Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 3592–3598
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Existence for fractional Dirichlet boundary value problem under barrier strip conditions

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Communicated by X.-J. Yang

Abstract

In this paper, a fixed-point theorem is used to establish existence results for fractional Dirichlet boundary value problem

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha-1}x(t)), \quad x(0) = A, \ \ x(1) = B,$$

where $1 < \alpha \leqslant 2, D^{\alpha}x(t)$ is the conformable fractional derivative, and $f : [0,1] \times R^2 \to R$ is a continuous function. The main condition is sign condition. The method used is based upon the theory of fixed-point index.

Keywords: Barrier strips, fixed-point index, conformable fractional derivative.
2010 MSC: 34A08, 34B18, 35J05.

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1. Introduction

Due to the extensive application of fractional calculus in non-Newtonian fluid mechanics, signal analysis, image processing, and other disciplines [2, 3, 35–39], the fractional differential equations have been widely studied and many interesting results have been obtained. For example, in 2005, by using fixed point theorem of cone extension and compression, the existence of positive solution of the following problem are obtained [8]

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \\ u(0) = u(1) = 0. \end{cases}$$

After that, the idea was developed to deal with various fractional boundary value problems such as fractional boundary value problem at resonance [5, 6, 9, 10], Caputo fractional derivative problem [41], impulsive problem [7, 31, 33], nonlocal problem [4], integral boundary value problem [27], variational structure problem [21], fractional p-Laplace problem [15, 20, 25, 26, 34, 40], fractional lower and upper

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doi:10.22436/jnsa.010.07.19

Received 2017-04-14

solution problem [11, 12, 42], fractional delay problems, [30, 32, 44], solitons [16], Biological mathematics [17, 29, 43], etc. However, all above works were obtained with standard Riemann-Liouville or Caputo fractional derivatives. The unusual properties of these fractional derivatives lead to some difficulties in application of fractional derivatives in physics and mechanics. Recently, the new conformable fractional derivative definition given by [1, 14, 24] has many good properties which inspired us study problems with conformable fractional derivative.

In the research of integer order boundary value problem, Kelevedjiev got the existence of the solutions by using the technique of barrier strips in [22, 23]. These ideas were developed by Ma and Luo [28] and Gao [18] to other problems. Very recently, we obtained the existence of solutions for fractional differential equation

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha-1}x(t)),$$

with one of the following boundary value conditions

$$x(0) = A, \quad D^{\alpha-1}x(1) = B, \quad D^{\alpha-1}x(0) = A, \quad x(1) = B,$$

where $D^{\alpha}x(t)$ is the conformable fractional derivative. The main tool used is the topological transversality theorem [19].

In this paper, by using the fixed-point index theory, the barrier strips technique and a priori estimation, we consider the following problem

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha-1}x(t)), \quad x(0) = A, \quad x(1) = B, \tag{1.1}$$

where $1 < \alpha \leqslant 2, D^{\alpha}x(t)$ is the conformable fractional order derivative, and $f : [0, 1] \times R^2 \to R$ is a continuous function. The existence results of solutions to the problem are obtained under f satisfying some sign conditions.

The rest of the paper is organized as follows. In Section 2, the definitions and some preliminaries of the conformable fractional derivative and integral are given. In Section 3, by the use of the fixed-point theorem and the technique of barrier strips, the existence of the solution is obtained. An example is presented to illustrate the main results.

2. Conformable fractional order calculus

For the convenience of the reader, we recall some definitions and lemmas which can be found in [1, 13, 24].

Definition 2.1. Suppose $\alpha \in (n, n+1], u : [0, \infty) \to R$, and u n-order differentiable for $t > 0$, the α-order fractional derivative of u is defined as

$$D^{\alpha}u(t) = \lim_{\varepsilon \to 0} \frac{u^{(n)}(t + \varepsilon t^{n+1-\alpha}) - u^{(n)}(t)}{\varepsilon},$$

provided the limits of the right side exists.

