Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 3694–3707
Research Article
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Determinant and inverse of a Gaussian Fibonacci skew-Hermitian Toeplitz matrix

Zhaolin Jiang[a], Jixiu Sun[a,b,*]

[a]*Department of Mathematics, Linyi University, Linyi 276000, P. R. China.*
[b]*School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, P. R. China.*

Communicated by S.-M. Jung

Abstract

In this paper, we consider the determinant and the inverse of the Gaussian Fibonacci skew-Hermitian Toeplitz matrix. We first give the definition of the Gaussian Fibonacci skew-Hermitian Toeplitz matrix. Then we compute the determinant and inverse of the Gaussian Fibonacci skew-Hermitian Toeplitz matrix by constructing the transformation matrices.

Keywords: Gaussian Fibonacci number, skew-Hermitian Toeplitz matrix, determinant, inverse.
2010 MSC: 15B05, 15A09, 15A15.

1. Introduction

The Gaussian Fibonacci sequence [9, 10] is defined by the following recurrence relations:

$$G_{n+1} = G_n + G_{n-1}, \quad n \geqslant 1,$$

with the initial condition $G_0 = i$, $G_1 = 1$. The $\{G_n\}$ is given by the formula

$$G_n = \frac{(1-i\beta)\alpha^n + (i\alpha - 1)\beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})i}{\alpha - \beta},$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$. We can observe that $\bar{G}_{n+1} = \bar{G}_n + \bar{G}_{n-1}$, $n \geqslant 1$ and $G_n = F_n + iF_{n-1}$, where F_n is Fibonacci number [1].

On the other hand, skew-Hermitian Toeplitz matrix [3, 4, 21] and Hermitian Toeplitz matrix have important applications in various disciplines including image processing, signal processing, and solving least squares problems [7, 8, 20].

It is an ideal research area and hot topic for the inverses of the Toeplitz matrices [15, 25]. Some scholars showed the explicit determinants and inverses of the special matrices involving famous numbers.

*Corresponding author
Email addresses: `jzh1208@sina.com` (Zhaolin Jiang), `sunjixiu1314@163.com` (Jixiu Sun)

doi:10.22436/jnsa.010.07.27

Received 2017-02-17

Dazheng showed the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [6]. Circulant matrices with Fibonacci and Lucas numbers are discussed and their explicit determinants and inverses are proposed in [22]. The authors provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [2]. The explicit determinants of circulant and left circulant matrices including Tribonacci numbers and generalized Lucas numbers are shown based on Tribonacci numbers and generalized Lucas numbers in [17]. In [12], circulant type matrices with the k-Fibonacci and k-Lucas numbers are considered and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. Jiang et al. [11] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. In [13], Jiang and Hong gave the exact determinants of the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci, and the generalized Lucas numbers by the inverse factorization of polynomial. It should be noted that Jiang and Zhou [16] obtained the explicit formula for spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [26] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. Furthermore, in [14] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices. The authors [24] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. The determinants and inverses of Tribonacci circulant type matrices are discussed in [18]. Sun and Jiang [23] gave the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix by constructing the transformation matrices. Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are given by constructing the special transformation matrices in [5].

The purpose of this paper is to obtain better results for the determinant and inverse of Gaussian Fibonacci skew-Hermitian Toeplitz type matrix. In this paper we adopt the following two conventions $0^0 = 1$, $i^2 = -1$, and we define a kind of special matrix as follows.

Definition 1.1. A Gaussian Fibonacci skew-Hermitian Toeplitz matrix is a square matrix of the form

$$\mathbf{T}_{G,n} = \begin{pmatrix} G_0 & G_1 & G_2 & \cdots & G_{n-3} & G_{n-2} & G_{n-1} \\ -\bar{G}_1 & G_0 & G_1 & \ddots & \ddots & G_{n-3} & G_{n-2} \\ -\bar{G}_2 & -\bar{G}_1 & G_0 & \ddots & \ddots & \ddots & G_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -\bar{G}_{n-3} & \ddots & \ddots & \ddots & G_0 & G_1 & G_2 \\ -\bar{G}_{n-2} & -\bar{G}_{n-3} & \ddots & \ddots & -\bar{G}_1 & G_0 & G_1 \\ -\bar{G}_{n-1} & -\bar{G}_{n-2} & -\bar{G}_{n-3} & \cdots & -\bar{G}_2 & -\bar{G}_1 & G_0 \end{pmatrix}_{n\times n},$$

where $G_0, G_1, \cdots, G_{n-1}$ are the Gaussian Fibonacci numbers. It is evidently determined by its first row.

2. Preliminaries

The determinant and the inverse of some special structured matrices are discussed in this section. These matrices can be met in course of Gaussian Fibonacci skew-Hermitian Toeplitz matrix determinant-calculation (or inverse-calculation).

Lemma 2.1 ([19])**.** *A Toeplitz-Hessenberg matrix is an $n \times n$ matrix of the form*

$$T_H = \begin{pmatrix} \kappa_1 & \kappa_0 & 0 & \cdots & \cdots & 0 \\ \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & \vdots \\ \kappa_4 & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \kappa_{n-1} & \cdots & \cdots & \kappa_2 & \kappa_1 & \kappa_0 \\ \kappa_n & \cdots & \cdots & \cdots & \kappa_2 & \kappa_1 \end{pmatrix}_{n\times n}, \quad \kappa_0 \neq 0,$$

where $\kappa_i \neq 0$ *for at least one* $i > 0$.

