(L, M)-fuzzy convex structures

Fu-Gui Shi[a], Zhen-Yu Xiu[b,*]

[a]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China.

[b]College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610000, China.

Communicated by S. K. Samanta

Abstract

In this paper, the notion of (L, M)-fuzzy convex structures is introduced. It is a generalization of L-convex structures and M-fuzzifying convex structures. In our definition of (L, M)-fuzzy convex structures, each L-fuzzy subset can be regarded as an L-convex set to some degree. The notion of convexity preserving functions is also generalized to lattice-valued case. Moreover, under the framework of (L, M)-fuzzy convex structures, the concepts of quotient structures, substructures and products are presented and their fundamental properties are discussed. Finally, we create a functor ω from MYCS to LMCS and show that MYCS can be embedded in LMCS as a coreflective subcategory, where MYCS and LMCS denote the category of M-fuzzifying convex structures and the category of (L, M)-fuzzy convex structures, respectively. ©2017 All rights reserved.

Keywords: (L, M)-fuzzy convex structure, (L, M)-fuzzy convexity preserving function, quotient structures, substructures, products.
2010 MSC: 03E72, 52A01.

1. Introduction

Convexity theory has been accepted to be of increasing importance in recent years in the study of extremum problems in many areas of applied mathematics. The concept of convexity which was mainly defined and studied in $\mathbb{R}^n$ in the pioneering works of Newton, Minkowski, and others as described in [2], now finds a place in several other mathematical structures such as vector spaces, posets, lattices, metric spaces, graphs, and median algebras. This development is motivated by not only the need for an abstract theory of convexity generalizing the classical theorems in $\mathbb{R}^n$ due to Helly, Caratheodory, etc.; but also by the necessity to unify geometric aspects of all these mathematical structures. Abstract convexity theory is a branch of mathematics dealing with set-theoretic structures satisfying axioms similar to that usual convex sets fulfill. Here, by "usual convex sets", we mean convex sets in real linear spaces. In a general setting, the axioms of abstract convexity are the following:

(1) the empty set and the universe set are convex;
(2) the intersection of a nonempty collection of convex sets is convex;

---

*Corresponding author
Email addresses: fuguishi@bit.edu.cn (Fu-Gui Shi), xiuzhenyu112@sohu.com (Zhen-Yu Xiu)

(3) the union of a chain of convex sets is convex.

Clearly, usual convex sets have properties (1)-(3), but there are many other collections of sets, coming from various types of mathematical objects, that satisfy conditions (1)-(3), such as convexities in lattices and in Boolean algebras [22, 24], convexities in metric spaces and graphs [9, 20]. Also, convex structures appeared naturally in topology, especially in the theory of supercompact spaces [23].

The notion of a fuzzy subset was introduced by Zadeh [28] and then fuzzy subsets have been applied to various branches of mathematics. In 1994, Rosa generalized the notion of a convex structure to a fuzzy convex structure $(X, \mathfrak{C})$ in [14, 15] and $\mathfrak{C}$ was defined as a crisp family of fuzzy subsets of a set X satisfying certain axioms. For convenience, we call this fuzzy convex structure an I-convex structure. In 2009, Maruyama generalized I-convex structures to L-convex structures in [10], where L is a completely distributive lattice. Recently, Pang and Shi [12] introduced several types of L-convex structures, including stratified L-convex structures, convex-generated L-convex structures, weakly induced L-convex structures and induced L-convex structures and discussed their relations from a categorical aspect. Pang and Zhao [13] presented several characterizations of L-convex structures. When L is a continuous lattice, in [6], Jin and Li discussed the relations between the category of stratified L-convex structures and the category of convex structures.

In 2014, Shi and Xiu [19] introduced a new approach to the fuzzification of convex structures, which is called an M-fuzzifying convex structure. An M-fuzzifying convex structure is a pair $(X, \mathscr{C})$, where $\mathscr{C}$ is a mapping from $\mathbf{2}^X$ to M satisfying three axioms. Shi and Li [18] generalized the notion of restricted hull operators in classical convex structures to M-fuzzifying restricted hull operators and used it to characterize M-fuzzifying convex structures. Xiu and Shi [26] introduced the concept of M-fuzzifying interval operators and established its relations with M-fuzzifying convex structures from a categorical aspect.

In this paper, based on the idea of [8] and [21], combining L-convex structures and M-fuzzifying convex structures and based on complete distributive lattices L and M, we present a more general approach to the fuzzification of convex structures. More specifically, we define an (L, M)-fuzzy convexity on a nonempty set X by means of a mapping $\mathcal{C} : L^X \to M$ satisfying three axioms. It is a generalization of L-convex structures and M-fuzzifying convex structures. Each L-fuzzy subset of X can be regarded as an L-convex set to some degree.

Throughout this paper, unless otherwise stated, both L and M denote complete distributive lattices, $I = [0, 1]$, $\mathbf{2} = \{0, 1\}$ and X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. We often do not distinguish a crisp subset A of X and its characteristic function χ_A. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X, respectively. The smallest element and the largest element in $M(L)$ are denoted by $\perp_M(\perp_L)$ and $\top_M(\top_L)$, respectively. We also adopt the convention that $\bigwedge \emptyset = \top_M$.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$, [3]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [25], denoted by $\beta(b)$. Moreover, the binary relation $\prec^{op}$ in M is defined as follows: for $a, b \in M$, $a \prec^{op} b$ if and only if for every subset $D \subseteq M$, the relation $\wedge D \leqslant a$ always implies the existence of $d \in D$ with $d \leqslant b$. $\{b \in M : a \prec^{op} b\}$ is called the greatest maximal family of a in the sense of [25], denoted by $\alpha(a)$. In a completely distributive lattice M, there exist $\alpha(b)$ and $\beta(b)$ for each $b \in M$, and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ (see [25]).

For $a \in L$ and $A \in L^X$, we use the following notations:

(1) $A_{[a]} = \{x \in X : a \leqslant A(x)\}$;
(2) $A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}$;
(3) $A_{(a)} = \{x \in X : a \in \beta(A(x))\}$.

Some properties of these cut sets can be found in [5, 11, 16, 17].

Let $f : X \to Y$ be a mapping. Define $f_L^\rightarrow : L^X \to L^Y$ and $f_L^\leftarrow : L^Y \to L^X$ by $f_L^\rightarrow(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f_L^\leftarrow(B) = B \circ f$ for $B \in L^Y$, respectively.

Definition 1.1 ([16])**.** Let $a \in M$ and $D \subseteq X$, define two M-fuzzy sets $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D, \\ \bot, & x \notin D, \end{cases} \quad (a \vee D)(x) = \begin{cases} \top, & x \in D, \\ a, & x \notin D. \end{cases}$$

Theorem 1.2 ([25])**.** *For* $\{a_i : i \in \Omega\} \subseteq M$,

(1) $\alpha \left(\bigwedge_{i \in \Omega} a_i \right) = \bigcup_{i \in \Omega} \alpha(a_i)$, *i.e.,* α *is a* $\bigwedge - \bigcup$ *mapping;*

(2) $\beta \left(\bigvee_{i \in \Omega} a_i \right) = \bigcup_{i \in \Omega} \beta(a_i)$, *i.e.,* β *is a* $\bigvee - \bigcup$ *mapping.*

Theorem 1.3 ([5, 17])**.** *For each* L-*fuzzy set* A *in* L^X, *we have:*

(1) $A = \bigvee_{a \in L} (a \wedge A_{[a]}) = \bigwedge_{a \in L} (a \vee A^{[a]})$;

(2) $\forall a \in L, A_{[a]} = \bigcap_{b \in \beta(a)} A_{[b]} = \bigcap_{b \in \beta(a)} A_{(b)}$;

(3) $\forall a \in L, A^{[a]} = \bigcap_{a \in \alpha(b)} A^{[b]}$;

(4) $\forall a \in L, A_{(a)} = \bigcup_{a \in \beta(b)} A_{[b]}$.

Theorem 1.4 ([16])**.** *For a family of* L-*fuzzy sets* $\{A_i : i \in \Omega\}$ *in* L^X *and* $a \in L$, *we have:*

(1) $\left(\bigwedge_{i \in \Omega} A_i \right)_{[a]} = \bigcap_{i \in \Omega} (A_i)_{[a]}$;

(2) $\left(\bigvee_{i \in \Omega} A_i \right)_{(a)} = \bigcup_{i \in \Omega} (A_i)_{(a)}$;

(3) $\left(\bigwedge_{i \in \Omega} A_i \right)^{[a]} = \bigcap_{i \in \Omega} (A_i)^{[a]}$.

Definition 1.5 ([22])**.** A subset $\mathbb{C}$ of 2^X is called a convexity if it satisfies the following conditions:

(C1) $\emptyset, X \in \mathbb{C}$;

(C2) if $\{A_i : i \in \Omega\} \subseteq \mathbb{C}$ is nonempty, then $\bigcap_{i \in \Omega} A_i \in \mathbb{C}$;

(C3) if $\{A_i : i \in \Omega\} \subseteq \mathbb{C}$ is nonempty and totally ordered by inclusion, then $\bigcup_{i \in \Omega} A_i \in \mathbb{C}$.

The pair $(X, \mathbb{C})$ is called a convex structure and the elements in $\mathbb{C}$ are called convex sets.

