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Modified hybrid iterative methods for generalized mixed equilibrium, variational inequality and fixed point problems

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Abstract

In this paper, we introduce two modified hybrid iterative methods (one implicit method and one explicit method) for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of solutions of a variational inequality problem for a continuous monotone mapping and the set of fixed points of a continuous pseudocontractive mapping in Hilbert spaces, and show under suitable control conditions that the sequences generated by the proposed iterative methods converge strongly to a common element of three sets, which solves a certain variational inequality. As a direct consequence, we obtain the unique minimum-norm common point of three sets. The results in this paper substantially improve upon, develop and complement the previous well-known results in this area.

Keywords: Hybrid iterative method, generalized mixed equilibrium problem, continuous monotone mapping, continuous pseudocontractive mapping, variational inequality, fixed point, ρ-Lipschitzian and η-strongly monotone mapping, metric projection.
2010 MSC: 49J30, 49J40, 47H09, 47H10, 47J20, 47J25, 47J05, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

Let $B : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R}$ be a function and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Then, we consider the following generalized mixed equilibrium problem (for short, GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C, \tag{1.1}$$

which was introduced by Peng and Yao [22] (also see [18, 42]). The set of solutions of the problem (1.1) is denoted by $GMEP(\Theta, \varphi, B)$. Here some special cases of the problem (1.1) are stated as follows:

If $\varphi = 0$, then the problem (1.1) reduces the following generalized equilibrium problem (for short, GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x \rangle \geqslant 0, \quad \forall y \in C, \tag{1.2}$$

which was studied by Takahashi and Takahashi [29]. The set of solutions of the problem (1.2) is denoted

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by $\mathrm{GEP}(\Theta, B)$.

If $B = 0$, then the problem (1.1) reduces the following mixed equilibrium problem (for short, MEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Ceng and Yao [5] (see also [39]). The set of solutions of the problem (1.3) is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

If $\varphi = 0$ and $B = 0$, then the problem (1.1) reduces the following equilibrium problem (for short, EP) of finding $x \in C$ such that

$$\Theta(x, y) \geqslant 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of the problem (1.4) is denoted by $\mathrm{EP}(\Theta)$.

If $\varphi = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, the problem (1.1) reduces the following variational inequality problem (for short, VIP) of finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geqslant 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problem (1.5) is denoted by $\mathrm{VI}(C, B)$.

The GMEP (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, minmax problems, Nash equilibrium problems in noncooperative games and others, see for example [2, 5, 9, 10].

As we all know, the convex feasibility problem (CFP) is the problem of finding a point in the (nonempty) intersection $C = \cap_{i=1}^{m} C_i$ of a finite number of closed convex sets C_i $(i = 1, \cdots, m)$. The split common fixed point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the problem (CFP). Several iterative methods for solving the problem (SCFP) for nonlinear mappings were developed; see for example [7, 31, 33, 40] and the references therein.

Recently, many authors considered iterative methods for finding a common point of solution sets of the problems GMEP (1.1), GEP (1.2), MEP (1.3), EP (1.4) and VIP (1.5) and fixed point sets of nonlinear mappings as special cases of the problem (CFP). In particular, in order to study the EP (1.4) coupled with the fixed point problem, many authors have introduced iterative methods for finding a common element of the set of the solutions of the EP (1.4) and the set of fixed points of a countable family of nonexpansive mappings; see [6, 8, 13, 23, 24, 27, 28, 32, 37, 38] and the references therein.