Let n *be a positive integer. Then the determinant of an* $n \times n$ *Toeplitz-Hessenberg matrix is*

$$\det T_H = \sum_{t_1+2t_2+\cdots+nt_n=n} \begin{pmatrix} t_1+\cdots+t_n \\ t_1,\cdots,t_n \end{pmatrix} (-\kappa_0)^{n-t_1-\cdots-t_n} \kappa_1^{t_1}\kappa_2^{t_2}\cdots\kappa_n^{t_n},$$

where

$$\begin{pmatrix} t_1+\cdots+t_n \\ t_1,\cdots,t_n \end{pmatrix} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$$

is the multinomial coefficient and $n = t_1 + 2t_2 + \cdots + nt_n$ *is a partition of the positive integer* n *where each positive integer* i *appears* t_i *times.*

Lemma 2.2. *Define an* $i \times i$ *lower-triangular Toeplitz-like matrix by*

$$\nabla_i([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-1}) = \begin{pmatrix} \mu_i & \mu_{i-1} & \mu_{i-2} & \mu_{i-3} & \mu_{i-4} & \cdots & \mu_2 & \mu_1 \\ \kappa_1 & \kappa_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & & & \vdots \\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & & \vdots \\ \kappa_4 & \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & 0 \\ \kappa_{i-1} & \cdots & \cdots & \kappa_4 & \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 \end{pmatrix}_{i\times i}, \quad k_0 \neq 0,$$

we have

$$\begin{aligned} &\det \nabla_1(\mu_1) = \mu_1, \\ &\det \nabla_i([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-1}) \\ &= \mu_i\kappa_0^{i-1} + \sum_{j=1}^{i-1}(-1)^{2+j}\mu_{i-j}\kappa_0^{i-j-1}\left(\sum_{t_1+2t_2+\cdots+jt_j=j}\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix}(-\kappa_0)^{j-t_1-\cdots-t_j}\kappa_1^{t_1}\cdots\kappa_j^{t_j}\right), \ i \geqslant 2, \end{aligned}$$

where $\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} = \dfrac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$.

Proof. By using the Laplace expansion of matrix $\nabla_i([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-1})$ $(i \geqslant 2)$ along the first row and Lemma 2.1, it is easy to check that

$$\begin{aligned} &\det \nabla_i([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-1}) \\ &= (-1)^{1+1}\mu_i\kappa_0^{i-1} \\ &\quad + (-1)^{1+2}\mu_{i-1}\kappa_0^{i-2}\left(\sum_{t_1=1}\begin{pmatrix} t_1 \\ t_1 \end{pmatrix}(-\kappa_0)^{1-t_1}\kappa_1^{t_1}\right) \\ &\quad + (-1)^{1+3}\mu_{i-2}\kappa_0^{i-3}\left(\sum_{t_1+2t_2=2}\begin{pmatrix} t_1+t_2 \\ t_1,t_2 \end{pmatrix}(-\kappa_0)^{2-t_1-t_2}\kappa_1^{t_1}\kappa_2^{t_2}\right) \\ &\quad \vdots \\ &\quad + (-1)^{1+i}\mu_1\kappa_0^0\left(\sum_{t_1+2t_2+\cdots+(i-1)t_{i-1}=i-1}\begin{pmatrix} t_1+\cdots+t_{i-1} \\ t_1,\cdots,t_{i-1} \end{pmatrix}(-\kappa_0)^{i-1-t_1-\cdots-t_{i-1}}\kappa_1^{t_1}\cdots\kappa_{i-1}^{t_{i-1}}\right) \end{aligned}$$

$$= \mu_i \kappa_0^{i-1} + \sum_{j=1}^{i-1} (-1)^{2+j} \mu_{i-j} \kappa_0^{i-j-1} \left(\sum_{t_1+2t_2+\cdots+jt_j=j} \begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} (-\kappa_0)^{j-t_1-\cdots-t_j} \kappa_1^{t_1} \cdots \kappa_j^{t_j} \right),$$

where $\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} = \frac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$. □

Lemma 2.3. *Define an $i \times i$ lower-triangular Toeplitz-like matrix by*

$$\mathcal{J}_i([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-2}) = \begin{pmatrix} \mu_i & \mu_{i-1} & \mu_{i-2} & \mu_{i-3} & \mu_{i-4} & \cdots & \mu_2 & \mu_1 \\ \kappa_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \kappa_1 & \kappa_0 & \ddots & & & & & \vdots \\ \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & & & \vdots \\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \kappa_{i-3} & \cdots & \cdots & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & \vdots \\ \kappa_{i-2} & \cdots & \cdots & \cdots & \kappa_2 & \kappa_1 & \kappa_0 & 0 \end{pmatrix}_{i\times i},$$

with $\mu_1\kappa_0 \neq 0$, then its inverse $\mathcal{J}_i^{-1}([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-2})$ is given by

$$\mathcal{J}_i^{-1}([\mu_k]_{k=1}^i, \kappa_0, \kappa_1, \cdots, \kappa_{i-2}) = \begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_2 & b_1 & \ddots & & & & \vdots \\ 0 & b_3 & b_2 & b_1 & \ddots & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & b_{i-1} & b_{i-2} & \cdots & \cdots & b_3 & b_2 & b_1 \\ B_1 & B_2 & B_3 & \cdots & \cdots & B_{i-2} & B_{i-1} & B_i \end{pmatrix}_{i\times i},$$

where

$$b_1 = \frac{1}{\kappa_0},$$

$$b_j = (-1)^{j-1}\kappa_0^{-j} \left(\sum_{t_1+2t_2+\cdots+(j-1)t_{j-1}=j-1} \begin{pmatrix} t_1+\cdots+t_{j-1} \\ t_1,\cdots,t_{j-1} \end{pmatrix} (-\kappa_0)^{(j-1)-t_1-\cdots-t_{j-1}} \kappa_1^{t_1} \cdots \kappa_{j-1}^{t_{j-1}} \right),$$

$$2 \leqslant j \leqslant i-1,$$

$$B_1 = \frac{1}{\mu_1},$$

$$B_j = \frac{(-1)^{2j-1} \det \nabla_{i+1-j}([\mu_k]_{k=2}^{i+2-j}, \kappa_0, \kappa_1, \cdots, \kappa_{i-j})}{\mu_1 \kappa_0^{i+1-j}}, \ (2 \leqslant j \leqslant i),$$

with

$$\det \nabla_1(\mu_2) = \mu_2,$$

$$\det \nabla_i([\mu_k]_{k=2}^{i+1}, \kappa_0, \kappa_1, \cdots, \kappa_{i-1})$$

$$= \mu_{i+1}\kappa_0^{i-1} + \sum_{j=1}^{i-1} (-1)^{2+j} \mu_{i-j+1} \kappa_0^{i-j-1} \left(\sum_{t_1+2t_2+\cdots+jt_j=j} \begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} (-\kappa_0)^{j-t_1-\cdots-t_j} \kappa_1^{t_1} \cdots \kappa_j^{t_j} \right), \ i \geqslant 2,$$

and $\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} = \dfrac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$.