Definition 1.6 ([10])**.** For a nonempty set X and a subset $\mathfrak{C}$ of L^X, $\mathfrak{C}$ is called an L-convexity if it satisfies the following conditions:

(LC1) $\chi_\emptyset, \chi_X \in \mathfrak{C}$;

(LC2) if $\{A_i : i \in \Omega\} \subseteq \mathfrak{C}$ is nonempty, then $\bigwedge_{i \in \Omega} A_i \in \mathfrak{C}$;

(LC3) if $\{A_i : i \in \Omega\} \subseteq \mathfrak{C}$ is nonempty and totally ordered by inclusion, then $\bigvee_{i \in \Omega} A_i \in \mathfrak{C}$.

If $\mathfrak{C}$ is an L-convexity on X, then the pair $(X, \mathfrak{C})$ is called an L-convex structure. When $L = I$, an L-convexity is exactly a fuzzy convex structure in [14, 15].

Definition 1.7 ([19])**.** A mapping $\mathscr{C} : 2^X \to M$ is called an M-fuzzifying convexity on X if it satisfies the following conditions:

(MYC1) $\mathscr{C}(\emptyset) = \mathscr{C}(X) = \top_M$;

(MYC2) if $\{A_i : i \in \Omega\} \subseteq 2^X$ is nonempty, then $\mathscr{C}(\bigcap\limits_{i \in \Omega} A_i) \geqslant \bigwedge\limits_{i \in \Omega} \mathscr{C}(A_i)$;

(MYC3) if $\{A_i : i \in \Omega\} \subseteq 2^X$ is nonempty and totally ordered by inclusion, then $\mathscr{C}(\bigcup\limits_{i \in \Omega} A_i) \geqslant \bigwedge\limits_{i \in \Omega} \mathscr{C}(A_i)$.

If $\mathscr{C}$ is an M-fuzzifying convexity on X, then the pair $(X, \mathscr{C})$ is called an M-fuzzifying convex structure.

Theorem 1.8 ([19]). *A mapping $\mathscr{C} : 2^X \to M$ is an M-fuzzifying convexity if and only if for each $\mathfrak{a} \in M \backslash \{\perp_M\}$, $\mathscr{C}_{[\mathfrak{a}]}$ is a convexity.*

Definition 1.9 ([19]). Let $\varphi : 2^X \to M$ be a mapping. The M-fuzzifying convex structure $(X, \mathscr{C})$ generated by φ is given by

$$\forall A \in 2^X, \ \mathscr{C}_\varphi(A) = \bigwedge \{\mathscr{D}(A) : \varphi \leqslant \mathscr{D} \in \mathfrak{G}\},$$

where $\mathfrak{G}$ denotes all the M-fuzzifying convexities on X. Then φ is called a subbase of the M-fuzzifying convexity $\mathscr{C}$. Alternatively, we say that φ generates the convexity $\mathscr{C}_\varphi$.

Definition 1.10 ([1]). Let L be a lattice and A a fuzzy subset of L. Then A is called a fuzzy sublattice of L if for all $x, y \in L$,
(i) $A(x \wedge y) \geqslant A(x) \wedge A(y)$,
(ii) $A(x \vee y) \geqslant A(x) \wedge A(y)$.
    A fuzzy sublattice A is said to be convex if for every interval $[a, b] \subseteq L$ and for all $x \in [a, b]$, $A(x) \geqslant A(a) \wedge A(b)$.

Definition 1.11 ([2]). Let G be a group. A fuzzy subset λ of G is said to be a fuzzy subgroup if
(1) $\lambda(xy) \geqslant \lambda(x) \wedge \lambda(y)$,
(2) $\lambda(x^{-1}) \geqslant \lambda(x)$.
    Let G be an ordered group. A fuzzy subgroup λ of G is said to be a fuzzy convex subgroup if for every interval $[a, b] \subseteq G$ and for all $x \in [a, b]$, we have $\lambda(x) \geqslant \lambda(a) \wedge \lambda(b)$.

2. (L, M)-fuzzy convex structures

In this section, combining the concepts of L-convex structures and M-fuzzifying convex structures, we introduce a general approach to the fuzzification of convex structures as follows.

Definition 2.1. A mapping $\mathcal{C} : L^X \to M$ is called an (L, M)-fuzzy convexity on X if it satisfies the following three conditions:

(LMC1) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$;

(LMC2) if $\{A_i : i \in \Omega\} \subseteq L^X$ is nonempty, then $\mathcal{C}\left(\bigwedge\limits_{i \in \Omega} A_i\right) \geqslant \bigwedge\limits_{i \in \Omega} \mathcal{C}(A_i)$;

(LMC3) if $\{A_i : i \in \Omega\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then $\mathcal{C}\left(\bigvee\limits_{i \in \Omega} A_i\right) \geqslant \bigwedge\limits_{i \in \Omega} \mathcal{C}(A_i)$.

If $\mathcal{C}$ is an (L, M)-fuzzy convexity, then $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex structure.
    An $(L, \mathbf{2})$-fuzzy convex structure is an L-convex structure. An $(I, \mathbf{2})$-fuzzy convex structure can be viewed as an I-convex structure. A $(\mathbf{2}, M)$-fuzzy convex structure is an M-fuzzifying convex structure. A crisp convex structure in [22] can be regarded as a $(\mathbf{2}, \mathbf{2})$-fuzzy convex structure.
    If $\mathcal{C}$ is an (L, M)-fuzzy convexity, then $\mathcal{C}(A)$ can be regarded as the degree to which A is an L-convex set.

    Next we give some examples of (L, M)-fuzzy convex structures, L-convex structures, and M-fuzzifying convex structures, respectively.

Example 2.2. Let a mapping $\mathcal{T}: L^X \to M$ be an (L, M)-fuzzy topology in [8, 21]. If it satisfies the following conditions:

(S) $\forall \{A_j\}_{j \in J} \subseteq L^X$, $\mathcal{T}(\bigwedge_{j \in J} A_j) \geqslant \bigwedge_{j \in J} \mathcal{T}(A_j)$,

then $\mathcal{T}$ is called a saturated (L, M)-fuzzy topology, and $(X, \mathcal{T})$ is called an Alexandroff (L, M)-fuzzy topological space.

 We can see that an Alexandroff (L, M)-fuzzy topological space $(X, \mathcal{T})$ is an (L, M)-fuzzy convex structure.

 When $L = \mathbf{2}$ and $M = I$, an Alexandroff (L, M)-fuzzy topological space $(X, \mathcal{T})$ is an Alexandroff fuzzifying topological space in [4, 27] and it is an example of M-fuzzifying convex structures.

Example 2.3 ([7])**.** An I-fuzzified set of all upper sets of a fuzzy preordered set (X, R) is a map $\nabla(R) : I^X \to I$ defined by

$$\forall U \in I^X, \ \nabla(R)(U) = \bigwedge_{(x,y) \in X \times X} R(x, y) \to (U(x) \to U(x)).$$

For a given fuzzy preorder R on X, $\nabla(R)$, the I-fuzzified set of all upper sets of (X, R) has the following properties: for all $\mathcal{F} \subseteq I^X$, $U, V \in I^X$, and $\lambda \in [0, 1]$,

 (i) $\nabla(R)(\lambda) = 1$ for every constant mapping λ from X to $[0, 1]$;
 (ii) $\bigwedge \nabla(R)(\mathcal{F}) \leqslant \nabla(R)(\bigwedge \mathcal{F})$ where $\nabla(R)(\mathcal{F}) = \{\nabla(R)(U) | U \in \mathcal{F}\}$;
(iii) $\bigwedge \nabla(R)(\mathcal{F}) \leqslant \nabla(R)(\bigvee \mathcal{F})$.

We can see that $\nabla(R)$ satisfies (LMC1)-(LMC3) and then $(X, \nabla(R))$ is an (L, M)-fuzzy convex structure, where $M = L = I$.

Example 2.4. Define a mapping $\mathcal{C} : L^{\mathbb{R}^n} \to M$ by

$$\forall A \in L^{\mathbb{R}^n}, \ \mathcal{C}(A) = \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} (A(x) \wedge A(y)) \to A(\lambda x + (1 - \lambda)y),$$

where the binary function $\to$ is defined as follows: for $a, b, c \in L$, $a \to b = \bigvee\{c \in L : a \wedge c \leqslant b\}$. Then $(X, \mathcal{C})$ is an (L, M)-fuzzy convex structure. Next we show that $\mathcal{C}$ satisfies (LMC1)-(LMC3).

(LMC1) Clearly, $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_{\mathbb{R}^n}) = \top_M$.

(LMC2) For any nonempty set $\{A_i : i \in \Omega\} \subseteq L^{\mathbb{R}^n}$, we have

$$\mathcal{C}(\bigwedge_{i \in \Omega} A_i) = \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} ((\bigwedge_{i \in \Omega} A_i)(x) \wedge (\bigwedge_{i \in \Omega} A_i)(y) \to (\bigwedge_{i \in \Omega} A_i)(\lambda x + (1 - \lambda)y)$$

$$= \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} (\bigwedge_{i \in \Omega} A_i(x) \wedge \bigwedge_{i \in \Omega} A_i(y)) \to \bigwedge_{i \in \Omega} A_i(\lambda x + (1 - \lambda)y)$$

$$= \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} \bigwedge_{i \in \Omega} ((\bigwedge_{i \in \Omega} A_i(x) \wedge \bigwedge_{i \in \Omega} A_i(y)) \to A_i(\lambda x + (1 - \lambda)y))$$

$$\geqslant \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} \bigwedge_{i \in \Omega} (A_i(x) \wedge A_i(y)) \to A_i(\lambda x + (1 - \lambda)y)$$

$$= \bigwedge_{i \in \Omega} \bigwedge_{\lambda \in [0,1]} \bigwedge_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} ((A_i(x) \wedge A_i(y)) \to A_i(\lambda x + (1 - \lambda)y))$$

$$= \bigwedge_{i \in \Omega} \mathcal{C}(A_i).$$

The proof of (LMC3) is similar to that of (LMC2) and is omitted.

