Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 3619–3629
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

On solving general split equality variational inclusion problems in Banach space

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Communicated by S. S. Chang

Abstract

In this paper, we are concerned with a new iterative scheme for general split equality variational inclusion problems in Banach spaces. We also show that the iteration converges strongly to a common solution of the general split equality variational inclusion problems (GSEVIP). The results obtained in this paper extend and improve some well-known results in the literature.

Keywords: General split equality variational problems, strong convergence, Banach space.
2010 MSC: 47J22, 34A60.

1. Introduction

Many problems in physics, optimization, and economics reduce to find a solution of an equilibrium problem. Some methods have been proposed to solve the equilibrium problem; see for instance [1, 4, 6, 9, 12, 14–17, 20, 24].

Let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. C_1 and C_2 are two nonempty closed convex subsets of H_1 and H_2, respectively. If $A : H_1 \to H_2$ is a bounded linear operator, the split feasibility problem (SFP) is defined as follows: find $x^* \in C_1$ such that

$$Ax^* \in C_2.$$

In 1994, Censor and Elfving [3] firstly introduced the (SFP) in finite-dimensional spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [4, 5]. The (SFP) in an infinite-dimensional real Hilbert space can be found in [22–24].

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doi:10.22436/jnsa.010.07.22

Received 2017-05-01

Recently, Moudafi and Al-Shemas introduced the following split equality feasibility problem (SEFP): to find $x \in C_1, y \in C_2$ such that

$$Ax = By, \tag{1.1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators.

In order to solve the split equality feasibility problem (1.1), Moudafi and Al-Shemas [19], introduced the following simultaneous iterative method:

$$\begin{cases} x_{n+1} = P_C[x_n - \gamma A^*(Ax_n - By_n)], \\ y_{n+1} = P_Q[y_n + \gamma A^*(Ax_n - By_n)], \end{cases}$$

and under suitable conditions they proved the weak convergence of the sequence (x_n, y_n) to a solution of (1.1) in Hilbert spaces.

Let H_1, H_2 be two real Hilbert spaces and F a real Banach space. $A : H_1 \to F$ and $B : H_2 \to F$ are two bounded linear operators and A^* and B^* are the adjoint mappings of A and B, respectively. For every $j = 1, 2, \cdots$, let C_j and Q_j be nonempty closed convex subsets of H_1 and H_2, respectively. i_{C_j} and i_{Q_j} denote the indicator functions of C_j and Q_j, while $N_{C_j}(x)$ and $N_{Q_j}(y)$ are the normal cone of C_j and Q_j at x and y, respectively, i.e.,

$$i_{C_j}(x) = \begin{cases} 0, & \text{if } x \in C_j, \\ +\infty, & \text{if } x \notin C_j, \end{cases} \qquad i_{Q_j}(y) = \begin{cases} 0, & \text{if } y \in Q_j, \\ +\infty, & \text{if } y \notin Q_j, \end{cases}$$

$$N_{C_j}(x) = \{z \in H_1 : \langle z, v - x\rangle \geqslant 0, \forall v \in C_j\}, \qquad N_{Q_j}(y) = \{z \in H_2 : \langle z, v - y\rangle \geqslant 0, \forall v \in Q_j\}.$$

It is well-known that i_{C_j} and i_{Q_j} are proper convex and lower semicontinuous functions on H_1 and H_2, respectively. And the sub-differentials ∂i_{C_j} and ∂i_{Q_j} are maximal monotone operators. For $j = 1, 2, \cdots$ and for all $\mu > 0$, we define the resolvent operator $J_\mu^{\partial i_{C_j}}$ of ∂i_{C_j} by

$$J_\mu^{\partial i_{C_j}}(\cdot) = (I + \mu \partial i_{C_j})^{-1}(\cdot) : H_1 \to H_1,$$

where

$$\begin{aligned} \partial i_{C_j}(x) = \{z \in H_1 : i_{C_j}(x) + \langle z, u - x\rangle \leqslant i_{C_j}(u), \forall u \in H_1\} &= \{z \in H_1 : \langle z, u - x\rangle \leqslant 0, \forall u \in C_j\} \\ &= N_{C_j}(x), x \in C_j. \end{aligned}$$

Hence we have

$$u = J_\mu^{\partial i_{C_j}}(x) \Leftrightarrow x - u \in \mu N_{C_j}(u) \Leftrightarrow \langle x - u, y - u\rangle \leqslant 0, \forall y \in C_j \Leftrightarrow u = P_{C_j}(x).$$