Proof. By using the Laplace expansion of matrix $\mathcal{J}_i([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2})$ along the last column, it is easy to check that

$$\det \mathcal{J}_i([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2}) = (-1)^{1+i}\mu_1\kappa_0^{i-1}\neq 0,$$

according to the sufficient and necessary conditions for the invertibility of matrices, we obtain that $\mathcal{J}_i([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2})$ is invertible, and the augmented matrix of $\mathcal{J}_i([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2})$ is as follows

$$\mathcal{J}_i^*([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2}) = \begin{pmatrix} 0 & b_1' & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_2' & b_1' & \ddots & & & & \vdots \\ 0 & b_3' & b_2' & b_1' & \ddots & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & b_{i-1}' & b_{i-2}' & \cdots & \cdots & b_3' & b_2' & b_1' \\ B_1' & B_2' & B_3' & \cdots & \cdots & B_{i-2}' & B_{i-1}' & B_i' \end{pmatrix}_{i\times i},$$

where

$$\begin{aligned} & b_1' = (-1)^{i+1}\mu_1\kappa_0^{i-2}, \\ & b_j' = (-1)^{i+j}\mu_1\kappa_0^{i-1-j}\left(\sum_{t_1+2t_2+\cdots+(j-1)t_{j-1}=j-1}\begin{pmatrix} t_1+\cdots+t_{j-1} \\ t_1,\cdots,t_{j-1} \end{pmatrix}(-\kappa_0)^{(j-1)-t_1-\cdots-t_{j-1}}\kappa_1^{t_1}\cdots\kappa_{j-1}^{t_{j-1}}\right), \\ & \quad 2\leqslant j\leqslant i-1, \\ & B_1' = (-1)^{1+i}\kappa_0^{i-1}, \\ & B_j' = (-1)^{2j+i}\kappa_0^{j-2}\det\nabla_{i+1-j}([\mu_k]_{k=2}^{i+2-j},\kappa_0,\kappa_1,\cdots,\kappa_{i-j}),\ 2\leqslant j\leqslant i, \end{aligned}$$

with

$$\begin{aligned} & \det\nabla_1(\mu_2) = \mu_2, \\ & \det\nabla_i([\mu_k]_{k=2}^{i+1},\kappa_0,\kappa_1,\cdots,\kappa_{i-1}) \\ & = \mu_{i+1}\kappa_0^{i-1} + \sum_{j=1}^{i-1}(-1)^{2+j}\mu_{i-j+1}\kappa_0^{i-j-1}\left(\sum_{t_1+2t_2+\cdots+jt_j=j}\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix}(-\kappa_0)^{j-t_1-\cdots-t_j}\kappa_1^{t_1}\cdots\kappa_j^{t_j}\right),\ i\geqslant 2, \end{aligned}$$

and $\begin{pmatrix} t_1+\cdots+t_j \\ t_1,\cdots,t_j \end{pmatrix} = \dfrac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$.

Then $\mathcal{J}_i^{-1}([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2}) = \dfrac{1}{(-1)^{1+i}\mu_1\kappa_0^{i-1}}\mathcal{J}_i^*([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2})$, and we obtain the inverse of $\mathcal{J}_i([\mu_k]_{k=1}^i,\kappa_0,\kappa_1,\cdots,\kappa_{i-2})$ as described in Lemma 2.3. □

3. Determinant and inverse of the Gaussian Fibonacci skew-Hermitian Toeplitz matrix

Let $\mathbf{T}_{G,n}$ be an invertible Gaussian Fibonacci skew-Hermitian Toeplitz matrix. In this section, we give the determinant and inverse of the matrix $\mathbf{T}_{G,n}$.

Theorem 3.1. *Let $\mathbf{T}_{G,n}$ be a Gaussian Fibonacci skew-Hermitian Toeplitz matrix, then we have*

$$\det \mathbf{T}_{G,1} = i,\ \det \mathbf{T}_{G,2} = 0,\ \det \mathbf{T}_{G,3} = 5i,\ \det \mathbf{T}_{G,4} = -5,$$

and

$$\begin{aligned}\det \mathbf{T}_{G,n} = {}&(-1)^{n-1}(2+i)^{n-3}G_0\{(\frac{\bar{G}_{n-2}G_1}{G_0} - \bar{G}_{n-3})[\frac{\bar{G}_{n-1}G_{n-1}}{G_0} + G_0\\ &+ \sum_{k=2}^{n-2}\Delta_{n-1-k}(\frac{\bar{G}_{n-1}G_k}{G_0} - \bar{G}_{n-1-k})] - (\frac{\bar{G}_{n-1}G_1}{G_0} - \bar{G}_{n-2})[\frac{\bar{G}_{n-2}G_{n-1}}{G_0} + G_1\\ &- \frac{2G_1}{2+i}(\frac{\bar{G}_{n-2}G_{n-2}}{G_0} + G_0) + \sum_{k=2}^{n-3}\Delta_{n-1-k}(\frac{\bar{G}_{n-2}G_k}{G_0} - \bar{G}_{n-2-k})]\},\ n > 4,\end{aligned} \tag{3.1}$$

where

$$\Delta_0 = 1,\ \Delta_1 = -\frac{2G_1}{2+i},\ \Delta_i = -\frac{2}{2+i}(G_i + \sum_{k=1}^{i-1} G_k\Delta_{i-k}),\ (2 \leqslant i \leqslant n-3).$$