When $M = 2$, we obtain the following example.

Example 2.5 ([10]). An L-fuzzy set μ on $\mathbb{R}^n$ is an L-fuzzy convex set on $\mathbb{R}^n$ iff $\mu(rx + (1 - r)y) \geqslant \mu(x) \wedge \mu(y)$ for any $x, y \in \mathbb{R}^n$ and for any $r \in [0, 1]$. $\mathfrak{C}_L$ denotes the set of all L-fuzzy convex sets on $\mathbb{R}^n$. Then $(\mathbb{R}^n, \mathfrak{C}_L)$ is an L-convex structure.

Example 2.6. Let $\mathfrak{C}$ denote the set of all fuzzy convex sublattices on L. It is easy to show that $\mathfrak{C}$ is an I-convexity and $(L, \mathfrak{C})$ is an I-convex structure.

Example 2.7. Let G be an ordered group, and let $\mathfrak{C}$ denote the set of all fuzzy convex subgroup on G. Then we can see that $\mathfrak{C}$ is an I-convexity and $(G, \mathfrak{C})$ is an I-convex structure.

The next two theorems give characterizations of an (L, M)-fuzzy convexity.

Theorem 2.8. *A mapping $\mathcal{C} : L^X \to M$ is an (L, M)-fuzzy convexity if and only if for each $a \in M \backslash \{\perp_M\}$, $\mathcal{C}_{[a]}$ is an L-convexity.*

Proof. The proof is obvious and is omitted. $\square$

Theorem 2.9. *A mapping $\mathcal{C} : L^X \to M$ is an (L, M)-fuzzy convexity if and only if for each $a \in \alpha(\perp_M)$, $\mathcal{C}^{[a]}$ is an L-convexity.*

Proof.

(LMC1) For each $a \in \alpha(\perp_M)$, $\chi_\emptyset, \chi_X \in \mathcal{C}^{[a]}$. We have $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$.

(LMC2) Let $\{A_i | i \in \Omega\} \subseteq L^X$ be nonempty, and for $a \in \alpha(\perp_M)$, $a \notin \alpha(\bigwedge_{i \in \Omega} \mathcal{C}(A_i))$. Thus $a \notin \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. We know that $a \notin \alpha(\mathcal{C}(A_i))$ and then $A_i \in \mathcal{C}^{[a]}$ for each $i \in \Omega$. Since for each $a \in \alpha(\perp_M)$, $\mathcal{C}^{[a]}$ is an L-convexity, $\bigwedge_{i \in \Omega} A_i \in \mathcal{C}^{[a]}$, that is, $a \notin \alpha(\mathcal{C}(\bigwedge_{i \in \Omega} A_i))$. Therefore $\mathcal{C}(\bigwedge_{i \in \Omega} A_i) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$.

(LMC3) Let $\{A_i | i \in \Omega\} \subseteq L^X$ be nonempty and totally ordered by inclusion, and let $a \notin \alpha(\bigwedge_{i \in \Omega} \mathcal{C}(A_i))$ for $a \in \alpha(\perp_M)$. Thus $a \notin \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. We know that $a \notin \alpha(\mathcal{C}(A_i))$ and then $A_i \in \mathcal{C}^{[a]}$ for each $i \in \Omega$. Since for each $a \in \alpha(\perp_M)$, $\mathcal{C}^{[a]}$ is an L-convexity, $\bigvee_{i \in \Omega} A_i \in \mathcal{C}^{[a]}$, that is, $a \notin \alpha(\mathcal{C}(\bigvee_{i \in \Omega} A_i))$. Therefore $\mathcal{C}(\bigvee_{i \in \Omega} A_i) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$.

Conversely, suppose that $\mathcal{C} : L^X \to M$ is an (L, M)-fuzzy convexity and $a \in \alpha(\perp_M)$. Now we prove that $\mathcal{C}^{[a]}$ is an L-convexity.

(LC1) By $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$ and $\alpha(\top_M) = \emptyset$, we know that $a \notin \alpha(\mathcal{C}(\chi_\emptyset))$ and $a \notin \alpha(\mathcal{C}(\chi_X))$. This implies $\chi_\emptyset, \chi_X \in \mathcal{C}^{[a]}$.

(LC2) If $\{A_i : i \in \Omega\} \subseteq \mathcal{C}^{[a]}$, then for each $i \in \Omega$, $a \notin \alpha(\mathcal{C}(A_i))$. Hence $a \notin \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. By $\mathcal{C}(\bigwedge_{i \in \Omega} A_i) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$, we know that $\alpha\left(\mathcal{C}\left(\bigwedge_{i \in \Omega} A_i\right)\right) \subseteq \alpha\left(\bigwedge_{i \in \Omega} \mathcal{C}(A_i)\right) = \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. This shows $a \notin \alpha\left(\mathcal{C}\left(\bigwedge_{i \in \Omega} A_i\right)\right)$. Therefore $\bigwedge_{i \in \Omega} A_i \in \mathcal{C}^{[a]}$.

(LC3) If $\{A_i : i \in \Omega\} \subseteq \mathcal{C}^{[a]}$ is nonempty and totally ordered by inclusion, then for each $i \in \Omega$, $a \notin \alpha(\mathcal{C}(A_i))$. Hence $a \notin \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. By $\mathcal{C}(\bigvee_{i \in \Omega} A_i) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$, we know that $\alpha\left(\mathcal{C}\left(\bigvee_{i \in \Omega} A_i\right)\right) \subseteq \alpha\left(\bigwedge_{i \in \Omega} \mathcal{C}(A_i)\right) = \bigcup_{i \in \Omega} \alpha(\mathcal{C}(A_i))$. This shows $a \notin \alpha\left(\mathcal{C}\left(\bigvee_{i \in \Omega} A_i\right)\right)$. Therefore $\bigvee_{i \in \Omega} A_i \in \mathcal{C}^{[a]}$. The proof is completed. $\square$

Now we consider the conditions that a family of L-convexities forms an (L, M)-fuzzy convexity. By Theorem 1.3, we can obtain the following result.

Corollary 2.10. *If $\mathcal{C}$ is an (L, M)-fuzzy convexity, then*

(1) $\mathcal{C}_{[b]} \subseteq \mathcal{C}_{[a]}$ *for any $a, b \in M\backslash\{\perp_M\}$ with $a \in \beta(b)$.*
(2) $\mathcal{C}^{[b]} \subseteq \mathcal{C}^{[a]}$ *for any $a, b \in \alpha(\perp_M)$ with $b \in \alpha(a)$.*

Theorem 2.11. *Let $\{\mathcal{C}^a : a \in \alpha(\perp_M)\}$ be a family of L-convexities. If $\mathcal{C}^a = \bigcap\{\mathcal{C}^b : a \in \alpha(b)\}$ for all $a \in \alpha(\perp_M)$, then there exists an (L, M)-fuzzy convexity $\mathcal{C}$ such that $\mathcal{C}^{[a]} = \mathcal{C}^a$.*

Proof. The proof is straightforward and is omitted. $\square$

Theorem 2.12. *Let $\{\mathcal{C}_a : a \in M\backslash\{\perp_M\}\}$ be a family of L-convexities. If $\mathcal{C}_a = \bigcap\{\mathcal{C}_b : b \in \beta(a)\}$ for all $a \in M\backslash\{\perp_M\}$, then there exists an (L, M)-fuzzy convexity $\mathcal{C}$ such that $\mathcal{C}_{[a]} = \mathcal{C}_a$.*

Proof. The proof is straightforward. $\square$

Definition 2.13. Let $\mathcal{C}, \mathcal{D}$ be (L, M)-fuzzy convexities on X. If $\mathcal{C}(A) \leqslant \mathcal{D}(A)$ for all $A \in L^X$, i.e., $\mathcal{C} \leqslant \mathcal{D}$, then $\mathcal{C}$ is coarser than $\mathcal{D}$ and $\mathcal{D}$ is finer than $\mathcal{C}$.

Theorem 2.14. *Let $\{\mathcal{C}_t : t \in T\}$ be a family of (L, M)-fuzzy convexities on X. Then $\bigwedge_{t \in T} \mathcal{C}_t$ is an (L, M)-fuzzy convexity on X, where $\bigwedge_{t \in T} \mathcal{C}_t : L^X \to M$ is defined by $\left(\bigwedge_{t \in T} \mathcal{C}_t\right)(A) = \bigwedge_{t \in T} \mathcal{C}_t(A)$ for each $A \in L^X$. Obviously, $\bigwedge_{t \in T} \mathcal{C}_t$ is coarser than $\mathcal{C}_t$ for all $t \in T$.*

Proof. The proof is straightforward. $\square$

3. (L, M)-fuzzy convexity preserving functions

In this section, we shall generalize the notion of convexity preserving functions to lattice-valued setting. We obtain the concept of (L, M)-fuzzy convexity preserving functions and give some of its characterizations.