Here P_{C_j} is the metric projection from H_1 onto C_j. Therefore, we get

$$J_\mu^{\partial i_{C_j}} = P_{C_j}, \text{ and } J_\mu^{\partial i_{Q_j}} = P_{Q_j}, \quad j = 1, 2, \cdots,$$

which implies

$$\partial i_{C_j}{}^{-1}(0) = F(J_\mu^{\partial i_{C_j}}) = F(P_{C_j}), \text{ and } \partial i_{Q_j}{}^{-1}(0) = F(J_\mu^{\partial i_{Q_j}}) = F(P_{Q_j}), \quad j = 1, 2, \cdots. \tag{1.2}$$

The general split equality variational inclusion problem (GSEVIP) in a Banach space is defined as follows: find $(p, q) \in H_1 \times H_2$, such that

$$p \in \bigcap_{j=1}^{\infty} \partial i_{C_j}{}^{-1}(0), \quad q \in \bigcap_{j=1}^{\infty} \partial i_{Q_j}{}^{-1}(0), \quad \text{and} \quad Ap = Bq. \tag{1.3}$$

The set of all solutions of (GSEVIP) (1.3) is denoted by Ω, i.e,

$$\Omega = \{(p,q) \in H_1 \times H_2 : p \in \bigcap_{j=1}^{\infty} \partial i_{C_j}{}^{-1}(0),\ q \in \bigcap_{j=1}^{\infty} \partial i_{Q_j}{}^{-1}(0), \text{ and } Ap = Bq\}.$$

In this paper, we introduce a new iterative algorithm for solving the general split equality variational inclusion problem (GSEVIP) (1.3) in a Banach space and show that the suggested the iteration algorithm converges strongly to a solution of (GSEVIP) (1.3). The results of this paper extend and improve the corresponding results announced by Chang et al. [8, 9], and Moudafi and Al-Shemas [19].

2. Preliminaries and lemmas

In this section, we give some definitions and preliminaries which will be used in the sequel. Let H be a real Hilbert space and C be a nonempty closed convex subset of H.

An operator $G : H \to H$ is said to be

(i) a nonexpansive mapping, if

$$\|Gx - Gy\| \leqslant \|x - y\|, \forall x, y \in H;$$

(ii) a firmly nonexpansive mapping, if

$$\|Gx - Gy\|^2 \leqslant \langle Gx - Gy, x - y\rangle, \forall x, y \in H.$$

We denote by P_C the Metric projection from H onto C. Obviously, P_C is a firmly nonexpansive mapping from H onto C. Further, for any $x \in H$, $z{=}P_C x$ if and only if $\langle x - z, z - y\rangle \geqslant 0$, for all $y \in C$.

Let F be a real smooth Banach space. J_F is the dual mapping of F defined by

$$J_F(x) = \{x^* \in F^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2, x \in F\}.$$

Lemma 2.1 ([6])**.** *Let H be a real Hilbert space. Then for all $x, y \in H$, we have*

$$\| x + y \|^2 \leqslant \| x \|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.2 ([13])**.** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of positive real numbers satisfying $a_n \leqslant (1 - b_n)a_n + c_n$ for all $n \geqslant 1$. If the following conditions are satisfied*

(1) $b_n \in [0, 1]$ *and* $\sum_{n=1}^{\infty} b_n = \infty$;
(2) $\sum_{n=1}^{\infty} c_n < \infty$, *or* $\limsup_{n\to\infty} \frac{c_n}{b_n} \leqslant 0$,

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([10])**.** *Let H be a real Hilbert space, $B : H \to 2^H$ be a maximal monotone mapping and J^B_β be the resolvent mapping of B defined by $J^B_\beta = (I + \beta B)^{-1}$,$\beta > 0$, then*

(1) *for each $\beta > 0$, J^B_β is a single-valued and firmly nonexpansive mapping;*
(2) $D(J^B_\beta) = H$ *and* $F(J^B_\beta) = B^{-1}(0)$;
(3) $(I - J^B_\beta)$ *is a firmly nonexpansive mapping for each $\beta > 0$;*
(4) *suppose that $B^{-1}(0) \neq \emptyset$, then*

$$\| x - J^B_\beta x \|^2 + \| J^B_\beta x - x^* \| \leqslant \| x - x^* \|^2$$

for each $x \in H$, each $x^* \in B^{-1}(0)$, and each $\beta > 0$;