Proof. Let $\mathbf{T}_{G,n}$ be a Gaussian Fibonacci skew-Hermitian Toeplitz matrix, and we can easily get the following conclusions:

$$\det \mathbf{T}_{G,1} = i,\ \det \mathbf{T}_{G,2} = 0,\ \det \mathbf{T}_{G,3} = 5i,\ \det \mathbf{T}_{G,4} = -5.$$

We can introduce the following two transformation matrices when $n > 4$,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ x & \vdots & & & & & \iddots & 1 \\ y & \vdots & & & & \iddots & 1 & 0 \\ 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & & \iddots & 1 & 1 & -1 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & & \vdots \\ 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

and

$$\mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \vdots & & & & \iddots & 1 \\ \vdots & \Delta_{n-3} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \Delta_{n-4} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\ 0 & \Delta_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

where

$$x = \frac{\bar{G}_{n-1}}{G_0},\ y = \frac{\bar{G}_{n-2}}{G_0},\ \Delta_0 = 1,\ \Delta_1 = -\frac{2G_1}{2+i},\ \Delta_i = -\frac{2}{2+i}(G_i + \sum_{k=1}^{i-1} G_k\Delta_{i-k}),\ (2 \leqslant i \leqslant n-3).$$

By using $\mathcal{M}_1$, $\mathcal{N}_1$, and the recurrence relations of the Gaussian Fibonacci sequences, the matrix $\mathbf{T}_{G,n}$ is changed into the following form,

$$\mathcal{M}_1\mathbf{T}_{G,n}\mathcal{N}_1 = \begin{pmatrix} G_0 & K_1 & G_{n-2} & G_{n-3} & \cdots & G_4 & G_3 & G_2 & G_1 \\ 0 & K_2 & \xi_{n-2} & \xi_{n-3} & \cdots & \xi_4 & \xi_3 & \xi_2 & \xi_1 \\ \vdots & K_3 & \eta_{n-2} & \eta_{n-3} & \cdots & \eta_4 & \eta_3 & \eta_2 & \eta_1 \\ \vdots & 0 & \alpha & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & 2G_1 & \alpha & \ddots & & & & \vdots \\ \vdots & \vdots & 2G_2 & 2G_1 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2G_{n-4} & \cdots & \cdots & 2G_2 & 2G_1 & \alpha & 0 \end{pmatrix}_{n\times n},$$

where

$$K_1 = \sum_{k=2}^{n-1} G_k\Delta_{n-1-k}, \qquad K_2 = \sum_{k=2}^{n-1} \xi_k\Delta_{n-1-k}, \qquad K_3 = \sum_{k=2}^{n-1} \eta_k\Delta_{n-1-k},$$
$$\xi_i = xG_i - \bar{G}_{n-i-1},\ (1 \leqslant i \leqslant n-2), \qquad \xi_{n-1} = xG_{n-1} + G_0,$$
$$\eta_i = yG_i - \bar{G}_{n-i-2},\ (1 \leqslant i \leqslant n-3), \qquad \eta_{n-2} = yG_{n-2} + G_0, \qquad \eta_{n-1} = yG_{n-1} + G_1,\ \alpha = 2+i.$$

By using the Laplace expansion of matrix $\mathcal{M}_1\mathbf{T}_{G,n}\mathcal{N}_1$ along the first column, we can get that

$$\begin{aligned} \det(\mathcal{M}_1\mathbf{T}_{G,n}\mathcal{N}_1) &= (-1)^{n-1}\alpha^{n-3}G_0(K_2\eta_1 - K_3\xi_1), \\ &= (-1)^{n-1}(2+i)^{n-3}G_0\{(\frac{\bar{G}_{n-2}G_1}{G_0} - \bar{G}_{n-3})[\frac{\bar{G}_{n-1}G_{n-1}}{G_0} + G_0 \\ &\quad + \sum_{k=2}^{n-2}\Delta_{n-1-k}(\frac{\bar{G}_{n-1}G_k}{G_0} - \bar{G}_{n-1-k})] - (\frac{\bar{G}_{n-1}G_1}{G_0} - \bar{G}_{n-2})[\frac{\bar{G}_{n-2}G_{n-1}}{G_0} + G_1 \\ &\quad - \frac{2G_1}{2+i}(\frac{\bar{G}_{n-2}G_{n-2}}{G_0} + G_0) + \sum_{k=2}^{n-3}\Delta_{n-1-k}(\frac{\bar{G}_{n-2}G_k}{G_0} - \bar{G}_{n-2-k})]\}, \end{aligned}$$

while

$$\det \mathcal{M}_1 = \det \mathcal{N}_1 = (-1)^{\frac{(n-1)(n-2)}{2}},$$

we can obtain $\det \mathbf{T}_{G,n}$ as (3.1), which completes the proof. □

Theorem 3.2. *Let $\mathbf{T}_{G,n}$ be an invertible Gaussian Fibonacci skew-Hermitian Toeplitz matrix and $n > 5$. Then we have*

$$\mathbf{T}_{G,n}^{-1} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ -\bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \ddots & \ddots & \psi_{2,n-1} & \psi_{1,n-1} \\ -\bar{\psi}_{1,3} & -\bar{\psi}_{2,3} & \psi_{3,3} & \ddots & \ddots & \ddots & \psi_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -\bar{\psi}_{1,n-2} & \ddots & \ddots & \ddots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ -\bar{\psi}_{1,n-1} & -\bar{\psi}_{2,n-1} & \ddots & \ddots & -\bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ -\bar{\psi}_{1,n} & -\bar{\psi}_{1,n-1} & -\bar{\psi}_{1,n-2} & \cdots & -\bar{\psi}_{1,3} & -\bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix}_{n\times n},$$