Definition 3.1. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. A function $f : X \to Y$ is called an (L, M)-fuzzy convexity preserving function if $\mathcal{C}\left(f_L^\leftarrow(B)\right) \geqslant \mathcal{D}(B)$ for all $B \in L^Y$.

A $(2, M)$-fuzzy convexity preserving function is an M-fuzzifying convexity preserving function in [19].

Theorem 3.2. *Let $(Y, \mathcal{D})$ be an (L, M)-fuzzy convex structure and $f : X \to Y$ a surjective function. Define a mapping $f_L^\leftarrow(\mathcal{D}) : L^X \to M$ by*

$$\forall A \in L^X, \ f_L^\leftarrow(\mathcal{D})(A) = \bigvee\{\mathcal{D}(B) : f_L^\leftarrow(B) = A\}.$$

Then $(X, f_L^\leftarrow(\mathcal{D}))$ is an (L, M)-fuzzy convex structure.

Proof.
(LMC1) holds from the following equalities:

$$f_L^\leftarrow(\mathcal{D})(\chi_\emptyset) = \bigvee\{\mathcal{D}(B) : f_L^\leftarrow(B) = \chi_\emptyset\} = \mathcal{D}(\chi_\emptyset) = \top_M,$$

and

$$f_L^\leftarrow(\mathcal{D})(\chi_X) = \bigvee\{\mathcal{D}(B) : f_L^\leftarrow(B) = \chi_X\} = \mathcal{D}(\chi_Y) = \top_M.$$

(LMC2) For any nonempty set $\{A_i : i \in \Omega\} \subseteq L^X$, let a be any element in M with the property of $\bigwedge_{i \in \Omega} f_L^{\leftarrow}(\mathcal{D})(A_i) \succ a$. For each $i \in \Omega$, $\bigvee \{\mathcal{D}(B) : f_L^{\leftarrow}(B) = A_i\} = f_L^{\leftarrow}(\mathcal{D})(A_i) \succ a$. Then for each $i \in \Omega$, there exists $B_i \in L^X$ such that $f_L^{\leftarrow}(B_i) = A_i$ and $\mathcal{D}(B_i) \geqslant a$. Note that $f_L^{\leftarrow}\left(\bigwedge_{i \in \Omega} B_i\right) = \bigwedge_{i \in \Omega} f_L^{\leftarrow}(B_i) = \bigwedge_{i \in \Omega} A_i$ and $\mathcal{D}\left(\bigwedge_{i \in \Omega} B_i\right) \geqslant \bigwedge_{i \in \Omega} \mathcal{D}(B_i) \geqslant a$. Finally we have

$$f_L^{\leftarrow}(\mathcal{D})\left(\bigwedge_{i \in \Omega} A_i\right) = \bigvee\left\{\mathcal{D}(B) : f_L^{\leftarrow}(B) = \bigwedge_{i \in \Omega} A_i\right\} \geqslant \mathcal{D}\left(\bigwedge_{i \in \Omega} B_i\right) \geqslant a.$$

This implies $f_L^{\leftarrow}(\mathcal{D})\left(\bigwedge_{i \in \Omega} A_i\right) \geqslant \bigwedge_{i \in \Omega} f_L^{\leftarrow}(\mathcal{D})(A_i)$.

(LMC3) For any nonempty set $\{A_i : i \in \Omega\} \subseteq L^X$, which is totally ordered by inclusion, let a be any element in M with the property of $\bigwedge_{i \in \Omega} f_L^{\leftarrow}(\mathcal{D})(A_i) \succ a$, that is, $\bigwedge_{i \in \Omega} \bigvee \{\mathcal{D}(B) : f_L^{\leftarrow}(B) = A_i\} \succ a$. Then for all $i \in \Omega$, $\bigvee \{\mathcal{D}(B) : f_L^{\leftarrow}(B) = A_i\} = f_L^{\leftarrow}(\mathcal{D})(A_i) \succ a$. For each $i \in \Omega$, there exists $B_i \in L^X$ such that $f_L^{\leftarrow}(B_i) = A_i$ and $\mathcal{D}(B_i) \geqslant a$. Since f is surjective and $\{A_i : i \in \Omega\}$ is totally ordered by inclusion, we have $\{B_i : i \in \Omega\}$ is totally ordered by inclusion. Note that $f_L^{\leftarrow}\left(\bigvee_{i \in \Omega} B_i\right) = \bigvee_{i \in \Omega} f_L^{\leftarrow}(B_i) = \bigvee_{i \in \Omega} A_i$ and $\mathcal{D}\left(\bigvee_{i \in \Omega} B_i\right) \geqslant \bigwedge_{i \in \Omega} \mathcal{D}(B_i) \geqslant a$. Finally we have

$$f_L^{\leftarrow}(\mathcal{D})\left(\bigvee_{i \in \Omega} A_i\right) = \bigvee\left\{\mathcal{D}(B) : f_L^{\leftarrow}(B) = \bigvee_{i \in \Omega} A_i\right\} \geqslant \mathcal{D}\left(\bigvee_{i \in \Omega} B_i\right) \geqslant a.$$

This implies $f_L^{\leftarrow}(\mathcal{D})\left(\bigvee_{i \in \Omega} A_i\right) \geqslant \bigwedge_{i \in \Omega} f_L^{\leftarrow}(\mathcal{D})(A_i)$. □

The following theorem gives a characterization of (L, M)-fuzzy convexity preserving functions.

Theorem 3.3. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be two (L, M)-fuzzy convex structures. A surjective function $f : X \to Y$ is an (L, M)-fuzzy convexity preserving function if and only if $f_L^{\leftarrow}(\mathcal{D})(A) \leqslant \mathcal{C}(A)$ for all $A \in L^X$.*

Proof. If $f : X \to Y$ is an (L, M)-fuzzy convexity preserving function, then $\mathcal{C}\left(f_L^{\leftarrow}(B)\right) \geqslant \mathcal{D}(B)$ for all $B \in L^Y$. Hence for all $A \in L^X$, we have

$$f_L^{\leftarrow}(\mathcal{D})(A) = \bigvee\{\mathcal{D}(B) : f_L^{\leftarrow}(B) = A\} \leqslant \bigvee\{\mathcal{C}(f_L^{\leftarrow}(B)) : f_L^{\leftarrow}(B) = A\} = \mathcal{C}(A).$$

Conversely, if $f_L^{\leftarrow}(\mathcal{D})(A) \leqslant \mathcal{C}(A)$ for all $A \in L^X$, then

$$\mathcal{D}(B) \leqslant \bigvee\{\mathcal{D}(G) : f_L^{\leftarrow}(G) = f_L^{\leftarrow}(B)\} = f_L^{\leftarrow}(\mathcal{D})(f_L^{\leftarrow}(B)) \leqslant \mathcal{C}(f_L^{\leftarrow}(B))$$

for all $B \in L^Y$. This shows that $f : X \to Y$ is an (L, M)-fuzzy convexity preserving function. □

The following theorems are trivial.

Theorem 3.4. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ and $g : (Y, \mathcal{D}) \to (Z, \mathcal{H})$ are (L, M)-fuzzy convexity preserving functions, then $g \circ f : (X, \mathcal{C}) \to (Z, \mathcal{H})$ is an (L, M)-fuzzy convexity preserving function.*

Theorem 3.5. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. Then a function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is an (L, M)-fuzzy convexity preserving function if and only if $f : (X, \mathcal{C}_{[a]}) \to (Y, \mathcal{D}_{[a]})$ is an L-convexity preserving function for any $a \in M \backslash \{\bot_M\}$.*

Theorem 3.6. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. Then a function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is an (L, M)-fuzzy convexity preserving function if and only if $f : (X, \mathcal{C}^{[a]}) \to (Y, \mathcal{D}^{[a]})$ is an L-convexity preserving function for any $a \in \alpha(\bot_M)$.*

4. Quotient (L, M)-fuzzy convex structures

In this section, the notions of quotient structures and quotient functions are generalized to lattice-valued setting. We obtain the concepts of quotient (L, M)-fuzzy convex structures, (L, M)-fuzzy quotient functions, and (L, M)-fuzzy convex-to-convex functions and give some of their properties.

Theorem 4.1. *Let* $(X, \mathcal{C})$ *be an* (L, M)*-fuzzy convex structure and* $f : X \to Y$ *a surjective function. Define a mapping* $\mathcal{C}_{/f} : L^Y \to M$ *by*

$$\forall B \in L^Y, \ \mathcal{C}_{/f}(B) = \mathcal{C}(f_L^{\leftarrow}(B)).$$

Then $(Y, \mathcal{C}_{/f})$ *is an* (L, M)*-fuzzy convex structure and we call* $\mathcal{C}_{/f}$ *a quotient* (L, M)*-fuzzy convexity on* Y *with respect to* f *and* $\mathcal{C}$*. Moreover, it is easy to see that* f *is an* (L, M)*-fuzzy convexity preserving function from* $(X, \mathcal{C})$ *to* $(Y, \mathcal{C}_{/f})$*.*

Proof.