If u is α-order differentiable on $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} D^{\alpha}u(t)$ exists, then define

$$D^{\alpha}u(0) = \lim_{t \to 0^+} D^{\alpha}u(t).$$

Lemma 2.2 ([24])**.** *Let $\alpha \in (n, n+1], u : [0, \infty) \to R$. Function $u(t)$ is α-order differentiable if and only if u is $(n+1)$-order differentiable, moreover,*

$$D^{\alpha}u(t) = t^{n+1-\alpha}u^{(n+1)}(t).$$

Definition 2.3. Let $\alpha \in (n, n+1], u : [0, \infty) \to R$. The α-order fractional integral of $u(t)$ is defined as

$$J_{0+}^{\alpha} u(t) = I^{n+1}[t^{\alpha-n-1}u(t)] = \frac{1}{n!}\int_0^t (t-s)^n s^{\alpha-n-1}u(s)ds,$$

where I^{n+1} is the $(n+1)$-order integral.

Remark 2.4. With Lemma 2.2 and Definition 2.3, for $\alpha \in (n, n+1], i = 0, 1, \cdots, n$, there hold

$$D^{\alpha-i}[J_{0+}^{\alpha}u(t)] = t^{n+1-\alpha}D^{n+1-i}[I^{n+1}(t^{\alpha-n-1}u(t))] = t^{n+1-\alpha}I^{i}[t^{\alpha-n-1}u(t)].$$

Lemma 2.5 ([24]). *Let $a \geqslant 0, f : [0, b] \to R$ satisfy*

(i) *f is continuous on $[0, b]$;*
(ii) *f is α-order differentiable on $(0, b)$,*

then, there exists $c \in (a, b)$ such that $D^{\alpha}f(c) = (f(b) - f(a))/(\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha})$.

Now, we introduce a function space. Given $\alpha \in (n, n+1]$, let

$$C^{\alpha}[0,1] = \{u \mid u(t) = J_{0+}^{\alpha}x(t) + C_n t^n + \cdots + C_1 t + C_0, C_i \in R, i = 0, 1 \cdots, n, x(t) \in C[0,1]\},$$
$$\|u\|_{\alpha} = \|D^{\alpha}u\|_0 + \|D^{\alpha-1}u\|_0 + \cdots + \|D^{\alpha-n}u\|_0 + \|u\|_0,$$

where $\|u\|_0 = \max\limits_{t\in[0,1]} | u(t) |$.

Lemma 2.6 ([19]). *$(C^{\alpha}[0,1], \|\cdot\|_{\alpha})$ is a Banach space.*

3. Main results

Theorem 3.1. *Let $f : [0,1] \times R^2 \to R$ be continuous. Suppose there are constants L_i, $i = 1, 2, \cdots, 8$ such that $L_2 > L_1 \geqslant E$, $L_4 > L_3 \geqslant E$, $L_5 < L_6 \leqslant E$, $L_7 < L_8 \leqslant E$, where $E = (\alpha - 1)(B - A)$ and*

$$f(t, x, p) \geqslant 0 \ \text{ for } \ (t, x, p) \in [0,1] \times R \times ([L_1, L_2] \cup [L_5, L_6]),$$
$$f(t, x, p) \leqslant 0 \ \text{ for } \ (t, x, p) \in [0,1] \times R \times ([L_3, L_4] \cup [L_7, L_8]).$$

Then problem (1.1) has at least one solution in $C^{\alpha}[0,1]$.

Proof. Consider the boundary value problem:

$$D^{\alpha}x(t) = \lambda f(t, x(t), D^{\alpha-1}x(t)), \quad t \in [0,1], \ \ \lambda \in [0,1], \tag{3.1}$$
$$x(0) = A, \quad x(1) = B. \tag{3.2}$$

Define a compact homotopy operator $T_\lambda : C_B^{\alpha}[0,1] \to C_B^{\alpha}[0,1]$ as

$$(T_\lambda x)(t) = \lambda \int_0^1 G(t,s) f(s, x(s), D^{\alpha-1}x(s))ds + \psi(t),$$

where

$$C_B^{\alpha}[0,1] = \{x \in C^{\alpha}[0,1] | x(0) = A, x(1) = B\}, \quad G(t,s) = \begin{cases} (1-t)s^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ ts^{\alpha-2}(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

is the Green function (see [13]) and $\psi(t) = A + Bt$ is the solution of the problem

$$\begin{cases} D^{\alpha}x(t) = 0, \\ x(0) = A, \quad x(1) = B. \end{cases}$$

Then $T_\lambda : C^\alpha_B[0,1] \to C^\alpha_B[0,1]$ is completely continuous (see [13]). The fixed point of T_λ is the solution of problem (3.1)-(3.2). For some $M > 0$, let

$$U = \{u \in C^\alpha_B[0,1] \mid \|u\|_\alpha < M + \|\psi\|_\alpha + 1\}.$$

We will find positive number M such that $T_\lambda : \overline{U} \to C^\alpha_B[0,1]$ has no fixed-point on ∂U for every $\lambda \in [0,1]$.