where

$$
\begin{aligned}
\psi_{1,1} &= \frac{1}{G_0} - \frac{K_1 x}{G_0 K_2} + D_1 B_1\left(-\frac{K_3 x}{K_2} + y\right),\\
\psi_{1,2} &= D_2 b_1 + D_1 B_{n-2},\\
\psi_{1,3} &= D_3 b_1 + D_2(b_2 + b_1) + D_1(B_{n-3} + B_{n-2}),\\
\psi_{1,j} &= D_j b_1 + D_{j-1}(b_2 + b_1) + D_1(B_{n-j} + B_{n-j+1} - B_{n-j+2})\\
&\quad + \sum_{k=2}^{j-2} D_k(b_{j-k+1} + b_{j-k} - b_{j-k-1}), \quad (4 \leqslant j \leqslant n-2),
\end{aligned}
$$

$$
\psi_{1,n-1} = D_{n-2} b_1 + D_1(B_1 + B_2 - B_3) + \sum_{k=2}^{n-3} D_k(b_{n-1-k} - b_{n-2-k}),
$$

$$
\begin{aligned}
\psi_{1,n} &= -\frac{K_1}{G_0 K_2} - D_1\left(\frac{K_3 B_1}{K_2} + B_2\right) - \sum_{k=2}^{n-2} D_k b_{n-1-k},\\
\psi_{2,2} &= B_{n-2},\ \psi_{2,3} = B_{n-3} + B_{n-2},\ \psi_{2,j} = B_{n-j} + B_{n-j+1} - B_{n-j+2},\ (4 \leqslant j \leqslant n-1),\\
\psi_{3,3} &= a_{n-3,3} b_1 + a_{n-3,2}(b_2 + b_1) + a_{n-3,1}(B_{n-3} + B_{n-2}),\\
\psi_{i,j} &= a_{n-i,j} b_1 + a_{n-i,j-1}(b_2 + b_1) + a_{n-i,1}(B_{n-j} + B_{n-j+1} - B_{n-j+2})\\
&\quad + \sum_{k=2}^{j-2} a_{n-i,k}(b_{j-k+1} + b_{j-k} - b_{j-k-1}), \quad \left(3 \leqslant i \leqslant \left[\frac{n+1}{2}\right];\ i \leqslant j \leqslant n+1-i;\ \textit{except}\ \psi_{3,3}\right),
\end{aligned}
$$

in which

$$
\begin{aligned}
D_i &= \frac{K_1 \xi_i}{G_0 K_2} - \frac{G_i}{G_0},\ (1 \leqslant i \leqslant n-2),\ \omega_i = -\frac{K_3}{K_2}\xi_i + \eta_i,\ (1 \leqslant i \leqslant n-2),\\
a_{i,j} &= \begin{cases} -\dfrac{\xi_j \Delta_i}{K_2}, & i+j \neq n-1,\\ -\dfrac{\xi_j \Delta_i}{K_2} + 1, & i+j = n-1, \end{cases} \quad (0 \leqslant i \leqslant n-3,\ 1 \leqslant j \leqslant n-2),\\
b_1 &= \frac{1}{\alpha},\\
b_i &= (-1)^{i-1}\alpha^{-i}\left(\sum_{t_1+2t_2+\cdots+(i-1)t_{i-1}=i-1} \binom{t_1+\cdots+t_{i-1}}{t_1,\cdots,t_{i-1}} (-\alpha)^{(i-1)-t_1-\cdots-t_{i-1}} 2G_1^{t_1}\cdots 2G_{i-1}^{t_{i-1}}\right),\\
&\quad (2 \leqslant i \leqslant n-3),\\
B_1 &= \frac{1}{\omega_1},\\
B_i &= \frac{(-1)^{2i-1}\det\nabla_{n-1-i}([\omega_k]_{k=2}^{n-i}, \alpha, 2G_1, 2G_2, \cdots, 2G_{n-2-i})}{\omega_1 \alpha^{n-1-i}},\ (2 \leqslant i \leqslant n-2),\\
\det\nabla_1(\omega_2) &= \omega_2,\\
\det\nabla_i([\omega_k]_{k=2}^{i+1}, \alpha, 2G_1, \cdots, 2G_{i-1}) &= \omega_{i+1}\alpha^{i-1} + \sum_{j=1}^{i-1}(-1)^{2+j}\omega_{i-j+1}\alpha^{i-j-1}\\
&\quad \cdot \left(\sum_{t_1+2t_2+\cdots+jt_j=j} \binom{t_1+\cdots+t_j}{t_1,\cdots,t_j} (-\alpha)^{j-t_1-\cdots-t_j} 2G_1^{t_1}\cdots 2G_j^{t_j}\right),\ 2 \leqslant i \leqslant n-3,
\end{aligned}
$$

with $\binom{t_1+\cdots+t_j}{t_1,\cdots,t_j} = \dfrac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$, x, y, α, K_1, K_2, K_3, $\Delta_i (0 \leqslant i \leqslant n-3)$, $\eta_i (1 \leqslant i \leqslant n-1)$, $\xi_i (1 \leqslant$

$i \leqslant n-1$) *as in Theorem* 3.1, $[x] = m$, $m \leqslant x < m+1$, m *is an integer.*

We can observe that $\mathbf{T}_{G,n}^{-1}$ is not only a skew-Hermitian matrix, but also a symmetric matrix along its secondary diagonal, i.e., sub-symmetric matrix.

Proof. We can introduce the following two transformation matrices when $n > 5$,

$$\mathcal{M}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & -\frac{K_3}{K_2} & 1 & \ddots & & & & \vdots \\ \vdots & 0 & 0 & 1 & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

$$\mathcal{N}_2 = \begin{pmatrix} 1 & -\dfrac{K_1}{G_0} & D_{n-2} & D_{n-3} & \cdots & D_2 & D_1 \\ 0 & 1 & -\dfrac{\xi_{n-2}}{K_2} & -\dfrac{\xi_{n-3}}{K_2} & \cdots & -\dfrac{\xi_2}{K_2} & -\dfrac{\xi_1}{K_2} \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

where $D_i = \dfrac{K_1\xi_i}{G_0K_2} - \dfrac{G_i}{G_0}$, $(1 \leqslant i \leqslant n-2)$.