(LMC1) holds from the following equalities:

$$\mathcal{C}_{/f}(\chi_\emptyset) = \mathcal{C}(f_L^{\leftarrow}(\chi_\emptyset)) = \mathcal{C}(\chi_\emptyset) = \top_M \quad \text{and} \quad \mathcal{C}_{/f}(\chi_Y) = \mathcal{C}(f_L^{\leftarrow}(\chi_Y)) = \mathcal{C}(\chi_X) = \top_M.$$

(LMC2) can be shown from the following fact: for any nonempty set $\{B_i : i \in \Omega\} \subseteq L^Y$,

$$\mathcal{C}_{/f}\left(\bigwedge_{i \in \Omega} B_i\right) = \mathcal{C}(f_L^{\leftarrow}(\bigwedge_{i \in \Omega} B_i)) = \mathcal{C}(\bigwedge_{i \in \Omega} f_L^{\leftarrow}(B_i)) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(f_L^{\leftarrow}(B_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_{/f}(B_i).$$

(LMC3) If $\{B_i : i \in \Omega\} \subseteq L^Y$ is nonempty and totally ordered by inclusion, then

$$\mathcal{C}_{/f}\left(\bigvee_{i \in \Omega} B_i\right) = \mathcal{C}(f_L^{\leftarrow}(\bigvee_{i \in \Omega} B_i)) = \mathcal{C}(\bigvee_{i \in \Omega} f_L^{\leftarrow}(B_i)) \geqslant \bigwedge_{i \in \Omega} \mathcal{C}(f_L^{\leftarrow}(B_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_{/f}(B_i).$$

$\square$

Theorem 4.2. *Let* $(X, \mathcal{C})$ *be an* (L, M)*-fuzzy convex structure and* $f : X \to Y$ *a surjective function. Then* $\mathcal{C}_{/f}$ *is the finest convexity on* Y *such that* f *is an* (L, M)*-fuzzy convexity preserving function.*

Proof. Let $\mathcal{D}$ be an (L, M)-fuzzy convexity on Y such that f is an (L, M)-fuzzy convexity preserving function from $(X, \mathcal{C})$ to $(Y, \mathcal{D})$. Then we have for all $B \in L^Y$, $\mathcal{C}(f_L^{\leftarrow}(B)) \geqslant \mathcal{D}(B)$ and thus $\mathcal{C}_{/f}(B) = \mathcal{C}(f_L^{\leftarrow}(B)) \geqslant \mathcal{D}(B)$. Therefore $\mathcal{C}_{/f} \geqslant \mathcal{D}$. $\square$

Definition 4.3. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. A function $f : X \to Y$ is called an (L, M)-fuzzy quotient function if f is surjective and $\mathcal{D}$ is a quotient (L, M)-fuzzy convexity with respect to f and $\mathcal{C}$.

Theorem 4.4. *If* $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ *is an* (L, M)*-fuzzy quotient function, then* $g : (Y, \mathcal{D}) \to (Z, \mathcal{H})$ *is an* (L, M)*-fuzzy convexity preserving function if and only if* $g \circ f : (X, \mathcal{C}) \to (Z, \mathcal{H})$ *is an* (L, M)*-fuzzy convexity preserving function.*

Proof. Since $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is an (L, M)-fuzzy quotient function, we know that f is surjective and for all $B \in L^Y$, $\mathcal{D}(B) = \mathcal{C}(f_L^{\leftarrow}(B))$.

Since $g : (Y, \mathcal{D}) \to (Z, \mathcal{H})$ is an (L, M)-fuzzy convexity preserving function, for all $A \in L^Z$, $\mathcal{D}(g_L^{\leftarrow}(A)) \geqslant \mathcal{H}(A)$. Thus for all $A \in L^Z$, $\mathcal{C}((g \circ f)_L^{\leftarrow}(A)) = \mathcal{C}(f_L^{\leftarrow}(g_L^{\leftarrow}(A))) = \mathcal{D}(g_L^{\leftarrow}(A)) \geqslant \mathcal{H}(A)$.

Conversely, since $g \circ f : (X, \mathcal{C}) \to (Z, \mathcal{H})$ is an (L, M)-fuzzy convexity preserving function, then for all $A \in L^Z$, $\mathcal{D}(g_L^{\leftarrow}(A)) = \mathcal{C}(f_L^{\leftarrow}(g_L^{\leftarrow}(A))) = \mathcal{C}((g \circ f)_L^{\leftarrow}(A)) \geqslant \mathcal{H}(A)$. $\square$

Definition 4.5. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. A function $f : X \to Y$ is called an (L, M)-fuzzy convex-to-convex function if $\mathcal{D}\left(f_L^{\to}(A)\right) \geqslant \mathcal{C}(A)$ for all $A \in L^X$.

Theorem 4.6. *If* $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ *is a surjective* (L, M)-*fuzzy convexity preserving function and an* (L, M)-*fuzzy convex-to-convex function, then* $\mathcal{D}$ *is a quotient* (L, M)-*fuzzy convexity. Moreover,* f *is an* (L, M)-*fuzzy quotient function with respect to* f *and* $\mathcal{C}$.

Proof. Since $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is a surjective (L, M)-fuzzy convexity preserving function and an (L, M)-fuzzy convex-to-convex function, we have for all $B \in L^Y$, $\mathcal{C}(f_L^{\leftarrow}(B)) \geqslant \mathcal{D}(B)$ and for all $A \in L^X$, $\mathcal{D}(f_L^{\to}(A)) \geqslant \mathcal{C}(A)$. Since f is surjective, we know for all $B \in L^Y$, $f_L^{\to}(f_L^{\leftarrow}(B)) = B$. Hence

$$\mathcal{D}(B) = \mathcal{D}(f_L^{\to}(f_L^{\leftarrow}(B))) \geqslant \mathcal{C}(f_L^{\leftarrow}(B)) \geqslant \mathcal{D}(B).$$

So $\mathcal{C}(f_L^{\leftarrow}(B)) = \mathcal{D}(B)$ for each $B \in L^Y$ and then $\mathcal{D}$ is a quotient (L, M)-fuzzy convexity with respect to f and $\mathcal{C}$. $\qquad\square$

The following corollary is a straight consequence of Theorem 4.1.

Corollary 4.7. *Let* $(X, \mathcal{C})$ *be an* (L, M)-*fuzzy convex structure and* R *be an equivalence relation defined on* X. *Let* X/R *be the usual quotient set and let* $\pi : X \longrightarrow X/R$ *be the projection map.*

(1) *Define* $\mathcal{D} : L^{(X/R)} \to M$ *by*

$$\forall B \in L^{(X/R)}, \ \mathcal{D}(B) = \mathcal{C}(\pi_L^{\leftarrow}(B)).$$

Then $\mathcal{D}$ *is an* (L, M)-*fuzzy convexity on* X/R *and* $(X/R, \mathcal{D})$ *is a quotient* (L, M)-*fuzzy convex structure of* $(X, \mathcal{C})$.

(2) *If* $(X, \mathcal{C})$ *is an* I-*convex structure, then one can define an* I-*convexity* $\mathfrak{D}$ *on* X/R *as follows:* $\mathfrak{D} = \{B \in I^{(X/R)} : \pi_I^{\leftarrow}(B) \in \mathcal{C}\}$. *Then* $\mathfrak{D}$ *is an* I-*convexity on* X/R *and* $(X/R, \mathfrak{D})$ *is called the quotient* I-*convex structure.*

5. Substructures and products of (L, M)-fuzzy convex structures

In this section, we give substructures and products of (L, M)-fuzzy convex structures and discuss some of their fundamental properties.

Lemma 5.1. *Let* $(X, \mathcal{C})$ *be an* L-*convex structure and* $\emptyset \neq Y \subseteq X$. *For* $A \in \mathcal{C}$, $co(A|Y)|Y = A|Y$.

Proof. On one hand, it is obvious that $A|Y \subseteq co(A|Y)$. Then $A|Y = (A|Y)|Y \subseteq co(A|Y)|Y$. On the other hand, $A|Y \subseteq A$. Hence $co(A|Y) \subseteq co(A) = A$ and then $co(A|Y)|Y \subseteq co(A)|Y = A|Y$. Therefore, $co(A|Y)|Y = A|Y$. $\qquad\square$

Theorem 5.2. *Let* $(X, \mathcal{C})$ *be an* (L, M)-*fuzzy convex structure,* $\emptyset \neq Y \subseteq X$. *Then* $(Y, \mathcal{C}|Y)$ *is an* (L, M)-*fuzzy convex structure on* Y, *where for each* $A \in L^Y$, $(\mathcal{C}|Y)(A) = \bigvee\{\mathcal{C}(B) : B \in L^X, \ B|Y = A\}$. *We call* $(Y, \mathcal{C}|Y)$ *an* (L, M)-*fuzzy substructure of* $(X, \mathcal{C})$.

Proof.

(1) Clearly, $(\mathcal{C}|Y)(\chi_{\emptyset}) = (\mathcal{C}|Y)(\chi_X) = \top_M$.