(5) *suppose that $B^{-1}(0) \neq \emptyset$, then $\langle x - J^B_\beta x, J^B_\beta x - w\rangle \geqslant 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.*

Lemma 2.4 ([7]). *Let* H *be a real Hilbert space and* $\{x_n\}$ *be a sequence in* H*. Then for any sequence* $\{\lambda_n\}$ $(\lambda_n \in (0,1))$ *with* $\sum_{n=1}^{\infty}\lambda_n = 1$*, the following inequality holds*

$$\| \sum_{n=1}^{\infty} \lambda_n x_n \|^2 \leqslant \sum_{n=1}^{\infty} \lambda_n \| x_n \|^2 - \lambda_i\lambda_j \| x_i - x_j \|^2, \forall i,j, i<j.$$

Lemma 2.5 ([18]). *Let* $\{t_n\}$ *be a sequence of real numbers. If there exists a subsequence* $\{n_i\}$ *of* $\{n\}$*, such that* $t_{n_i} < t_{n_i+1}$ *for all* $i \geqslant 1$*, then there exists a nondecreasing sequence* $\{\theta(n)\}$ *with* $\theta(n) \to \infty$ *as* $n \to \infty$*, such that for all (sufficiently large) positive integer number* n*, the following holds:*

$$t_{\theta(n)} \leqslant t_{\theta(n)+1}, \quad t_n \leqslant t_{\theta(n)+1}.$$

In fact,

$$\theta(n) = \max\{k \leqslant n : t_k \leqslant t_{k+1}\}.$$

Lemma 2.6 (demiclosedness principle). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. Then* $I - T$ *is said to be demi-closed at zero, if for any sequence* $\{x_n\} \subset C$ *with* $x_n \rightharpoonup x$ *and* $\|x_n - Tx_n\| \to 0$*, then* $x = Tx$*.*

3. The main results

In this section, we show some strong convergence theorems for finding a common element of the solution set of the general split equality variational inclusion problem (GSEVIP) (1.3) in a Banach space.

In order to solve problem (GSEVIP) (1.3), we propose the following simultaneous type iterative algorithm.

Algorithm 3.1. For any given $W_0 = (x_0, y_0)$, $V_0 = (v_{01}, v_{02}) \in H_1 \times H_2$, the iterative sequence $\{W_n\} \subset H_1 \times H_2$ is generated by

$$W_{n+1} = \alpha_n W_n + \beta_n V_0 + \sum_{j=1}^{\infty} \gamma_{n,j} J_{\mu}^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G) W_n, \quad n \geqslant 0, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,j}\}$ are the sequences of nonnegative numbers satisfying

$$\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{n,j} = 1, \quad n \geqslant 0,$$

$$J_{\mu}^{(\partial i_{C_j}, \partial i_{Q_j})} = \begin{pmatrix} J_{\mu}^{\partial i_{C_j}} \\ J_{\mu}^{\partial i_{Q_j}} \end{pmatrix}, G = \begin{pmatrix} A & -B \end{pmatrix}, G^* = \begin{pmatrix} A^* \\ -B^* \end{pmatrix}, G^* J_F G = \begin{pmatrix} A^* J_F A & -A^* J_F B \\ -B^* J_F A & B^* J_F B \end{pmatrix}.$$

We also need the following conclusion.

Lemma 3.2. *If the set of solutions of (GSEVIP)* (1.3) *is* $\Omega \neq \emptyset$*, then* $W^* = (x^*, y^*) \in H_1 \times H_2$ *is a solution of (GSEVIP)* (1.3) *if and only if for each* $j \geqslant 1$*, and for any given* $\mu > 0$

$$W^* = J_{\mu}^{(\partial i_{C_j}, \partial i Q_j)}(I - \mu G^* J_F G) W^*. \tag{3.2}$$

Proof. Indeed, $W^* = (x^*, y^*) \in H_1 \times H_2$ is a solution of (GSEVIP) (1.3), from Lemma 2.3 (2), for all $j \geqslant 1$, we have

$$x^* \in \partial i_{C_j}{}^{-1}(0) = F(J_{\mu}^{\partial i_{C_j}}), \; y^* \in \partial i_{Q_j}{}^{-1}(0) = F(J_{\mu}^{\partial i_{Q_j}}), \; Ax^* = By^*, \; GW^* = Ax^* - By^* = 0.$$