If we multiply $\mathcal{M}_1\mathbf{T}_n\mathcal{N}_1$ by $\mathcal{M}_2$ and $\mathcal{N}_2$, the $\mathcal{M}_1$ and $\mathcal{N}_1$ are as in the proof of Theorem 3.1, so we obtain

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_n\mathcal{N}_1\mathcal{N}_2 = \begin{pmatrix} G_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & K_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \omega_{n-2} & \omega_{n-3} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ \vdots & \vdots & \alpha & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & 2G_1 & \alpha & \ddots & & & & \vdots \\ \vdots & \vdots & 2G_2 & 2G_1 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2G_{n-4} & \cdots & \cdots & 2G_2 & 2G_1 & \alpha & 0 \end{pmatrix}_{n\times n},$$

where $\omega_i = -\dfrac{K_3}{K_2}\xi_i + \eta_i$, $(1 \leqslant i \leqslant n-2)$.

We have

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_n\mathcal{N}_1\mathcal{N}_2 = \Phi \oplus \Lambda,$$

where $\Phi = \begin{pmatrix} G_0 & 0 \\ 0 & K_2 \end{pmatrix}$ is a diagonal matrix, and Λ is a lower-triangular Toeplitz-like matrix

$$\Lambda = \begin{pmatrix} \omega_{n-2} & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ \alpha & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 2G_1 & \alpha & \ddots & & & & & & \vdots \\ 2G_2 & 2G_1 & \alpha & \ddots & & & & & \vdots \\ \vdots & 2G_2 & 2G_1 & \alpha & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 2G_1 & \alpha & \ddots & \vdots \\ 2G_{n-4} & \cdots & \cdots & \cdots & \cdots & 2G_2 & 2G_1 & \alpha & 0 \end{pmatrix}_{(n-2)\times(n-2)},$$

$\Phi \oplus \Lambda$ is the direct sum of Φ and Λ. Let $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2$, then we obtain

$$\mathbf{T}_{G,n}^{-1} = \mathcal{N}(\Phi^{-1} \oplus \Lambda^{-1})\mathcal{M},$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ x & \vdots & & & & \iddots & 1 \\ -\frac{K_3}{K_2}x + y & \vdots & & & \iddots & 1 & -\frac{K_3}{K_2} \\ 0 & \vdots & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$\mathcal{N} = \begin{pmatrix} 1 & -\frac{K_1}{G_0} & D_{n-2} & D_{n-3} & \cdots & D_2 & D_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \Delta_{n-3} & a_{n-3,n-2} & a_{n-3,n-3} & \cdots & a_{n-3,2} & a_{n-3,1} \\ \vdots & \Delta_{n-4} & a_{n-4,n-2} & a_{n-4,n-3} & \cdots & a_{n-4,2} & a_{n-4,1} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \Delta_1 & a_{1,n-2} & a_{1,n-3} & \cdots & a_{1,2} & a_{1,1} \\ 0 & 1 & a_{0,n-2} & a_{0,n-3} & \cdots & a_{0,2} & a_{0,1} \end{pmatrix}_{n\times n},$$

where

$$a_{i,j} = \begin{cases} -\frac{\xi_j\Delta_i}{K_2}, & i+j \neq n-1, \\ -\frac{\xi_j\Delta_i}{K_2} + 1, & i+j = n-1, \end{cases} \quad (0 \leqslant i \leqslant n-3,\ 1 \leqslant j \leqslant n-2).$$

We can observe that the inverse matrix of Φ is $\begin{pmatrix} \frac{1}{G_0} & 0 \\ 0 & \frac{1}{K_2} \end{pmatrix}_{2\times 2}$.

According to Lemma 2.3, we have

$$\Lambda^{-1}=\begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots & \cdots & 0\\ \vdots & b_2 & b_1 & \ddots & & & \vdots\\ \vdots & b_3 & b_2 & \ddots & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & & \ddots & b_2 & b_1 & 0\\ 0 & b_{n-3} & \cdots & \cdots & b_3 & b_2 & b_1\\ B_1 & B_2 & B_3 & \cdots & B_{n-4} & B_{n-3} & B_{n-2} \end{pmatrix}_{(n-2)\times(n-2)},$$

where

$$b_1=\frac{1}{\alpha},$$
$$b_i=(-1)^{i-1}\alpha^{-i}\left(\sum_{t_1+2t_2+\cdots+(i-1)t_{i-1}=i-1}\begin{pmatrix} t_1+\cdots+t_{i-1}\\ t_1,\cdots,t_{i-1}\end{pmatrix}(-\alpha)^{(i-1)-t_1-\cdots-t_{i-1}}2G_1^{t_1}\cdots 2G_{i-1}^{t_{i-1}}\right),$$
$$(2\leqslant i\leqslant n-3),$$
$$B_1=\frac{1}{\omega_1},$$
$$B_i=\frac{(-1)^{2i-1}\det\nabla_{n-1-i}([\omega_k]_{k=2}^{n-i},\alpha,2G_1,2G_2,\cdots,2G_{n-2-i})}{\omega_1\alpha^{n-1-i}},\ (2\leqslant i\leqslant n-2),$$

in which

$$\det\nabla_1(\omega_2)=\omega_2,$$
$$\det\nabla_i([\omega_k]_{k=2}^{i+1},\alpha,2G_1,\cdots,2G_{i-1})$$
$$=\omega_{i+1}\alpha^{i-1}+\sum_{j=1}^{i-1}(-1)^{2+j}\omega_{i-j+1}\alpha^{i-j-1}$$
$$\cdot\left(\sum_{t_1+2t_2+\cdots+jt_j=j}\begin{pmatrix} t_1+\cdots+t_j\\ t_1,\cdots,t_j\end{pmatrix}(-\alpha)^{j-t_1-\cdots-t_j}2G_1^{t_1}\cdots 2G_j^{t_j}\right),\ (2\leqslant i\leqslant n-3)$$

and $\begin{pmatrix} t_1+\cdots+t_j\\ t_1,\cdots,t_j\end{pmatrix}=\dfrac{(t_1+\cdots+t_j)!}{t_1!\cdots t_j!}$.

