y \in X$, *we have*

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, j_q(x + y)\rangle,$$

for all $j_q(x + y) \in J_q(x + y)$.

Lemma 2.3 ([31, Corollary 1]). *Let C be a closed convex subset of a uniformly smooth Banach space X, and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Let x belong to C. Define for each $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x_t \mapsto tx + (1-t)Tx_t$ converges strongly as $t \to 0$ to a fixed point of T.*

Lemma 2.4 ([23, Lemma 3.1]). *Let $\{a_n\}$ and $\{\eta_n\}$ be sequences of nonegative real numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \tau_n + \eta_n, \quad n \geqslant 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\tau_n\}$ is a real sequence. Assume $\sum\limits_{n=1}^{\infty} \eta_n < \infty$. Then the following results hold:

(i) *If $\tau_n \leqslant \gamma_n M$, for some $M \geqslant 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup\limits_{n\to\infty} \frac{\tau_n}{\gamma_n} \leqslant 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

Lemma 2.5 ([20, Lemma 8]). *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leqslant (1-\gamma_n)s_n + \gamma_n\tau_n, \quad n \geqslant 1,$$

and

$$s_{n+1} \leqslant s_n - \eta_n + \rho_n, \quad n \geqslant 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$.

(ii) $\lim\limits_{n\to\infty} \rho_n = 0$.

(iii) $\lim\limits_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup\limits_{k\to\infty} \tau_{n_k} \leqslant 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim\limits_{n\to\infty} s_n = 0$.

Lemma 2.6 ([25, p. 63]). *Let $q > 1$. Then the following inequality holds:*

$$ab \leqslant \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a and b.

Lemma 2.7 ([12, Proposition 3.1]). *Let $q > 1$ and let X be a real smooth Banach space with the generalized duality mapping j_q. Let $m \in \mathbb{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geqslant 0$ for all $i = 1, 2, \cdots, m$ with $\sum\limits_{i=1}^{m} t_i \leqslant 1$. Then we have*

$$\|\sum_{i=1}^{m} t_i x_i\|^q \leqslant \frac{\sum\limits_{i=1}^{m} t_i \|x_i\|^q}{q - (q-1)\sum\limits_{i=1}^{m} t_i}.$$

We define the domain and the range of an operator $A : X \to 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. The inverse of A, denoted by A^{-1}, is defined by $x \in A^{-1}y$ if and only if $y \in Ax$. A set-valued operator A is said to be accretive, if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \geqslant 0, \quad u \in Ax, \quad v \in Ay.$$

An accretive operator A is said to be m-accretive if $R(I+rA) = X$, for all $r > 0$.

Given $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is α-inverse strongly accretive (α-isa) of order q, if for each $x, y \in D(A)$, there exists $j_q(x-y) \in J(x-y)$ such that

$$\langle u-v, j_q(x-y)\rangle \geqslant \alpha\|u-v\|^q, \quad u \in Ax, \quad v \in Ay.$$

In what follows, we shall use the following notation:

$$T_r^{A,B} = J_r^B(I-rA) = (I+rB)^{-1}(I-rA), \quad r > 0.$$

Lemma 2.8 ([22, Lemma 3.1 and Lemma 3.2])**.** *Let X be a Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Then we have*

(i) *For $r > 0$, $\mathrm{Fix}(T_r^{A,B}) = (A+B)^{-1}(0)$.*

(ii) *For $0 < s \leqslant r$ and $x \in X$, $\|x - T_s^{A,B}x\| \leqslant 2\|x - T_r^{A,B}x\|$.*

Lemma 2.9 ([22, Lemma 3.3])**.** *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q in X. Then, for given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : R^+ \to R^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$,*

$$\begin{aligned}\|T_r^{A,B}x - T_r^{A,B}y\|^q \leqslant{} & \|x-y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q \\ & - \phi_q(\|(I-J_r^B)(I-rA)x - (I-J_r^B)(I-rA)y\|),\end{aligned}$$

where k_q is the q-uniform smoothness coefficient of X.