Thus for any $\mu > 0$,

$$J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W^* = J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})} W^* = (J_\mu^{\partial i_{C_j}} x^*, J_\mu^{\partial i_{Q_j}} y^*) = (x^*, y^*) = W^*.$$

Conversely, if $W^* = (x^*, y^*) \in H_1 \times H_2$ satisfies (3.2), we have

$$\begin{cases} x^* = J_\mu^{\partial i_{C_j}}(x^* - \mu A^* J_F(Ax^* - By^*)), \\ y^* = J_\mu^{\partial i_{Q_j}}(y^* + \mu B^* J_F(Ax^* - By^*)). \end{cases} \tag{3.3}$$

From Lemma 2.3 (5) and $\Omega \neq \emptyset$, we have

$$\langle x^* - (x^* - \mu A^* J_F(Ax^* - By^*)), x - x^* \rangle \geqslant 0, \quad \forall x \in \partial i_{C_j}{}^{-1}(0).$$

Since $\mu > 0$, we get

$$\langle J_F(Ax^* - By^*), Ax - Ax^* \rangle \geqslant 0, \quad \forall x \in \partial i_{Q_j}{}^{-1}(0). \tag{3.4}$$

Simplifying, from (3.3) and Lemma 2.3 (1), we have

$$\langle J_F(Ax^* - By^*), By^* - By \rangle \geqslant 0, \quad \forall y \in \partial i_{Q_j}{}^{-1}(0). \tag{3.5}$$

Adding up (3.4) and (3.5), we get

$$\langle J_F(Ax^* - By^*), Ax^* - By^* \rangle \leqslant \langle J_F(Ax^* - By^*), Ax - By \rangle \;\; \forall x \in \partial i_{C_j}{}^{-1}(0), \forall y \in \partial i_{Q_j}{}^{-1}(0).$$

So, we get

$$\|Ax^* - By^*\|^2 \leqslant \langle J_F(Ax^* - By^*), Ax - By \rangle, \quad \forall x \in \partial i_{C_j}{}^{-1}(0), \forall y \in \partial i_{Q_j}{}^{-1}(0).$$

Since the set of solutions of (GSEVIP) (1.3) is $\Omega \neq \emptyset$, taking $W = (x_0, y_0) \in \Omega$ and $x = x_0$, $y = y_0$, we get

$$\|Ax^* - By^*\| = 0, \;\; \text{i.e.,} \;\; Ax^* = By^*. \tag{3.6}$$

From (3.3), we have

$$\begin{cases} x^* = J_\mu^{\partial i_{C_j}} x^*, \\ y^* = J_\mu^{\partial i_{Q_j}} y^*, \end{cases} \quad \text{i.e.,} \; x^* \in F(J_\mu^{\partial i_{C_j}}) = \partial i_{C_j}{}^{-1}(0), y^* \in F(J_\mu^{\partial i_{Q_j}}) = \partial i_{Q_j}{}^{-1}(0), \forall j \geqslant 1, \tag{3.7}$$

which from (3.6) and (3.7) implies that $W^* \in \Omega$. □

Lemma 3.3. *If $\mu \in (0, \frac{2}{L})$, where $L = \|G\|^2$, then $I - \mu G^* J_F G : H_1 \times H_2 \to H_1 \times H_2$ is a nonexpansive mapping.*

Proof. For any given $w, u \in H_1 \times H_2$, we have

$$\begin{aligned} \|(I - \mu G^* J_F G)u - (I - \lambda G^* J_F G)w\|^2 &= \|(u - w) - \mu G^* J_F G(u - w)\|^2 \\ &= \|u - w\|^2 + \mu^2 \|G^* J_F G(u - w)\|^2 - 2\mu \langle u - w, G^* J_F G(u - w) \rangle \\ &\leqslant \|u - w\|^2 + \mu^2 L \|J_F G(u - w)\|^2 - 2\mu \langle G(u - w), J_F G(u - w) \rangle. \\ &= \|u - w\|^2 + \mu^2 L \|G(u - w)\|^2 - 2\mu \|G(u - w)\|^2. \\ &= \|u - w\|^2 - \mu(2 - \mu L)\|G(u - w)\|^2. \\ &\leqslant \|u - w\|^2. \end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $\{W_n\}$ be the sequence defined by (3.1). If the set of solutions of (GSEVIP) (1.3) is $\Omega \neq \emptyset$ and the following conditions are satisfied:*

(B1) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,j}\} \subset [0,1]$, *for any* $n \geqslant 0$, $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{n,j} = 1$;

(B2) $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\lim\limits_{n\to\infty} \beta_n = 0$;

(B3) $\mu \in (0, \frac{2}{L})$, *where* $L = \|G\|^2$;

(B4) $\liminf\limits_{n\to\infty} \alpha_n \gamma_{n,j} > 0, \forall j \geqslant 1$,

then the sequence $\{W_n\}$ converges strongly to $W^* = P_\Omega V_0$, which is a solution of (GSEVIP) (1.3).