In fact, all the possible solutions of problem (3.1)-(3.2) have a priori bound that do not depend on $\lambda \in [0,1]$. Let $x(t) \in C^\alpha[0,1]$ be a solution to problem (3.1)-(3.2). Firstly, we will estimate $D^{\alpha-1}x(t)$. By Lemma 2.5, there is $d \in (0,1)$ such that $D^{\alpha-1}x(d) = (\alpha-1)(B-A) = E$. Then

$$f(t,x,p) \geqslant 0 \text{ for } (t,x,p) \in [0,d] \times R \times [L_1, L_2],$$
$$f(t,x,p) \leqslant 0 \text{ for } (t,x,p) \in [0,d] \times R \times [L_7, L_8].$$

Assume that the sets

$$S_0 = \{t \in [0,d] | L_1 < D^{\alpha-1}x(t) \leqslant L_2\}, \quad S_1 = \{t \in [0,d] | L_3 < D^{\alpha-1}x(t) < L_4\},$$

are not empty. Let $t_0 \in S_0, t_1 \in S_1$ be fixed. Assume there are $t_0' \in (t_0, d]$ and $t_1' \in (t_1, d]$ such that

$$D^{\alpha-1}x(t_0') < D^{\alpha-1}x(t_0), \quad D^{\alpha-1}x(t_1') > D^{\alpha-1}x(t_1). \tag{3.3}$$

The continuity of $D^{\alpha-1}x(t)$ allows us to take $t_0' \in (t_0,d] \cap S_0$, $t_1' \in (t_1,d] \cap S_1$. But $D^\alpha x(t) = \lambda f(t,x(t), D^{\alpha-1}x(t)) \geqslant 0$, for $t \in S_0$ and $D^\alpha x(t) \leqslant 0$, for $t \in S_1$. Consequently,

$$D^{\alpha-1}x(t_0') \geqslant D^{\alpha-1}x(t_0), \quad D^{\alpha-1}x(t_1') \leqslant D^{\alpha-1}x(t_1).$$

From the contradiction to (3.3), it follows that

$$D^{\alpha-1}x(t) \geqslant D^{\alpha-1}x(t_0), \text{ for } t \in (t_0, d],$$
$$D^{\alpha-1}x(t) \leqslant D^{\alpha-1}x(t_0), \text{ for } t \in (t_1, d],$$

and in particular,

$$E = D^{\alpha-1}x(d) \geqslant D^{\alpha-1}x(t_0) > L_1 \geqslant E, \quad E = D^{\alpha-1}x(d) \leqslant D^{\alpha-1}x(t_1) < L_8 \leqslant E.$$

The contradictions obtained show that S_0, S_1 are empty. Since $D^{\alpha-1}x(t) \in C[0,1]$, $L_4 \leqslant D^{\alpha-1}x(t) \leqslant L_1$ for $t \in [0,d]$, so

$$|D^{\alpha-1}x(t)| \leqslant \max\{|L_1|, |L_8|\}, \quad t \in [0,d].$$

Similarly, the facts

$$f(t,x,p) \leqslant 0 \text{ for } (t,x,p) \in [d,1] \times R \times [L_3, L_4],$$
$$f(t,x,p) \geqslant 0 \text{ for } (t,x,p) \in [d,1] \times R \times [L_5, L_6],$$

yield that

$$|D^{\alpha-1}x(t)| \leqslant \max\{|L_3|, |L_6|\}, \quad t \in [d,1].$$

Consequently,

$$|D^{\alpha-1}x(t)| \leqslant \max\{|L_1|, |L_3|, |L_6|, |L_8|\} \triangleq G_1, \quad t \in [0,1].$$

On the other hand, by Lemma 2.5, for each $t \in (0,1]$, there exists $c \in (0,t)$ such that

$$x(t) - x(0) = D^{\alpha-1}x(c) \cdot \frac{1}{\alpha-1} t^{\alpha-1}.$$

From which we get $|x(t)| \leqslant G_2$, $G_2 = \max\{|A| + G_1/(\alpha-1)\}$. Finally the differential equation (3.1) together with the continuity of f and a priori estimations of $x(t)$ and $D^{\alpha-1}x(t)$ show that $|D^\alpha x(t)| \leqslant G_2$ for the constant G_2 independent of λ.