We obtain that

$$\Phi^{-1}\oplus\Lambda^{-1}=\begin{pmatrix} \frac{1}{G_0} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\ 0 & \frac{1}{K_2} & \vdots & \ddots & & & & \vdots\\ \vdots & 0 & \vdots & b_1 & \ddots & & & \vdots\\ \vdots & \vdots & \vdots & b_2 & b_1 & \ddots & & \vdots\\ \vdots & \vdots & \vdots & \vdots & b_2 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0\\ \vdots & \vdots & 0 & b_{n-3} & \cdots & \cdots & b_2 & b_1\\ 0 & 0 & B_1 & B_2 & B_3 & \cdots & B_{n-3} & B_{n-2} \end{pmatrix}_{n\times n},$$

then we have

$$\mathbf{T}_{G,n}^{-1} = \mathcal{N}(\Phi^{-1} \oplus \Lambda^{-1})\mathcal{M}$$
$$= \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ -\bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \ddots & \ddots & \psi_{2,n-1} & \psi_{1,n-1} \\ -\bar{\psi}_{1,3} & -\bar{\psi}_{2,3} & \psi_{3,3} & \ddots & \ddots & \ddots & \psi_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -\bar{\psi}_{1,n-2} & \ddots & \ddots & \ddots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ -\bar{\psi}_{1,n-1} & -\bar{\psi}_{2,n-1} & \ddots & \ddots & -\bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ -\bar{\psi}_{1,n} & -\bar{\psi}_{1,n-1} & -\bar{\psi}_{1,n-2} & \cdots & -\bar{\psi}_{1,3} & -\bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix}_{n\times n},$$

where $[\psi_{i,j}](1 \leqslant i \leqslant [\frac{n+1}{2}];\ i \leqslant j \leqslant n+1-i)$ are the same as in Theorem 3.2, which completes the proof. □

4. Numerical example

In this section, an example demonstrates the method which introduced above for the calculation of determinant and inverse of the Gaussian Fibonacci skew-Hermitian Toeplitz matrix. Here, we consider a 6×6 matrix:

$$\mathbf{T}_{G,6} = \begin{pmatrix} i & 1 & 1+i & 2+i & 3+2i & 5+3i \\ -1 & i & 1 & 1+i & 2+i & 3+2i \\ -1+i & -1 & i & 1 & 1+i & 2+i \\ -2+i & -1+i & -1 & i & 1 & 1+i \\ -3+2i & -2+i & -1+i & -1 & i & 1 \\ -5+3i & -3+2i & -2+i & -1+i & -1 & i \end{pmatrix}_{6\times 6}.$$

Using the corresponding formulas in Theorem 3.1, we get

$$\Delta_1 = -\frac{4}{5} + \frac{2}{5}i,\ \Delta_2 = -\frac{18}{25} - \frac{26}{25}i,\ \Delta_3 = \frac{14}{125} + \frac{48}{125}i,$$

From (3.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{G,6} = (-1)^5(2+i)^3 G_0\{(\frac{\bar{G}_4 G_1}{G_0} - \bar{G}_3)[\frac{\bar{G}_5 G_5}{G_0} + G_0 \\ + \sum_{k=2}^{4} \Delta_{5-k}(\frac{\bar{G}_5 G_k}{G_0} - \bar{G}_{5-k})] - (\frac{\bar{G}_5 G_1}{G_0} - \bar{G}_4)[\frac{\bar{G}_4 G_5}{G_0} + G_1 \\ + \Delta_1(\frac{\bar{G}_4 G_4}{G_0} + G_0) + \sum_{k=2}^{3} \Delta_{5-k}(\frac{\bar{G}_4 G_k}{G_0} - \bar{G}_{4-k})]\} = -45. \end{aligned}$$

As the inverse calculation, if we use the corresponding formulas in Theorem 3.2, we get

$$\begin{aligned} &\psi_{1,1} = -\frac{4}{9}i, && \psi_{1,2} = -\frac{2}{9} + \frac{5}{9}i, && \psi_{1,3} = -\frac{2}{9}, && \psi_{1,4} = \frac{2}{3} + \frac{2}{9}i, \\ &\psi_{1,5} = -\frac{4}{45} + \frac{22}{45}i, && \psi_{1,6} = -\frac{7}{45} - \frac{8}{15}i, && \psi_{2,2} = -\frac{5}{9}i, && \psi_{2,3} = -\frac{2}{9} - \frac{1}{9}i, \\ &\psi_{2,4} = -\frac{4}{9} - \frac{4}{9}i, && \psi_{2,5} = \frac{4}{15} - \frac{16}{45}i, && \psi_{3,3} = -\frac{1}{9}i, && \psi_{3,4} = -\frac{10}{9} + \frac{1}{3}i, \end{aligned}$$

so we can get

$$\mathbf{T}_{G,6}^{-1} = \begin{pmatrix} -\frac{4}{9}i & -\frac{2}{9}+\frac{5}{9}i & -\frac{2}{9} & \frac{2}{3}+\frac{2}{9}i & -\frac{4}{45}+\frac{22}{45}i & -\frac{7}{45}-\frac{8}{15}i \\ \frac{2}{9}+\frac{5}{9}i & -\frac{5}{9}i & -\frac{2}{9}-\frac{1}{9}i & -\frac{4}{9}-\frac{4}{9}i & \frac{4}{15}-\frac{16}{45}i & -\frac{4}{45}+\frac{22}{45}i \\ \frac{2}{9} & \frac{2}{9}-\frac{1}{9}i & -\frac{1}{9}i & -\frac{10}{9}+\frac{1}{3}i & -\frac{4}{9}-\frac{4}{9}i & \frac{2}{3}+\frac{2}{9}i \\ -\frac{2}{3}+\frac{2}{9}i & \frac{4}{9}-\frac{4}{9}i & \frac{10}{9}+\frac{1}{3}i & -\frac{1}{9}i & -\frac{2}{9}-\frac{1}{9}i & -\frac{2}{9} \\ \frac{4}{45}+\frac{22}{45}i & -\frac{4}{15}-\frac{16}{45}i & \frac{4}{9}-\frac{4}{9}i & \frac{2}{9}-\frac{1}{9}i & -\frac{5}{9}i & -\frac{2}{9}+\frac{5}{9}i \\ \frac{7}{45}-\frac{8}{15}i & \frac{4}{45}+\frac{22}{45}i & -\frac{2}{3}+\frac{2}{9}i & \frac{2}{9} & \frac{2}{9}+\frac{5}{9}i & -\frac{4}{9}i \end{pmatrix}_{6\times 6}.$$