(2) For any nonempty set $\{A_i : i \in \Omega\} \subseteq L^Y$, we have

$$\bigwedge_{i \in \Omega} (\mathcal{C}|Y)(A_i) = \bigwedge_{i \in \Omega} \bigvee\{\mathcal{C}(B) : B \in L^X, \ B|Y = A_i\} = \bigvee_{f \in \Pi_{i \in \Omega} H_i} \bigwedge_{i \in \Omega} \mathcal{C}(f(i)) \leqslant \bigvee_{f \in \Pi_{i \in \Omega} H_i} \mathcal{C}\left(\bigwedge_{i \in \Omega} f(i)\right),$$

where $H_i = \{B : B \in L^X, \ B|Y = A_i\} \ (i \in \Omega)$. Since $\left(\bigwedge_{i \in \Omega} f(i)\right)|Y = \bigwedge_{i \in \Omega} (f(i)|Y) = \bigwedge_{i \in \Omega} A_i$, we have

$$(\mathcal{C}|Y)\left(\bigwedge_{i \in \Omega} A_i\right) \geqslant \bigwedge_{i \in \Omega} (\mathcal{C}|Y)(A_i).$$

(3) For any $\{A_i : i \in \Omega\} \subseteq L^Y$, which is nonempty and totally ordered by inclusion, we have

$$\bigwedge_{i\in\Omega} (\mathcal{C}|Y)(A_i) = \bigwedge_{i\in\Omega} \bigvee \{\mathcal{C}(B) : B \in L^X, \ B|Y = A_i\} = \bigvee_{f\in\Pi_{i\in\Omega}\mu_i} \bigwedge_{i\in\Omega} \mathcal{C}(f(i)) \leqslant \bigvee_{f\in\Pi_{i\in\Omega}\mu_i} \mathcal{C}\left(\bigvee_{i\in\Omega} f(i)\right),$$

where $\mu_i = \{B : B \in L^X, \ B|Y = A_i\}$ $(i \in \Omega)$. Since $\left(\bigvee_{t\in T} f(i)\right)|Y = \bigvee_{i\in\Omega} (f(i)|Y) = \bigvee_{i\in\Omega} A_i$, we have

$(\mathcal{C}|Y)\left(\bigvee_{i\in\Omega} A_i\right) \geqslant \bigwedge_{i\in\Omega} (\mathcal{C}|Y)(A_i).$

(4) For any set $\{A_i : i \in \Omega\} \subseteq L^Y \subseteq L^X$, which is nonempty and totally ordered by inclusion, let a be any element in $M\setminus\{\perp\}$ with the property of $\bigwedge_{i\in\Omega} (\mathcal{C}|Y)(A_i) \succ a$, that is, $\bigwedge_{i\in\Omega} \bigvee\{\mathcal{C}(B) : B \in L^X, \ B|Y = A_i\} \succ a$.

Then for each $i \in \Omega$, there exists $B_i \in L^X$ such that $B_i|Y = A_i$ and $\mathcal{C}(B_i) \geqslant a$, i.e., $B_i \in \mathcal{C}_{[a]}$. By Theorem 1.8, for each $a \in M\setminus\{\perp\}$, $(X, \mathcal{C}_{[a]})$ is a convex structure. Let co_a denote the hull operator of $(X, \mathcal{C}_{[a]})$ for each $a \in M\setminus\{\perp\}$. Then $co_a(A_i) \in \mathcal{C}_{[a]}$ for all $i \in \Omega$. Since $\{A_i : i \in \Omega\} \subseteq L^Y$ is nonempty and totally ordered by inclusion, $\{co_a(A_i) : i \in \Omega\}$ is nonempty and totally ordered by inclusion. Hence, $\bigvee_{i\in\Omega} co_a(A_i) \in \mathcal{C}_{[a]}$, that is, $\mathcal{C}(\bigvee_{i\in\Omega} co_a(A_i)) \geqslant a$. By Lemma 5.1,

$$(\bigvee_{i\in\Omega} co_a(A_i))|Y = (\bigvee_{i\in\Omega} co_a(B_i|Y))|Y = \bigvee_{i\in\Omega} co_a(B_i|Y)|Y = \bigvee_{i\in\Omega} (B_i|Y) = \bigvee_{i\in\Omega} A_i.$$

So we have $(\mathcal{C}|Y)\left(\bigvee_{i\in\Omega} A_i\right) \geqslant a$. This implies $(\mathcal{C}|Y)\left(\bigvee_{i\in\Omega} A_i\right) \geqslant \bigwedge_{i\in\Omega} (\mathcal{C}|Y)(A_i)$. $\square$

Corollary 5.3 ([14, 15]). *Let $(X, \mathfrak{C})$ be an I-convex structure, $\emptyset \neq Y \subseteq X$. Then an I-convexity $\mathfrak{C}|Y$ on Y is given by the fuzzy sets of the form $\{B|Y : B \in \mathfrak{C}\}$. The pair $(Y, \mathfrak{C}|Y)$ is an I-convex substructure of $(X, \mathfrak{C})$.*

By Theorem 2.14, we can give the following definition.

Definition 5.4. Let $\varphi : L^X \to M$ be a mapping. The (L, M)-fuzzy convex structure $(X, \mathcal{C})$ generated by φ is given by

$$\forall A \in L^X, \ \mathcal{C}(A) = \bigwedge\{\mathcal{D}(A) : \varphi \leqslant \mathcal{D} \in \mathfrak{H}\},$$

where $\mathfrak{H}$ denotes all the (L, M)-fuzzy convexities on X. Then φ is called a subbase of the (L, M)-fuzzy convexity $\mathcal{C}$. Alternatively, we say that φ generates the convexity $\mathcal{C}$.

Based on Definition 5.4, we can define the product of (L, M)-fuzzy convex structures as follows.

Definition 5.5. Let $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ be a family of (L, M)-fuzzy convex structures. Let X be the product of the sets of X_t for $t \in T$, and let $\pi_t : X \to X_t$ denote the projection for each $t \in T$. Define a mapping $\varphi : L^X \to M$ by $\varphi(A) = \bigvee_{t\in T} \bigvee_{(\pi_t)^{\leftarrow}_L(B)=A} \mathcal{C}_t(B)$ for each $A \in L^X$. Then the product convexity $\mathcal{C}$ of X is the one generated by the subbase φ. The resulting (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is called the product of $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ and is donated by $\prod_{t\in T} (X_t, \mathcal{C}_t)$.

When $L = [0, 1]$ and $M = \mathbf{2}$, we can obtain the following definition.

Definition 5.6 ([14, 15]). Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of I-convex structures. Let X be the product of the sets of X_t for $t \in T$, and let $\pi_t : X \to X_t$ denote the projection for each $t \in T$. Then X can be equipped with the I-convexity $\mathfrak{C}$ generated by the convex fuzzy sets of the form $\{(\pi_t)^{\leftarrow}_I(B) : B \in \mathfrak{C}_t, t \in T\}$. Then $\mathfrak{C}$ is called the product I-convexity for X and $(X, \mathfrak{C})$ is called the product I-convex structure.

Theorem 5.7. *Let $(X, \mathcal{C})$ be the product of $\{(X_t, \mathcal{C}_t)\}_{t \in T}$. Then for all $t \in T$, $\pi_t : X \to X_t$ is an (L, M)-fuzzy convexity preserving function. Moreover, $\mathcal{C}$ is the coarsest (L, M)-fuzzy convex structure such that $\{\pi_t : t \in T\}$ are (L, M)-fuzzy convexity preserving functions.*

Proof. Let $t_0 \in T$ and $B \in L^{X_{t_0}}$. By

$$\mathcal{C}_{t_0}(B) \leqslant \bigvee_{t \in T} \bigvee_{(\pi_t)_L^{\leftarrow}(B) = (\pi_{t_0})_L^{\leftarrow}(B)} \mathcal{C}_t(B) = \varphi((\pi_{t_0})_L^{\leftarrow}(B)) \leqslant \mathcal{C}((\pi_{t_0})_L^{\leftarrow}(B)),$$

it implies that $\pi_{t_0} : X \to X_{t_0}$ is an (L, M)-fuzzy convexity preserving function. By the arbitrariness of t_0, we know for each $t \in T$, $\pi_t : X \to X_t$ is an (L, M)-fuzzy convexity preserving function. If there is an (L, M)-fuzzy convex structure $\mathcal{D}$ on X such that for each $t \in T$, $\pi_t : X \to X_t$ is an (L, M)-fuzzy convexity preserving function, then we need to prove $\mathcal{D} \geqslant \mathcal{C}$. For each $B \in L^X$, $t \in T$, if $(\pi_t)_L^{\leftarrow}(G) = B$, $\mathcal{D}(B) = \mathcal{D}((\pi_t)_L^{\leftarrow}(G)) \geqslant \mathcal{C}_t(G)$. Note that $\varphi(B) = \bigvee_{t \in T} \bigvee_{(\pi_t)_L^{\leftarrow}(G) = B} \mathcal{C}_t(G)$. We have $\mathcal{D}(B) \geqslant \varphi(B)$ for all $B \in L^X$. Hence $\mathcal{D} \geqslant \mathcal{C}$. □

6. Relation between MYCS and LMCS

In this section, we discuss the relation between (L, M)-fuzzy convex structures and M-fuzzifying convex structures from a categorical aspect. (L, M)-fuzzy convex structures and their (L, M)-fuzzy convexity preserving functions form a category which is denoted by LMCS and M-fuzzifying convex structures and their M-fuzzifying convexity preserving functions form a category which is denoted by MYCS. We show that MYCS can be embedded in LMCS as a coreflective subcategory. In this section, we always suppose that L satisfies the following further property: $\forall a, b \in L$, $\beta(a \wedge b) = \beta(a) \cap \beta(b)$.