Proof. We shall divide the proof into three steps.

Step (I). Showing that $\{W_n\}$ is bounded.

For any $p \in \Omega$, from Lemma 3.2, we have

$$p = J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)p.$$

Form Lemma 3.3, Lemma 2.3 (1), and condition (B3), we get

$$\begin{aligned}
&\|W_{n+1} - p\| \\
&= \|\alpha_n W_n + \beta_n V_0 + \textstyle\sum_{j=1}^{\infty} \gamma_{n,j} J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n - p\| \\
&\leqslant \alpha_n \|W_n - p\| + \beta_n \|V_0 - p\| + \textstyle\sum_{j=1}^{\infty} \gamma_{n,j} \|J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n - p\| \\
&\leqslant \alpha_n \|W_n - p\| + \beta_n \|V_0 - p\| + \textstyle\sum_{j=1}^{\infty} \gamma_{n,j} \|(I - \mu G^* J_F G)W_n - p\| \\
&\leqslant \alpha_n \|W_n - p\| + \beta_n \|V_0 - p\| + \textstyle\sum_{j=1}^{\infty} \gamma_{n,j} \|W_n - p\| \\
&= (1 - \beta_n)\|W_n - p\| + \beta_n \|V_0 - p\| \\
&\leqslant \max\{\|W_n - p\|, \|V_0 - p\|\}.
\end{aligned} \tag{3.8}$$

By induction, we have

$$\|W_n - p\| \leqslant \max\{\|W_0 - p\|, \|V_0 - p\|\}, \quad \forall n \geqslant 0,$$

which implies that $\{W_n\}$ is bounded.

Step (II). We show that the following inequality holds

$$\alpha_n \gamma_{n,j} \|W_n - J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n\|^2 \leqslant \|W_n - p\|^2 - \|W_{n+1} - p\|^2 + \beta_n \|V_0 - p\|^2. \tag{3.9}$$

From Lemma 2.4 and (3.1), we have

$$\begin{aligned}
\|W_{n+1} - p\|^2 &= \|\alpha_n W_n + \beta_n V_0 + \sum_{j=1}^{\infty} \gamma_{n,j} J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n - p\|^2 \\
&= \|\alpha_n (W_n - p) + \beta_n (V_0 - p) + \sum_{j=1}^{\infty} \gamma_{n,j} (J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n - p)\|^2 \\
&\leqslant \alpha_n \|W_n - p\|^2 + \beta_n \|V_0 - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j} \|J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n - p\|^2 \\
&\quad - \alpha_n \gamma_{n,j} \|W_n - J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n\|^2 \\
&\leqslant \alpha_n \|W_n - p\|^2 + \beta_n \|V_0 - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j} \|(I - \mu G^* J_F G)W_n - p\|^2 \\
&\quad - \alpha_n \gamma_{n,j} \|W_n - J_\mu^{(\partial i_{C_j}, \partial i_{Q_j})}(I - \mu G^* J_F G)W_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha_n\|W_n-p\|^2+\beta_n\|V_0-p\|^2+\sum_{j=1}^{\infty}\gamma_{n,j}\|W_n-p\|^2\\
&\quad-\alpha_n\gamma_{n,j}\|W_n-J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n\|^2\\
&\leqslant (1-\beta_n)\|W_n-p\|^2+\beta_n\|V_0-p\|^2-\alpha_n\gamma_{n,j}\|W_n-J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n\|^2\\
&\leqslant \|W_n-p\|^2+\beta_n\|V_0-p\|^2-\alpha_n\gamma_{n,j}\|W_n-J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n\|^2.
\end{aligned}
$$

Inequality (3.9) is proved.

Step (III). Since the set of solutions of (GSEVIP) (1.3) is $\Omega\neq\emptyset$ and C_j and Q_j $(j=1,2,\cdots)$ are nonempty closed convex subsets of H_1 and H_2, respectively, from (1.2) and lemma 2.3, we can get the set of solutions of (GSEVIP) (1.3), Ω, is nonempty closed convex. Setting $W^*=P_\Omega V_0$, we have $W^*\in\Omega$. We will prove that $\{W_n\}$ converges strongly to W^*.