3. Main results

To complete our proof, we need the following proposition:

Proposition 3.1. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\beta \in [0, 1)$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}^B(I - r_n A)\Big(\frac{x_{n+1}+x_n}{2}\Big) + e_n, \quad n \geqslant 1,$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $0 < r_n < (\alpha q/k_q)^{1/(q-1)}$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. If $\sum_{n=1}^{\infty}\|e_n\| < \infty$ or $\lim_{n\to\infty}\|e_n\|/\alpha_n = 0$, then $\{x_n\}$ is bounded.

Proof. For each $n \in \mathbb{N}$, we put $T_n = J_{r_n}^B(I - r_n A)$ and let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n f(y_n) + \lambda_n y_n + \delta_n T_n\Big(\frac{y_{n+1}+y_n}{2}\Big). \tag{3.1}$$

Firstly, we compute the following:

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &= \|\alpha_n(f(x_n) - f(y_n)) + \lambda_n(x_n - y_n) \\ &\quad + \delta_n\Big(T_n\Big(\frac{x_{n+1}+x_n}{2}\Big) - T_n\Big(\frac{y_{n+1}+y_n}{2}\Big)\Big) + e_n\|, \\ &\leqslant \alpha_n\|f(x_n) - f(y_n)\| + \lambda_n\|x_n - y_n\| \\ &\quad + \delta_n\Big\|T_n\Big(\frac{x_{n+1}+x_n}{2}\Big) - T_n\Big(\frac{y_{n+1}+y_n}{2}\Big)\Big\| + \|e_n\| \\ &\leqslant \alpha_n\beta\|x_n - y_n\| + \lambda_n\|x_n - y_n\| \\ &\quad + \frac{1}{2}\delta_n(\|x_n - y_n\| + \|x_{n+1} - y_{n+1}\|) + \|e_n\|.\end{aligned}$$

After simplifying, it follows that

$$(1-\frac{1}{2}\delta_n)\|x_{n+1}-y_{n+1}\| \leqslant (\alpha_n\beta+\lambda_n+\frac{1}{2}\delta_n)\|x_n-y_n\|+\|e_n\|.$$

Therefore

$$\begin{aligned}\|x_{n+1}-y_{n+1}\| &\leqslant \frac{2\alpha_n\beta+2\lambda_n+\delta_n}{2-\delta_n}\|x_n-y_n\|+\frac{2}{2-\delta_n}\|e_n\|\\ &=(1-\frac{2\alpha_n(1-\beta)}{2-\delta_n})\|x_n-y_n\|+\frac{2}{2-\delta_n}\|e_n\|.\end{aligned}$$

By the assumptions and Lemma 2.4 (ii), we conclude that $\lim_{n\to\infty}\|x_n-y_n\|=0$. Let $p\in \mathrm{Fix}(T_n)$. We next show that $\{y_n\}$ is bounded. Indeed

$$\begin{aligned}\|y_{n+1}-p\| &= \|\alpha_n(f(y_n)-p)+\lambda_n(y_n-p)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-p)\|,\\ &\leqslant \alpha_n\|f(y_n)-p\|+\lambda_n\|y_n-p\|+\delta_n\|T_n(\frac{y_{n+1}+y_n}{2})-p\|\\ &\leqslant \alpha_n(\|f(y_n)-f(p)\|+\|f(p)-p\|)+\lambda_n\|y_n-p\|\\ &\quad+\frac{1}{2}\delta_n(\|y_n-p\|+\|y_{n+1}-p\|)\\ &\leqslant \alpha_n\beta\|y_n-p\|+\alpha_n\|f(p)-p\|+\lambda_n\|y_n-p\|\\ &\quad+\frac{1}{2}\delta_n(\|y_n-p\|+\|y_{n+1}-p\|).\end{aligned}$$

By simplifying, we have

$$(1-\frac{1}{2}\delta_n)\|y_{n+1}-p\| \leqslant (\alpha_n\beta+\lambda_n+\frac{1}{2}\delta_n)\|y_n-p\|+\alpha_n\|f(p)-p\|.$$

Hence

$$\begin{aligned}\|y_{n+1}-p\| &\leqslant \frac{2\alpha_n\beta+2\lambda_n+\delta_n}{2-\delta_n}\|y_n-p\|+\frac{2\alpha_n}{2-\delta_n}\|f(p)-p\|\\ &=(1-\frac{2\alpha_n(1-\beta)}{2-\delta_n})\|y_n-p\|+\frac{2\alpha_n}{2-\delta_n}\|f(p)-p\|.\end{aligned}$$

This shows that $\{y_n\}$ is bounded by Lemma 2.4 (i) and hence $\{x_n\}$ is also bounded. □

We are now ready to prove our main result.