5. Conclusion

We deal with two kinds of lower triangular Toeplitz-like matrices base on the determinant of the Toeplitz-Hessenberg matrix, the determinants and inverses of such special matrices are presented respectively. By constructing the special transformation matrices and using the determinant of the lower triangular Toeplitz-like matrix, we obtain the determinant and inverse of a Gaussian Fibonacci skew-Hermitian Toeplitz matrix.

Acknowledgment

The research was supported by National Natural Science Foundation of China (Grant No.11671187), Natural Science Foundation of Shandong Province (Grant No. ZR2016AM14) and the AMEP of Linyi University, China.

References

[1] M. Akbulak, D. Bozkurt, *On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers*, Hacet. J. Math. Stat., **37** (2008), 89–95. 1

[2] D. Bozkurt, T.-Y. Tam, *Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas Numbers*, Appl. Math. Comput., **219** (2012), 544–551. 1

[3] A. Buckley, *On the solution of certain skew symmetric linear systems*, SIAM J. Numer. Anal., **14** (1977), 566–570. 1

[4] R. H. Chan, X.-Q. Jin, *Circulant and skew-circulant preconditioners for skew-Hermitian type Toeplitz systems*, BIT, **31** (1991), 632–646. 1

[5] X.-T. Chen, Z.-L. Jiang, J.-M. Wang, *Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices*, British J. Math. Comput. Sci., **19** (2016), 1–21. 1

[6] L. Dazheng, *Fibonacci-Lucas quasi-cyclic matrices*, A special tribute to Calvin T. Long, Fibonacci Quart., **40** (2002), 280–286. 1

[7] U. Grenander, G. Szegö, *Toeplitz forms and their applications*, California Monographs in Mathematical Sciences University of California Press, Berkeley-Los Angeles, (1958). 1

[8] G. Heining, K. Rost, *Algebraic methods for Toeplitz-like matrices and operators*, Mathematical Research, Akademie-Verlag, Berlin, (1984). 1

[9] A. F. Horadam, *Further appearence of the Fibonacci sequence*, Fibonacci Quart., **1** (1963), 41–42. 1

[10] A. İpek, K. Arı, *On Hessenberg and pentadiagonal determinants related with Fibonacci and Fibonacci-like numbers*, Appl. Math. Comput., **229** (2014), 433–439. 1

[11] Z.-L. Jiang, Y.-P. Gong, Y. Gao, *Circulant type matrices with the sum and product of Fibonacci and Lucas numbers*, Abstr. Appl. Anal., **2014** (2014), 12 pages. 1

[12] Z.-L. Jiang, Y.-P. Gong, Y. Gao, *Invertibility and explicit inverses of circulant-type matrices with* k*-Fibonacci and* k*-Lucas numbers*, Abstr Appl. Anal., **2014** (2014), 10 pages. 1

[13] X.-Y. Jiang, K.-C. Hong, *Exact determinants of some special circulant matrices involving four kinds of famous numbers*, Abstr. Appl. Anal., **2014** (2014), 12 pages. 1

[14] X.-Y. Jiang, K.-C. Hong, *Explicit inverse matrices of Tribonacci skew circulant type matrices*, Appl. Math. Comput., **268** (2015), 93–102. 1
[15] X.-Y. Jiang, K.-C. Hong, *Skew cyclic displacements and inversions of two innovative patterned matrices*, Appl. Math. Comput., **308** (2017), 174–184. 1
[16] Z.-L. Jiang, J.-W. Zhou, *A note on spectral norms of even-order r-circulant matrices*, Appl. Math. Comput., **250** (2014), 368–371. 1
[17] J. Li, Z.-L. Jiang, F.-L. Lu, *Determinants, norms, and the spread of circulant matrices with Tribonacci and generalized Lucas numbers*, Abstr. Appl. Anal., **2014** (2014), 9 pages. 1
[18] L. Liu, Z.-L. Jiang, *Explicit form of the inverse matrices of Tribonacci circulant type matrices*, Abstr. Appl. Anal., **2015** (2015), 10 pages. 1
[19] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Spec. Matrices, **2** (2014), 10–16. 2.1
[20] B. N. Mukherjee, S. S. Maiti, *On some properties of positive definite Toeplitz matrices and their possible applications*, Linear Algebra Appl., **102** (1988), 211–240. 1
[21] L. Rodman, *Pairs of Hermitian and skew-Hermitian quaternionic matrices: canonical forms and their applications*, Linear Algebra Appl., **429** (2008), 981–1019. 1
[22] S.-Q. Shen, J.-M. Cen, Y. Hao, *On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers*, Appl. Math. Comput., **217** (2011), 9790–9797. 1
[23] J.-X. Sun, Z.-L. Jiang, *Computing the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix*, British J. Mathe. Comput. Sci., **19** (2016), 1–16. 1
[24] Y.-P. Zheng, S.-G. Shon, *Exact determinants and inverses of generalized Lucas skew circulant type matrices*, Appl. Math. Comput., **270** (2015), 105–113. 1
[25] Y.-P. Zheng, S.-G. Shon, J.-Y. Kim, *Cyclic displacements and decompositions of inverse matrices for CUPL Toeplitz matrices*, J. Math. Anal. Appl., **455** (2017), 727–741. 1
[26] J.-W. Zhou, Z.-L. Jiang, *The spectral norms of g-circulant matrices with classical Fibonacci and Lucas numbers entries*, Appl. Math. Comput., **233** (2014), 582–587. 1