We consider two cases: (I) Suppose that the sequence $\{\|W_n-W^*\|\}$ is monotone.

Since $\{\|W_n-W^*\|\}$ is monotone, following Step (I), it is obvious that $\{\|W_n-W^*\|\}$ is convergent. From conditions (B2), (B4), and (3.9), we have

$$
\lim_{n\to\infty}\|W_n-J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n\|=0, \tag{3.10}
$$

and

$$
\begin{aligned}
\|W_{n+1}-W^*\|^2&=\|\alpha_nW_n+\beta_nV_0+\sum_{j=1}^{\infty}\gamma_{n,j}J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n-W^*\|^2\\
&=\|\alpha_n(W_n-W^*)+\beta_n(V_0-W^*)+\sum_{j=1}^{\infty}\gamma_{n,j}(J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n-W^*)\|^2\\
&\leqslant\|\alpha_n(W_n-W^*)+\sum_{j=1}^{\infty}\gamma_{n,j}(J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_n-W^*)\|^2\\
&\quad+2\beta_n\langle V_0-W^*,W_{n+1}-W^*\rangle\quad\text{(by Lemma 2.1)}\\
&\leqslant\{\alpha_n\|W_n-W^*\|+\sum_{j=1}^{\infty}\gamma_{n,j}\|W_n-W^*\|\}^2+2\beta_n\langle V_0-W^*,W_{n+1}-W^*\rangle\\
&=(1-\beta_n)^2\|W_n-W^*\|^2+2\beta_n\langle V_0-W^*,W_{n+1}-W^*\rangle.
\end{aligned}
$$

Since $\beta_n\in[0,1]$. Then we have

$$
\|W_{n+1}-W^*\|^2\leqslant(1-\beta_n)\|W_n-W^*\|^2+2\beta_n\langle V_0-W^*,W_{n+1}-W^*\rangle. \tag{3.11}
$$

Since $\{W_n\}$ is bounded, there exists a subsequence $\{W_{n_k}\}\subset\{W_n\}$ such that $W_{n_k}\rightharpoonup W\in H_1\times H_2$. From (3.10), we get

$$
\lim_{n\to\infty}\|W_{n_k}-J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W_{n_k}\|=0.
$$

Since $J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)$ is a nonexpansive mapping, by Lemma 2.6, we have

$$
W=J_\mu^{(\partial i_{C_j},\partial i_{Q_j})}(I-\mu G^*J_FG)W.
$$

From Lemma 3.2, this shows that $W\in\Omega$. We get

$$
\limsup_{n\to\infty}\langle V_0-W^*,W_n-W^*\rangle=\lim_{k\to\infty}\langle V_0-W^*,W_{n_k}-W^*\rangle\leqslant\langle V_0-W^*,W-W^*\rangle\leqslant 0.
$$

Taking $a_n = \|W_n - W^*\|$, $b_n = \beta_n$, and $c_n = \beta_n\langle V_0 - W^*, W_{n+1} - W^*\rangle$, all conditions of Lemma 2.2 are satisfied, so $\lim_{n\to\infty} \|W_{n+1} - W^*\| = 0$.

(II) Suppose that the sequence $\{\|W_n - W^*\|\}$ is not monotone.

By Lemma 2.5, there exists a sequence of positive integers, $\{\theta(n)\}$, $n \geqslant n_0$ (where n_0 is large enough) such that

$$\theta(n) = \max\{k \leqslant n : \|W_k - W^*\| \leqslant \|W_{k+1} - W^*\|\}. \tag{3.12}$$

Clearly, $\{\theta(n)\}$ is nondecreasing and $\theta(n) \to \infty$ as $n \to \infty$ and for all $n \geqslant n_0$,

$$\|W_{\theta(n)} - W^*\| \leqslant \|W_{\theta(n)+1} - W^*\|, \quad \|W_n - W^*\| \leqslant \|W_{\theta(n)+1} - W^*\|.$$

Hence, $\{\|W_{\theta(n)} - W^*\|\}$ is a nondecreasing sequence. By virtue of Case (I), $\lim_{n\to\infty} \|W_{\theta(n)} - W^*\| = 0$ and $\lim_{n\to\infty} \|W_{\theta(n)+1} - W^*\| = 0$, we have

$$0 \leqslant \|W_n - W^*\| \leqslant \max\{\|W_n - W^*\|, \|W_{\theta(n)} - W^*\|\} \leqslant \|W_{\theta(n)+1} - W^*\| \to 0, \text{ as } n \to \infty.$$

This implies that $W_n \to W^*$ and $W^* = P_\Omega V_0$ is a solution of (GSEVIP) (1.3). □

4. Applications

In this section, two examples will be illustrated to verify the validity of the proposed algorithm in Section 3.