Theorem 3.2. *Let X be a uniformly convex and q-uniformly smooth Banach space, $q\in(1,2]$. Let $A: X\to X$ be an α-isa of order q and $B: X\to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0)\neq\emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\beta\in[0,1)$. Let $\{x_n\}$ be generated by $x_1\in X$ and*

$$x_{n+1}=\alpha_n f(x_n)+\lambda_n x_n+\delta_n J^B_{r_n}(I-r_nA)(\frac{x_{n+1}+x_n}{2})+e_n, \quad n\geqslant 1,$$

where $J^B_{r_n}=(I+r_nB)^{-1}$, $\{r_n\}\subset(0,\infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n+\lambda_n+\delta_n=1$. Assume that

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}\alpha_n=0$;

(ii) $0<\liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < (\alpha q/k_q)^{1/(q-1)}$;

(iii) $\liminf_{n\to\infty}\delta_n>0$;

(iv) $\sum_{n=1}^{\infty}\|e_n\|<\infty$ *or* $\lim_{n\to\infty}\|e_n\|/\alpha_n=0$.

Then $\{x_n\}$ *strongly converges to some* $z\in\Omega$.

Proof. Let $z\in \mathrm{Fix}(T_n)$, from Lemma 2.2 and Lemma 2.6, we have

$$\begin{aligned}
\|y_{n+1}-z\|^q &= \|\alpha_n(f(y_n)-z)+\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\leqslant \|\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q+q\alpha_n\langle(f(y_n)-z),j_q(y_{n+1}-z)\rangle\\
&\leqslant \|\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\quad+q\alpha_n\langle f(y_n)-f(z),j_q(y_{n+1}-z)\rangle+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle\\
&\leqslant \|\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\quad+q\alpha_n\beta\|y_n-z\|\|y_{n+1}-z\|^{q-1}+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle\\
&\leqslant \|\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\quad+q\alpha_n\beta(\frac{1}{q}\|y_n-z\|^q+\frac{q-1}{q}\|y_{n+1}-z\|^q)+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle\\
&\leqslant \|\lambda_n(y_n-z)+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\quad+\alpha_n\beta\|y_n-z\|^q+(q-1)\alpha_n\beta\|y_{n+1}-z\|^q+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle.
\end{aligned}\tag{3.2}$$

On the other hand, by Lemma 2.7 and Lemma 2.9, we obtain

$$\begin{aligned}
\|\lambda_n(y_n-z)&+\delta_n(T_n(\frac{y_{n+1}+y_n}{2})-z)\|^q\\
&\leqslant \frac{1}{\alpha_n q+1-\alpha_n}(\lambda_n\|y_n-z\|^q+\delta_n\|T_n(\frac{y_{n+1}+y_n}{2})-z\|^q)\\
&\leqslant \frac{1}{\alpha_n q+1-\alpha_n}(\lambda_n\|y_n-z\|^q\\
&\quad+\delta_n(\|\frac{y_{n+1}+y_n}{2}-z\|^q-r_n(\alpha q-r_n^{q-1}k_q)\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&\quad-\phi_q(\|\frac{y_{n+1}+y_n}{2}-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|)))\\
&\leqslant \frac{1}{\alpha_n q+1-\alpha_n}(\lambda_n\|y_n-z\|^q\\
&\quad+\delta_n((\frac{1}{2}\|y_{n+1}-z\|^q+\frac{1}{2}\|y_n-z\|^q)-r_n(\alpha q-r_n^{q-1}k_q)\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&\quad-\phi_q(\|\frac{y_{n+1}+y_n}{2}-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|)))\\
&\leqslant \frac{\lambda_n+\frac{1}{2}\delta_n}{\alpha_n q+1-\alpha_n}\|y_n-z\|^q+\frac{\frac{1}{2}\delta_n}{\alpha_n q+1-\alpha_n}\|y_{n+1}-z\|^q\\
&\quad-\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{\alpha_n q+1-\alpha_n}\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&\quad-\frac{\delta_n}{\alpha_n q+1-\alpha_n}\phi_q(\|(\frac{y_{n+1}+y_n}{2})-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|).
\end{aligned}\tag{3.3}$$