From the above arguments, let $M = G_1 + G_2 + G_3$, we know

$$\|x\|_\alpha = \|D^\alpha x\|_0 + \|D^{\alpha-1}x\|_0 + \|x\|_0 < M < \infty.$$

The above results indicate that T_λ has no fixed point on ∂U and $T_\lambda : [0,1] \times \overline{U} \to C^\alpha[0,1]$ is completely continuous. Denote by $X = C_B^\alpha[0,1]$. With the normality of the fixed-point index, the index of the constant operator $T_0(x) \equiv \psi(t)$ on U with respect to $C_B^\alpha[0,1]$, $i(T_0, U, X) = 1$. It follows from the homotopy invariant property of the fixed-point index that

$$i(T_1, U, X) = i(T_\lambda, U, X) = i(T_0, U, X) = 1.$$

With the solvability of fixed-point index, T_1 has a fixed-point in U, and so problem (1.1) has a solution in $\overline{U}$. □

The conditions imposed on $f(t,x,p)$ above are local with respect to t and p. In the next theorem they are also localized with respect to x.

Theorem 3.2. *Let $f : [0,1] \times R^2 \to R$ be continuous. Suppose there are constants L_i, $i = 1,2,\cdots,8$ such that $L_2 > L_1 \geqslant E$, $L_4 > L_3 \geqslant E$, $L_5 < L_6 \leqslant E$, $L_7 < L_8 \leqslant E$, where $E = (\alpha-1)(B-A)$ and*

(1) *there is $M \geqslant \max\{|A|,|B|\}$ such that $xf(t,x,0) > 0$ for $|x| > M$;*

(2) *$f(t,x,p) \geqslant 0$ for $(t,x,p) \in [0,1] \times [-M,M] \times ([L_1,L_2] \cup [L_5,L_6])$, $f(t,x,p) \leqslant 0$ for $(t,x,p) \in [0,1] \times [-M,M] \times ([L_3,L_4] \cup [L_7,L_8])$.*

Then the problem (1.1) has at least one solution in $C^\alpha[0,1]$.

Proof. If $x \in C^\alpha[0,1]$ is a solution of problem (1.1), then $|x(t)| \leqslant M$. In fact, if there is $t_0 \in (0,1)$ such that $x(t_0) = \max\{x(t)|t \in (0,1)\} > M$, and then by Lemma 2.2, u is α-order differentiable if and only if u is second-order differentiable and

$$D^\alpha u(t) = t^{2-\alpha}u''(t), \quad D^{\alpha-1}u(t) = t^{2-\alpha}u'(t).$$

Assuming that $u(t)$ achieves its maximum at $t_0 \in (0,1)$, then

$$u''(t_0) \leqslant 0, \quad u'(t_0) = 0,$$

consequently

$$D^\alpha x(t_0) \leqslant 0, \quad D^{\alpha-1}x(t_0) = 0.$$

But according to the condition (1), there is

$$x(t_0)D^\alpha x(t_0) = x(t_0)\lambda f(t_0, x(t_0), 0) > 0.$$

Thus $D^\alpha x(t_0) > 0$, a contradiction. So $x(t) \leqslant M$. Proving by the same methods we can get $x(t) \geqslant -M$. In conclusion $|x(t)| \leqslant M$. Furthermore, the proof is not essentially different from the proof of Theorem 3.1. □

Example 3.3. Consider the following boundary value problem:

$$D^{\frac{3}{2}}x(t) = x(t) - \frac{5}{2}D^{\frac{1}{2}}x(t) + [D^{\frac{1}{2}}x(t)]^2 - 1, \quad x(0) = 0, \quad x(1) = 1. \tag{3.4}$$

Choose $E = 0.5, M = 1, L_1 = 4, L_2 = 5, L_3 = 1, L_4 = 2, L_5 = -2, L_6 = -1, L_7 = 0, L_8 = 0.5$, then,

$$f(t,x,p) \geqslant 0 \text{ for } (t,x,p) \in [0,1] \times [-1,1] \times ([4,5] \cup [-2,-1]),$$
$$f(t,x,p) \leqslant 0 \text{ for } (t,x,p) \in [0,1] \times [-1,1] \times ([1,2] \cup [0,0.5]).$$

By Theorem 3.2, problem (3.4) has at least one solution in $C^{\frac{3}{2}}[0,1]$.

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