4.1. *Application to the general split equality equilibrium problem in Banach space*

Let H_1, H_2 be two real Hilbert spaces and F is a real smooth Banach space. Let C_1 and C_2 are nonempty closed convex subsets of H_1 and H_2, respectively, and $A : H_1 \to F$, $B : H_2 \to F$ are bounded linear operator. let F_i: $C_1 \times C_1 \to R$ and G_i: $C_2 \times C_2 \to R$, $i = 1, 2, \cdots$ be two equilibrium functions, where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively.

Assumption 4.1. Let F: $C \times C \to R$ be an equilibrium function satisfying the following assumptions:

(1) $F(x, x)=0, \ \forall x \in C$;
(2) F is monotone, i.e., $F(x, y) + F(y, x) \leqslant 0, \quad \forall x, y \in C$;
(3) for each $x, y, z \in C$, $\limsup_{t\to 0^+} F(tz + (1 - t)x, y) \leqslant F(x, y)$;
(4) for each $x \in C, y \longmapsto F(x, y)$ is convex and lower semi-continuous.

For Assumption 4.1, see [1, 21].

Lemma 4.2 ([11])**.** *Let C be a nonempty closed convex subset of a Hilbert space* H *and* F: $C \times C \to R$ *be a equilibrium which satisfies Assumption 4.1. For all* $r > 0$ *and* $x \in H$*, the resolvent of the equilibrium function* F *is the operator* $T_r^F : H \to C$ *defined by*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \ \forall y \in C\}.$$

Then T_r^F *is well-defined and the followings hold:*

(1) T_r^F *is nonempty and single-valued;*
(2) T_r^F *is firmly nonexpansive, i.e., for any x,y* $\in H_1$

$$\|T_r^F(x) - T_r^F(y)\|^2 \leqslant \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

(3) *F(*T_r^F*)=EP(*F*);*
(4) *EP(*F*) is closed and convex.*

Let H_1, H_2 be two real Hilbert spaces. Let C_1 and C_2 be nonempty closed convex subsets of H_1 and H_2, respectively. The general split equality equilibrium problem (GSEEP) is defined as follows: to find $x^* \in C_1$, $y^* \in C_2$, such that

$$F_i(x^*, x) \geqslant 0, \ \ \forall x \in C_1, G_i(y^*, y) \geqslant 0, \ \ \forall\, y \in C_2 \ \text{ and } \ Ax^* = By^*. \tag{4.1}$$

We know that the general split equality equilibrium problem (GSEEP) (4.1) is equivalent to find $x^* \in C_1$, $y^* \in C_2$, such that for each $\lambda > 0$,

$$x^* \in \bigcap_{i=1}^{\infty} EP(F_i, C_1) = \bigcap_{i=1}^{\infty} F(T_r^{F_i}), y^* \in \bigcap_{i=1}^{\infty} EP(G_i, C_2) = \bigcap_{i=1}^{\infty} F(T_r^{G_i}), \text{ such that } Ax^* = By^*.$$

Letting $C = \bigcap_{i=1}^{\infty} F(T_r^{F_i})$, $Q = \bigcap_{i=1}^{\infty} F(T_r^{G_i})$, by Lemma 4.2, C (resp.Q) is a nonempty closed and convex subset of C (res.Q).

The general split equality equilibrium problem (GSEEP) in Banach space is defined as follows:

$$p \in C, q \in Q, \ \ \text{such that} \ \ Ap = Bq. \tag{4.2}$$

In order to solve problem (GSEEP) (4.2), we propose the following simultaneous type iterative algorithm.