Replacing (3.3) into (3.2), it follows that

$$\begin{aligned}\|y_{n+1}-z\|^q \leqslant& \frac{\lambda_n+\frac{1}{2}\delta_n}{\alpha_n q+1-\alpha_n}\|y_n-z\|^q+\frac{\frac{1}{2}\delta_n}{\alpha_n q+1-\alpha_n}\|y_{n+1}-z\|^q\\
&-\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{\alpha_n q+1-\alpha_n}\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&-\frac{\delta_n}{\alpha_n q+1-\alpha_n}\varphi_q(\|(\frac{y_{n+1}+y_n}{2})-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|)\\
&+\alpha_n\beta\|y_n-z\|^q+(q-1)\alpha_n\beta\|y_{n+1}-z\|^q+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle\\
\leqslant& \frac{\lambda_n+\frac{1}{2}\delta_n+\alpha_n\beta(\alpha_n q+1-\alpha_n)}{\alpha_n q+1-\alpha_n}\|y_n-z\|^q\\
&+\frac{\frac{1}{2}\delta_n+(q-1)\alpha_n\beta(\alpha_n q+1-\alpha_n)}{\alpha_n q+1-\alpha_n}\|y_{n+1}-z\|^q\\
&-\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{\alpha_n q+1-\alpha_n}\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&-\frac{\delta_n}{\alpha_n q+1-\alpha_n}\varphi_q(\|(\frac{y_{n+1}+y_n}{2})-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|)\\
&+q\alpha_n\langle f(z)-z,j_q(y_{n+1}-z)\rangle.\end{aligned}$$

After simplifying it follows that

$$\begin{aligned}\|y_{n+1}-z\|^q \leqslant& (1-\frac{\alpha_n q(1-\beta-\beta\alpha_n(q-1))}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n})\|y_n-z\|^q\\
&-\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&-\frac{\delta_n}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\\
&\times\varphi_q(\|(\frac{y_{n+1}+y_n}{2})-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|)\\
&+\frac{q\alpha_n(\alpha_n q+1-\alpha_n)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\langle f(z)-z,j_q(y_{n+1}-z)\rangle.\end{aligned}\tag{3.4}$$

We can check that $\frac{\alpha_n q(1-\beta-(q-1)\alpha_n\beta)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}$ is in $(0,1)$ since $1<q\leqslant 2$, $\{\alpha_n\}\subset(0,1)$ and $\lim\limits_{n\to\infty}\alpha_n=0$. Moreover, by condition (ii), $\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}$ and $\frac{\delta_n}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}$ are positive.

For each $n\geqslant 1$, we set

$$\begin{aligned}s_n&=\|y_n-z\|^q,\gamma_n=\frac{\alpha_n q(1-\beta-(q-1)\alpha_n\beta)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n},\\
\tau_n&=\frac{\alpha_n q+1-\alpha_n}{1-\beta-(q-1)\alpha_n\beta}\langle f(z)-z,j_q(y_{n+1}-z)\rangle,\\
\eta_n&=-\frac{\delta_n r_n(\alpha q-r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\|A(\frac{y_{n+1}+y_n}{2})-Az\|^q\\
&\quad-\frac{\delta_n}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\\
&\quad\times\varphi_q(\|(\frac{y_{n+1}+y_n}{2})-r_nA(\frac{y_{n+1}+y_n}{2})-T_n(\frac{y_{n+1}+y_n}{2})+r_nAz\|),\\
\rho_n&=\frac{q\alpha_n(\alpha_n q+1-\alpha_n)}{(1-(q-1)\alpha_n\beta)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\langle f(z)-z,j_q(y_{n+1}-z)\rangle.\end{aligned}$$