Lemma 6.1. *For any $b \in L$ and $A \in L^X$, $\{A_{[c]} : b \in \beta(c)\}$ is up-directed.*

Proof. Let $A_{[c_1]}, A_{[c_2]} \in \{A_{[c]} : b \in \beta(c)\}$. Then $b \in \beta(c_1)$ and $b \in \beta(c_2)$. We have $b \in \beta(c_1) \cap \beta(c_2) = \beta(c_1 \wedge c_2)$. Hence $A_{[c_1 \wedge c_2]} \in \{A_{[c]} : b \in \beta(c)\}$. Moreover, $A_{[c_1]}, A_{[c_2]} \subseteq A_{[c_1 \wedge c_2]}$. Therefore, $\{A_{[c]} : b \in \beta(c)\}$ is up-directed. □

Theorem 6.2. *Let $(X, \mathscr{C})$ be an M-fuzzifying convex structure. Define a mapping $\omega(\mathscr{C}) : L^X \to M$ by*

$$\forall A \in L^X, \ \omega(\mathscr{C})(A) = \bigwedge_{a \in L} \mathscr{C}(A_{[a]}).$$

Then $\omega(\mathscr{C})$ is an (L, M)-fuzzy convexity.

Proof.

(LMC1) Obviously, $\omega(\mathscr{C})(\chi_X) = \omega(\mathscr{C})(\chi_\emptyset) = \top_M$.

(LMC2) For any nonempty set $\{A_i : i \in \Omega\} \subseteq L^X$, we have

$$\omega(\mathscr{C})(\bigwedge_{i \in \Omega} A_i) = \bigwedge_{a \in L} \mathscr{C}((\bigwedge_{i \in \Omega} A_i)_{[a]}) = \bigwedge_{a \in L} \mathscr{C}(\bigwedge_{i \in \Omega} (A_i)_{[a]})$$

$$\geqslant \bigwedge_{a \in L} \bigwedge_{i \in \Omega} \mathscr{C}((A_i)_{[a]}) = \bigwedge_{i \in \Omega} \bigwedge_{a \in L} \mathscr{C}((A_i)_{[a]}) = \bigwedge_{i \in \Omega} \omega(\mathscr{C})(A_i).$$

(LMC3) For any set $\{A_i : i \in \Omega\} \subseteq L^X$, which is nonempty and totally ordered by inclusion, we need to prove that $\omega(\mathscr{C}) \left(\bigvee_{i \in \Omega} A_i \right) \geqslant \bigwedge_{i \in \Omega} \omega(\mathscr{C})(A_i)$, that is, $\bigwedge_{a \in L} \mathscr{C}((\bigvee_{i \in \Omega} A_i)_{[a]}) \geqslant \bigwedge_{i \in \Omega} \bigwedge_{a \in L} \mathscr{C}((A_i)_{[a]})$. Let $h \in M \setminus \{\perp_M\}$ and $\bigwedge_{i \in \Omega} \bigwedge_{a \in L} \mathscr{C}((A_i)_{[a]}) \geqslant h$. Then we have for any $i \in \Omega$ and for any $a \in L$, $\mathscr{C}((A_i)_{[a]}) \geqslant h$, i.e.,

$(A_i)_{[a]} \in \mathscr{C}_{[h]}$. Since $(X, \mathscr{C})$ is an M-fuzzifying convex structure, by Theorem 1.8, for each $h \in M \backslash \{\perp_M\}$, $(X, \mathscr{C}_{[h]})$ is a convex structure. By Theorems 1.3 and 1.4, we know that

$$\left(\bigvee_{i \in \Omega} A_i \right)_{[a]} = \bigcap_{b \in \beta(a)} \left(\bigvee_{i \in \Omega} A_i \right)_{(b)} = \bigcap_{b \in \beta(a)} \bigcup_{i \in \Omega} (A_i)_{(b)} = \bigcap_{b \in \beta(a)} \bigcup_{i \in \Omega} \bigcup_{b \in \beta(c)} (A_i)_{[c]}.$$

By Lemma 6.1, for each $b \in \beta(a)$ and for each $i \in \Omega$, $\{(A_i)_{[c]} : b \in \beta(c)\} \subseteq \mathscr{C}_{[h]}$ is up-directed. Then by Definition 1.5, we have $\bigcup_{b \in \beta(c)} (A_i)_{[c]} \in \mathscr{C}_{[h]}$. Let $B_i = \bigcup_{b \in \beta(c)} (A_i)_{[c]}$ for each $i \in \Omega$. Since $\{A_i : i \in \Omega\}$ is totally ordered, we obtain $\{B_i : i \in \Omega\}$ is totally ordered. Then $\bigcup_{i \in \Omega} \bigcup_{b \in \beta(c)} (A_i)_{[c]} \in \mathscr{C}_{[h]}$. Therefore

$\left(\bigvee_{i \in \Omega} A_i \right)_{[a]} = \bigcap_{b \in \beta(a)} \bigcup_{i \in \Omega} \bigcup_{b \in \beta(c)} (A_i)_{[c]} \in \mathscr{C}_{[h]}$. Hence $\omega(\mathscr{C}) \left(\bigvee_{i \in \Omega} A_i \right) \geqslant h$. By the arbitrariness of h, we

have $\omega(\mathscr{C}) \left(\bigvee_{i \in \Omega} A_i \right) \geqslant \bigwedge_{i \in \Omega} \omega(\mathscr{C})(A_i)$. $\square$

Theorem 6.3. *Let* $(X, \mathscr{C})$ *and* $(Y, \mathscr{D})$ *be two M-fuzzifying convex structures and* $f : X \to Y$ *a function. Then* $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ *is an M-fuzzifying convexity preserving function if and only if* $f : (X, \omega(\mathscr{C})) \to (Y, \omega(\mathscr{D}))$ *is an* (L, M)*-fuzzy convexity preserving function.*

Proof. Suppose that $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an M-fuzzifying convexity preserving function. Then $\mathscr{C}(f^{-1}(A)) \geqslant \mathscr{D}(A)$ for any $A \in \mathbf{2}^Y$. In order to prove that $f : (X, \omega(\mathscr{C})) \to (Y, \omega(\mathscr{D}))$ is an (L, M)-fuzzy convexity preserving function, we need to prove $\omega(\mathscr{C}) \left(f_L^\leftarrow(A) \right) \geqslant \omega(\mathscr{D})(A)$ for any $A \in L^Y$. For any $A \in L^Y$ and for any $a \in L$, we have $f_L^\leftarrow(A)_{[a]} = f^{-1}(A_{[a]})$. In fact, for any $A \in L^Y$, by

$$\omega(\mathscr{C}) \left(f_L^\leftarrow(A) \right) = \bigwedge_{a \in L} \mathscr{C}(f_L^\leftarrow(A)_{[a]}) = \bigwedge_{a \in L} \mathscr{C}(f^{-1}(A_{[a]})) \geqslant \bigwedge_{a \in L} \mathscr{D}(A_{[a]}) = \omega(\mathscr{D})(A),$$

we can prove the necessity.

Conversely, suppose that $f : (X, \omega(\mathscr{C})) \to (Y, \omega(\mathscr{D}))$ is an (L, M)-fuzzy convexity preserving function. Then $\omega(\mathscr{C}) \left(f_L^\leftarrow(A) \right) \geqslant \omega(\mathscr{D})(A)$ for any $A \in L^Y$. In particular, it follows that $\omega(\mathscr{C}) \left(f_L^\leftarrow(A) \right) \geqslant \omega(\mathscr{D})(A)$ for any $A \in \mathbf{2}^Y$. In order to prove that $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an M-fuzzifying convexity preserving function, we need to prove $\mathscr{C} \left(f^{-1}(A) \right) \geqslant \mathscr{D}(A)$ for any $A \in \mathbf{2}^Y$. In fact, for any $A \in \mathbf{2}^Y$, we have

$$\mathscr{C} \left(f^{-1}(A) \right) = \mathscr{C} \left(f^{-1}(A_{[a]}) \right) = \mathscr{C} \left(f_L^\leftarrow(A)_{[a]} \right)$$
$$= \bigwedge_{a \in L} \mathscr{C}(f_L^\leftarrow(A)_{[a]}) = \omega(\mathscr{C}) \left(f_L^\leftarrow(A) \right) \geqslant \omega(\mathscr{D}) (A) = \bigwedge_{a \in L} \mathscr{D}(A_{[a]}) = \mathscr{D}(A).$$

This shows that $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an M-fuzzifying convexity preserving function. $\square$

Theorem 6.4. *Suppose that* $(X, \mathcal{C})$ *is an* (L, M)*-fuzzy convex structure. We can obtain an M-fuzzifying convex structure* $\iota(\mathcal{C})$ *on* X *generated by the subbase* $\varphi_{\mathcal{C}}(U) : \mathbf{2}^X \to M$ *defined as follows:*

$$\forall U \in \mathbf{2}^X, \ \varphi_{\mathcal{C}}(U) = \bigvee_{a \in L} \bigvee \{\mathcal{C}(B) : B \in L^X, B_{[a]} = U\}.$$

Then $\iota \circ \omega = \mathrm{id}$ *and* $\omega \circ \iota \geqslant \mathrm{id}$.

Proof. We observe that for every M-fuzzifying convex structure $\mathscr{C}$ on X the relation $\varphi_{\omega(\mathscr{C})}(U) \geqslant \mathscr{C}(U)$ holds for all $U \in \mathbf{2}^X$. In fact, it could be showed by for each $U \in \mathbf{2}^X$,

$$\varphi_{\omega(\mathscr{C})}(U) = \bigvee_{a \in L} \bigvee \{\omega(\mathscr{C})(B) : B \in L^X, B_{[a]} = U\} \geqslant \omega(\mathscr{C})(U) = \bigwedge_{a \in L} \mathscr{C}(U_{[a]}) = \mathscr{C}(U).$$

Thus, $\iota(\omega(\mathscr{C})) \geqslant \mathscr{C}$, i.e., $\iota \circ \omega \geqslant \mathrm{id}$.