Algorithm 4.3. For any given $W_0 = (x_0, y_0)$, $V_0 = (v_{01}, v_{02}) \in H_1 \times H_2$, the iterative sequence $W_n \in H_1 \times H_2$ is generated by

$$W_{n+1} = \alpha_n W_n + \beta_n V_0 + P_{C\times Q}(I - \mu_n G^* J_F G)W_n \quad n \geqslant 0, \tag{4.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,j}\}$ are the sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geqslant 0$,

$$P_{C\times Q} = \begin{pmatrix} P_C \\ P_Q \end{pmatrix}, G = \begin{pmatrix} A & -B \end{pmatrix}, G^* = \begin{pmatrix} A^* \\ -B^* \end{pmatrix}, G^* J_F G = \begin{pmatrix} A^* J_F A & -A^* J_F B \\ -B^* J_F A & B^* J_F B \end{pmatrix}.$$

Theorem 4.4. *Let H_1, H_2, F, C, Q, A, B, A^*, B^*, $P_{C\times Q}$, G, G^*, be the same as above. Let $\{W_n\}$ be the sequence defined by (4.3). If the set of solutions of (GSEEP) (4.2) is $\Omega \neq \emptyset$, where $\Omega = \{(x^*, y^*) \in H_1 \times H_2 : (x^*, y^*) \in C \times Q : Ax^* = By^*\}$, the following conditions are satisfied:*

(B1) $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$, *for any* $n \geqslant 0$

$$\alpha_n + \beta_n + \gamma_n = 1;$$

(B2) $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;

(B3) $\mu_n \in (0, \frac{2}{L})$, *where* $L = \|G\|^2$;

(B4) $\liminf_{n\to\infty} a_n \gamma_n > 0$.

Then the sequence $\{W_n\}$ converges strongly to $W^* = P_\Omega V_0$, which is a solution of (GSEEP) (4.1).

4.2. Application to the general split equality feasibility problem in Banach space

Let H_1, H_2 be two real Hilbert spaces and F is a real smooth Banach space. Let $C_{1,j}$ and $C_{2,j}$, $j = 1, 2, \cdots$ be nonempty closed convex subsets of H_1 and H_2, respectively. The general split equality feasibility problem (GSEFP) in Banach space is defined as follows:

$$p \in \bigcap_{j=1}^{\infty} C_{1,j}, q \in \bigcap_{j=1}^{\infty} Q_{1,j} \ \ \text{such that} \ \ Ap = Bq. \tag{4.4}$$

In order to solve problem (GSEFP) (4.4), we propose the following simultaneous type iterative algorithm.

Algorithm 4.5. For any given $W_0 = (x_0, y_0)$, $V_0 = (v_{01}, v_{02}) \in H_1 \times H_2$, the iterative sequence $W_n \in H_1 \times H_2$ is generated by

$$W_{n+1} = \alpha_n W_n + \beta_n V_0 + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j \times Q_j}(I - \mu_n G^* J_F G) W_n, \quad n \geqslant 0, \tag{4.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in [0,1] with $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{n,j} = 1$, for all $n \geqslant 0$,

$$P_{C_j \times Q_j} = \begin{pmatrix} P_{C_j} \\ P_{Q_j} \end{pmatrix}, G = \begin{pmatrix} A & -B \end{pmatrix}, G^* = \begin{pmatrix} A^* \\ -B^* \end{pmatrix}, G^* J_F G = \begin{pmatrix} A^* J_F A & -A^* J_F B \\ -B^* J_F A & B^* J_F B \end{pmatrix}.$$

Theorem 4.6. *Let H_1, H_2, F, C_j, Q_j, A, B, A^*, B^*, G, G^*, P_{C_j}, P_{Q_j} be the same as above. Let $\{W_n\}$ be the sequence defined by (4.5). Assum the set of solution of (SEFPP) (4.4) is $\Omega \neq \emptyset$, and the following conditions are satisfied*

(B1) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,j}\} \subset [0, 1]$, *for any* $n \geqslant 0$

$$\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{n,j} = 1;$$

(B2) $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;

(B3) $\mu_n \in (0, \frac{2}{L})$, *where* $L = \|G\|^2$;

(B4) $\liminf_{n\to\infty} a_n \gamma_{n,j} > 0$, $\forall j \geqslant 1$.

Then the sequence $\{W_n\}$ converges strongly to $W^* = P_\Omega V_0$, which is a solution of (SEFPP) (4.4).

Acknowledgment

The work was supported by the National Natural Sciences Foundation of China (Grant NO: 11661012, 11671101), the National Natural Sciences Foundation of Guangxi (Grant NO: 2015GXNSFAA309018) and Special Funds of Guangxi Distinguished Experts Construction Engineering.

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