From (3.4), we then have

$$s_{n+1} \leqslant (1-\gamma_n)s_n + \gamma_n \tau_n, \quad n \geqslant 1,$$

and

$$s_{n+1} \leqslant s_n - \eta_n + \rho_n, \quad n \geqslant 1.$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. By the boundedness of $\{y_n\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we see that $\lim_{n\to\infty} \rho_n = 0$. In order to complete the proof, using Lemma 2.5, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leqslant 0$ for any subsequence $\{n_k\} \subset \{n\}$.

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\begin{aligned}
&\lim_{k\to\infty} \|A(\frac{y_{n_k+1}+y_{n_k}}{2}) - Az\| \\
&\quad = \lim_{k\to\infty} \|(\frac{y_{n_k+1}+y_{n_k}}{2}) - \gamma_{n_k} A(\frac{y_{n_k+1}+y_{n_k}}{2}) - T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) + r_{n_k} Az\| = 0.
\end{aligned}$$

This gives, by the triangle inequality, that

$$\lim_{k\to\infty} \|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - \frac{y_{n_k+1}+y_{n_k}}{2}\| = 0. \tag{3.5}$$

By (3.1), we have

$$\begin{aligned}
\|y_{n_k+1} - y_{n_k}\| &= \|\alpha_{n_k} f(y_{n_k}) + \lambda_{n_k} y_{n_k} + \delta_{n_k} T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - y_{n_k}\| \\
&\leqslant \alpha_{n_k}\|f(y_{n_k}) - y_{n_k}\| + \delta_{n_k}\|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - y_{n_k}\| \\
&\leqslant \alpha_{n_k}\|f(y_{n_k}) - y_{n_k}\| + \delta_{n_k}\|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - \frac{y_{n_k+1}+y_{n_k}}{2}\| \\
&\quad + \delta_{n_k}\|\frac{y_{n_k+1}+y_{n_k}}{2} - y_{n_k}\|.
\end{aligned}$$

By simplifying we have

$$\|y_{n_k+1} - y_{n_k}\| \leqslant \frac{\alpha_{n_k}}{1-\frac{1}{2}\delta_{n_k}}\|f(y_{n_k}) - y_{n_k}\| + \frac{\delta_{n_k}}{1-\frac{1}{2}\delta_{n_k}}\|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - \frac{y_{n_k+1}+y_{n_k}}{2}\|.$$

By Proposition 3.1, $\{y_n\}$ is bounded, and so is $\{f(x_n)\}$, by condition (i) and (3.5), we obtain

$$\lim_{k\to\infty} \|y_{n_k+1} - y_{n_k}\| = 0. \tag{3.6}$$

By the triangle inequality, we have

$$\begin{aligned}
\|T_{n_k} y_{n_k} - y_{n_k}\| &\leqslant \|T_{n_k} y_{n_k} - T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2})\| \\
&\quad + \|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - \frac{y_{n_k+1}+y_{n_k}}{2}\| \\
&\quad + \|\frac{y_{n_k+1}+y_{n_k}}{2} - y_{n_k}\| \\
&\leqslant \|y_{n_k+1} - y_{n_k}\| + \|T_{n_k}(\frac{y_{n_k+1}+y_{n_k}}{2}) - \frac{y_{n_k+1}+y_{n_k}}{2}\|.
\end{aligned}$$

By (3.5) and (3.6), we have

$$\lim_{k\to\infty} \|T_{n_k} y_{n_k} - y_{n_k}\| = 0. \tag{3.7}$$

Since $\liminf_{k\to\infty} r_n > 0$, there is $r > 0$ such that $r_n \geqslant r$ for all $n \geqslant 1$. In particular, $r_{n_k} \geqslant r$ for all $k \geqslant 1$. Lemma 2.8 (ii) yields that

$$\|T_r^{A,B}y_{n_k} - y_{n_k}\| \leqslant 2\|T_{n_k}y_{n_k} - y_{n_k}\|.$$

Then, by (3.7), we obtain

$$\limsup_{k\to\infty} \|T_r^{A,B}y_{n_k} - y_{n_k}\| \leqslant 2\lim_{k\to\infty} \|T_{n_k}y_{n_k} - y_{n_k}\|.$$