Conversely, let $U \in \mathbf{2}^X$ and take any $a \in L$. Then for each $B \in L^X$ with $B_{[a]} = U$,

$$\omega(\mathscr{C})(B) = \bigwedge_{b \in L} \mathscr{C}(B_{[b]}) \leqslant \mathscr{C}(U).$$

Hence, $\varphi_{\omega(\mathscr{C})}(U) = \bigvee\limits_{a \in L} \bigvee \{\omega(\mathscr{C})(B) : B \in L^X, B_{[a]} = U\} \leqslant \mathscr{C}(U)$. It means that $\iota(\omega(\mathscr{C})) \leqslant \mathscr{C}$, i.e., $\iota \circ \omega \leqslant \mathrm{id}$. Finally, we obtain $\iota \circ \omega = \mathrm{id}$ by all proofs above.

Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure. Then

$$\varphi_{\mathcal{C}}(U) = \bigvee_{a \in L} \bigvee \{\mathcal{C}(B) : B \in L^X, B_{[a]} = U\}$$

for all $U \in \mathbf{2}^X$ and $\iota(\mathcal{C}) = \bigwedge\{\mathcal{D} : \varphi_{\mathcal{C}} \leqslant \mathcal{D} \in \mathfrak{H}\}$, where $\mathfrak{H}$ denotes all the M-fuzzifying convexities on X. For each $A \in L^X$, by

$$(\omega \circ \iota(\mathcal{C}))(A) = \bigwedge_{a \in L} \iota(\mathcal{C})(A_{[a]}) \geqslant \bigwedge_{a \in L} \varphi_{\mathcal{C}}(A_{[a]}) = \bigwedge_{a \in L} \bigvee_{b \in L} \bigvee \{\mathcal{C}(B) : B \in L^X, B_{[b]} = A_{[a]}\} \geqslant \mathcal{C}(A),$$

we have $\omega \circ \iota(\mathcal{C}) \geqslant \mathcal{C}$, i.e., $\omega \circ \iota \geqslant \mathrm{id}$. $\qquad\square$

Theorem 6.5. *Let $(X, \mathscr{C})$ be an M-fuzzifying convex structure, $(X, \mathcal{D})$ be an (L, M)-fuzzy convex structure, and $f : (X, \mathscr{C}) \to (Y, \iota(\mathcal{D}))$ be an M-fuzzifying convexity preserving function. Then $f : (X, \omega(\mathscr{C})) \to (Y, \mathcal{D})$ is an (L, M)-fuzzy convexity preserving function.*

Proof. Since $f : (X, \mathscr{C}) \to (Y, \iota(\mathcal{D}))$ is an M-fuzzifying convexity preserving function, for each $A \in \mathbf{2}^Y$, $\mathscr{C}(f^{-1}(A)) \geqslant \iota(\mathcal{D})(A)$. For each $B \in L^Y$, by

$$\mathcal{D}(B) \leqslant \omega \circ \iota(\mathcal{D})(B) = \omega(\iota(\mathcal{D}))(B) = \bigwedge_{a \in L} \iota(\mathcal{D})(B_{[a]}) \leqslant \bigwedge_{a \in L} \mathscr{C}(f^{-1}(B_{[a]})) = \bigwedge_{a \in L} \mathscr{C}(f_L^{\leftarrow}(B)_{[a]}) = \omega(\mathscr{C})(f_L^{\leftarrow}(B)),$$

we obtain $f : (X, \omega(\mathscr{C})) \to (Y, \mathcal{D})$ is an (L, M)-fuzzy convexity preserving function. $\qquad\square$

Based on the above results, we finally obtain the following theorem.

Theorem 6.6. *MYCS can be embedded in LMCS as a coreflective subcategory.*

7. Conclusion

In this paper, combining L-convex structures [10, 14, 15] and M-fuzzifying convex structures [19] and based on complete distributive lattices L and M, we present a more general approach to the fuzzification of convex structures. It is a generalization of L-convex structures and M-fuzzifying convex structures. Under the framework of (L, M)-fuzzy convex structures, the concepts of quotient structures, substructures and products are presented and their fundamental properties are discussed.

The notion of convexity preserving functions is also generalized to lattice-valued fuzzy setting and then an (L, M)-fuzzy convexity preserving function is obtained. Thus there are two categories LMCS and MYCS, where LMCS consists of all (L, M)-fuzzy convex structures and of all (L, M)-fuzzy convexity preserving functions, and MYCS consists of all M-fuzzifying convex structures and of all M-fuzzifying convexity preserving functions. Moreover, we show that MYCS can be embedded in LMCS as a coreflective subcategory.

The above facts will be useful to help further investigations and it is possible that the fuzzification of convex structure would be applied to some problems related to the theory of abstract convexity in the future.

Acknowledgment

References

[1] N. Ajmal, K. V. Thomas, *Fuzzy lattices*, Inform. Sci., **79** (1994), 271–291. 1.10

[2] M. Berger, *Convexity*, Amer. Math. Monthly, **97** (1990), 650–678. 1, 1.11

[3] P. Dwinger, *Characterization of the complete homomorphic images of a completely distributive complete lattice*, I, Nederl. Akad. Wetensch. Indag. Math., **85** (1982), 403–414. 1

[4] J.-M. Fang, P.-W. Chen, *One-to-one correspondence between fuzzifying topologies and fuzzy preorders*, Fuzzy Sets and Systems, **158** (2007), 1814–1822. 2.2

[5] H.-L. Huang, F.-G. Shi, *L-fuzzy numbers and their properties*, Inform. Sci., **178** (2008), 1141–1151. 1, 1.3

[6] Q. Jin, L.-Q. Li, *On the embedding of convex spaces in stratified L-convex spaces*, SpringerPlus, **5** (2016), 10 pages. 1

[7] F. Jinming, *I-fuzzy Alexandrov topologies and specialization orders*, Fuzzy Sets and Systems, **158** (2007), 2359–2374. 2.3

[8] T. Kubiak, *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz University, Poznan, Poland, (1985). 1, 2.2

[9] M. Lassak, *On metric B-convexity for which diameters of any set and its hull are equal*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **25** (1977), 969–975. 1

[10] Y. Maruyama, *Lattice-valued fuzzy convex geometry*, RIMS Kokyuroku, **1641** (2009), 22–37. 1, 1.6, 2.5, 6

[11] C. V. Negoiţă, D. A. Ralescu, *Applications of fuzzy sets to systems analysis*, Translated from the Romanian, ISR—Interdisciplinary Systems Research, Birkhäuser Verlag, Basel-Stuttgart, **11** (1975), 187 pages. 1

[12] B. Pang, F.-G. Shi, *Subcategories of the category of L-convex spaces*, Fuzzy Sets and Systems, **313** (2017), 61–74. 1

[13] B. Pang, Y. Zhao, *Characterizations of L-convex spaces*, Iran. J. Fuzzy Syst., **13** (2016), 51–61. 1

[14] M. V. Rosa, *A study of fuzzy convexity with special reference to separation properties*, Ph.D. Thesis, Cochin University of Science and Technology, Kerala, India, (1994). 1, 1.6, 5.3, 5.6, 6

[15] M. V. Rosa, *On fuzzy topology fuzzy convexity spaces and fuzzy local convexity*, Fuzzy Sets and Systems, **62** (1994), 97–100. 1, 1.6, 5.3, 5.6, 6

[16] F.-G. Shi, *Theory of L_β-nested sets and L_α-nested sets and its applications*, (Chinese) Fuzzy Syst. Math., **4** (1995), 65–72. 1, 1.1, 1.4

[17] F.-G. Shi, *L-fuzzy relations and L-fuzzy subgroups*, J. Fuzzy Math., **8** (2000), 491–499. 1, 1.3

[18] F.-G. Shi, E.-Q. Li, *The restricted hull operator of M-fuzzifying convex structures*, J. Intell. Fuzzy Syst., **30** (2015), 409–421. 1

[19] F.-G. Shi, Z.-Y. Xiu, *A new approach to the fuzzification of convex structures*, J. Appl. Math., **2014** (2014), 12 pages. 1, 1.7, 1.8, 1.9, 3, 6

[20] V. P. Soltan, *d-convexity in graphs*, (Russian) Dokl. Akad. Nauk SSSR, **272** (1983), 535–537. 1

[21] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo, **11** (1985), 89–103. 1, 2.2

[22] M. L. J. van de Vel, *Theory of convex structures*, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, (1993). 1, 1.5, 2.1

[23] J. van Mill, *Supercompactness and Wallman spaces*, Mathematical Centre Tracts, Mathematisch Centrum, Amsterdam, (1977). 1

[24] J. C. Varlet, *Remarks on distributive lattices*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **23** (1975), 1143–1147. 1

[25] G.-J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems, **47** (1992), 351–376. 1, 1.2

[26] Z.-Y. Xiu, F.-G. Shi, *M-fuzzifying interval spaces*, Iran. J. Fuzzy Syst., **14** (2017), 145–162. 1

[27] M.-S. Ying, *A new approach for fuzzy topology*, I, Fuzzy Sets and Systems, **39** (1991), 303–321. 2.2

[28] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353. 1