It follows that

$$\lim_{k\to\infty} \|T_r^{A,B}y_{n_k} - y_{n_k}\| = 0. \tag{3.8}$$

Let $z_t = tf(z_t) + (1-t)T_r^{A,B}z_t,\ t \in (0,1)$. Employing Lemma 2.3, we have $z_t \to z \in \Omega$ as $t \to 0$, from Lemma 2.2 we have that

$$\begin{aligned}
\|z_t - y_{n_k}\|^q &= \|t(f(z_t) - y_{n_k}) + (1-t)(T_r^{A,B}z_t - y_{n_k})\|^q \\
&\leqslant (1-t)^q\|T_r^{A,B}z_t - y_{n_k}\|^q + qt\langle f(z_t) - y_{n_k}, j_q(z_t - y_{n_k})\rangle \\
&\leqslant (1-t)^q(\|T_r^{A,B}z_t - T_r^{A,B}y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_{n_k})\rangle + qt\langle z_t - y_{n_k}, j_q(z_t - y_{n_k})\rangle \\
&\leqslant (1-t)^q(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_{n_k})\rangle + qt\|z_t - y_{n_k}\|^q.
\end{aligned}$$

This shows that

$$\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle \leqslant \frac{(1-t)^q}{qt}(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q + \frac{qt-1}{qt}\|z_t - y_{n_k}\|^q.$$

From (3.8), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle &\leqslant \frac{(1-t)^q}{qt}M^q + \frac{qt-1}{qt}M^q \\
&= \frac{(1-t)^q + qt - 1}{qt}M^q,
\end{aligned} \tag{3.9}$$

where $M = \limsup_{k\to\infty} \|z_t - y_{n_k}\|,\ t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Lemma 2.1 (ii), we know that j_q is norm-to-norm uniformly continuous on bounded subsets of X. Since $z_t \to z$ as $t \to 0$, we have $\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\| \to 0$ as $t \to 0$. Observe that

$$\begin{aligned}
&|\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle - \langle z - f(z), j_q(z - y_{n_k})\rangle| \\
&\quad\leqslant |\langle z_t - z + z - f(z) + f(z) - f(z_t), j_q(z_t - y_{n_k})\rangle - \langle z - f(z), j_q(z - y_{n_k})\rangle| \\
&\quad\leqslant |\langle z_t - z, j_q(z_t - y_{n_k})\rangle| + |\langle z - f(z), j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\rangle| \\
&\qquad + |\langle f(z) - f(z_t), j_q(z_t - y_{n_k})\rangle| \\
&\quad\leqslant (1+\beta)\|z_t - z\|\|z_t - y_{n_k}\|^{q-1} + \|z - f(z)\|\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\|^{q-1}.
\end{aligned}$$

So, as $t \to 0$, we get

$$\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle \to \langle z - f(z), j_q(z - y_{n_k})\rangle.$$

From (3.9), as $t \to 0$, it follows that

$$\limsup_{k\to\infty}\langle z - f(z), j_q(z - y_{n_k})\rangle \leqslant 0. \tag{3.10}$$

Combining (3.6) and (3.10), we get that

$$\limsup_{k\to\infty}\langle z-f(z), j_q(z-y_{n_k+1})\rangle \leqslant 0.$$

It also follows that $\limsup_{k\to\infty}\tau_{n_k} \leqslant 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.5. Hence $y_n \to z$ as $n\to\infty$, by Proposition 3.1, $\lim_{n\to\infty}\|x_n-y_n\|=0$, so $\lim_{n\to\infty} x_n = z\in\Omega$. We thus complete the proof. □

By setting $\lambda_n = 0$ for all $n \geqslant 1$, we obtain the following result:

Corollary 3.3. *Let X be a uniformly convex and q-uniformly smooth Banach space, $q\in(1,2]$. Let $A : X\to X$ be an α-isa of order q and $B : X\to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0)\neq\emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\beta\in[0,1)$. Let $\{x_n\}$ be generated by $x_1\in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J^B_{r_n}(I-r_nA)\Big(\frac{x_{n+1}+x_n}{2}\Big)+e_n, \quad n\geqslant 1,$$

where $J^B_{r_n} = (I+r_nB)^{-1}$, $\{r_n\}\subset(0,\infty)$ and $\{\alpha_n\}$ is a sequences in $[0,1]$. Assume that

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}\alpha_n=0$;

(ii) $0<\liminf_{n\to\infty} r_n\leqslant\limsup_{n\to\infty} r_n<(\alpha q/k_q)^{1/(q-1)}$;

(iii) $\sum_{n=1}^{\infty}\|e_n\|<\infty$ *or* $\lim_{n\to\infty}\|e_n\|/\alpha_n=0$.

Then $\{x_n\}$ strongly converges to some $z\in\Omega$.

4. Applications

Firstly, we apply Theorem 3.2 to the convex minimization problem. Let H be a real Hilbert space. Let $F : H\to R$ be a convex smooth function and $G : H\to R$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $x^*\in H$ such that

$$F(x^*)+G(x^*)\leqslant F(x)+G(x), \tag{4.1}$$

for all $x\in H$. This problem (4.1) is equivalent, by Fermats rule, to the problem of finding $x^*\in H$ such that

$$0\in\nabla F(x^*)+\partial G(x^*),$$

where ∇F is the gradient of F and ∂G is the subdifferential of G. In this point of view, we can set $A=\nabla F$ and $B=\partial G$ in Theorem 3.2. This is because if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone [5, Corollary 10]. Moreover, ∂G is maximal monotone [32, Theorem A]. So we obtain the following result.

Theorem 4.1. *Let H be real Hilbert space. Let $F : H\to R$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇F and $G : H\to R$ be a convex and lower semi-continuous function which $F+G$ attains a minimizer. Let $\{e_n\}$ be a sequence in H and f be a contraction on X with coefficient $\beta\in[0,1)$. Let $\{x_n\}$ be generated by $x_1\in H$ and*

$$x_{n+1}=\alpha_n f(x_n)+\lambda_n x_n+\delta_n J_{r_n}(I-r_n\nabla F)\Big(\frac{x_{n+1}+x_n}{2}\Big)+e_n,\quad n\geqslant 1,$$

where $J_{r_n}=(I+r_n\partial G)^{-1}$, $\{r_n\}\subset(0,\infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n+\lambda_n+\delta_n=1$. Assume that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < 2L$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ *strongly converges to a minimizer of* $F + G$.

Secondly, we apply Theorem 3.2 to solve the unconstrained linear system

$$Cx = d, \tag{4.2}$$

where C is a bounded linear operator on H and $d \in H$. For each $x \in H$, we define $F : H \to R$ by

$$F(x) = \frac{1}{2}\|Cx - d\|^2.$$

From [8] we know that $\nabla F(x) = C^T(Cx - d)$ and ∇F is K-Lipschitz continuous with K the largest eigenvalue of $C^T C$. So we obtain the following result.

Theorem 4.2. *Let* H *be real Hilbert space. Let* $C : H \to H$ *be a bounded linear operator and* $d \in H$ *with* K *the largest eigenvalue of* $C^T C$. *Let* $\{e_n\}$ *be a sequence in* H *and* f *be a contraction on* X *with coefficient* $\beta \in [0, 1)$. *Let* $\{x_n\}$ *be generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n (I - r_n C^T(C - dI))\left(\frac{x_{n+1} + x_n}{2}\right) + e_n, \quad n \geqslant 1,$$

where $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \lambda_n + \delta_n = 1$. *Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < 2/K$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

If (4.2) *is consistent, then* $\{x_n\}$ *strongly converges to a solution of a linear system.*

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