In 2008, Su et al. [25] gave an iterative method for the EP (1.4), the VIP (1.5) for an inverse-strongly monotone mapping F and nonexpansive mapping S and proved strong convergence to a point z in $\mathrm{EP}(\Theta) \cap \mathrm{VI}(C, F) \cap \mathrm{Fix}(S)$. In 2009, Yao et al. [36] considered an iterative method for the MEP (1.3), the VIP (1.5) for a Lipschitz and relaxed-cocoercive mapping F and a sequence $\{S_n\}$ of nonexpansive mappings, and proved strong convergence to a point $z \in \cap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{MEP}(\Theta, \varphi) \cap \mathrm{VI}(C, F)$. In 2008, Peng and Yao [22] studied an iterative method for the GMEP (1.1) related to an α-inverse-strongly monotone mapping B, the VIP (1.5) for a monotone and Lipschitz continuous mapping F and a nonexpansive mapping S, and proved strong convergence to a point $z \in \mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, F) \cap \mathrm{Fix}(S)$. In 2010, by using the method of Yao et al. [39], Jaiboon and Kumam [12] also introduced an iterative method related to optimization problem for the MEP (1.3), the VIP (1.5) for an α-inverse-strongly monotone mapping F and a sequence $\{S_n\}$ of nonexpansive mappings, and showed strong convergence to a point $z \in \cap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{MEP}(\Theta, \varphi) \cap \mathrm{VI}(C, F)$.

In 2007, Tada and Takahashi [27] considered an iterative method for the EP (1.4) and nonexpansive mapping S and proved weak convergence to a point $w \in \mathrm{EP}(\Theta) \cap \mathrm{Fix}(S)$. In 2008, Moudafi [21] proposed an iterative method for the GEP (1.2) related to an α-inverse-strongly monotone mapping B and nonexpansive mapping S and showed weak convergence to a point $w \in \mathrm{GEP}(\Theta, B) \cap \mathrm{Fix}(S)$. In 2009, Ceng et al. [3] provided an iterative method for the EP (1.4) and k-strictly pseudocontractive mapping T and proved weak convergence to a point $w \in \mathrm{EP}(\Theta) \cap \mathrm{Fix}(T)$. In 2015, Lv [19] also studied an iterative method for the GEP (1.2) and k-strictly pseudocontractive mapping T and proved weak convergence to a point $w \in \mathrm{GEP}(\Theta) \cap \mathrm{Fix}(T)$.

In 2003, Takahashi and Toyoda [30] introduced an iterative method for the VIP (1.4) related to an α-inverse-strongly monotone mapping F and nonexpansive mapping S and established weak convergence to a point $w \in \mathrm{Fix}(S) \cap VI(C, F)$.

In 2012, Jung [16] considered an iterative method for GMEP (1.1) related to a β-inverse-strongly monotone mapping B, the VIP (1.5) for an α-inverse-strongly monotone mapping and k-strictly pseudocontractive mapping T and proved weak convergence to a point $w \in GMEP(\Theta, \varphi, B) \cap VI(C, F) \cap \mathrm{Fix}(T)$. In 2015 Jung [17] also proposed an iterative method for GMEP (1.1) related to a continuous monotone mapping B, the VIP (1.5) for a continuous monotone mapping F and a continuous pseudocontractive mapping T and proved weak convergence to a point $w \in GMEP(\Theta, \varphi, B) \cap VI(C, F) \cap \mathrm{Fix}(T)$.

In 2012, by using Yamada's hybrid steepest-descent method [35] and Jung's viscosity iterative method [14], Jung [15] introduced new implicit and explicit iterative methods for finding a common element of the set of solutions of the MEP (1.3) and the set of fixed points of a k-strictly pseudocontractive mapping T and proved strong convergence to a point $z \in MEP(\Theta, \varphi) \cap \mathrm{Fix}(T)$. In particular, in 2012, by combining Colao et al.'s hybrid viscosity iterative method [8] and Yamada's hybrid steepest-descent method [35], Ceng et al. [4] proposed a hybrid iterative method for finding a common element of the set of solutions of the GMEP (1.1) related to an α-inverse-strongly monotone mapping B and the set of fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and showed strong convergence to a point $z \in \cap_{i=1}^N \mathrm{Fix}(T_i) \cap GMEP(\Theta, \varphi, B)$ which is a unique solution of certain variational inequality related to Lipschitzian and strongly monotone mapping G.

In this paper, inspired and motivated by above-mentioned results, we introduce two new modified hybrid iterative methods (one implicit method and one explicit method) for finding a common element of the solution set $GMEP(\Theta, \varphi, B)$ of the GMEP (1.1) related to a continuous monotone mapping B, the solution set $VI(C, F)$ of the VIF (1.5) for a continuous monotone mapping F and the fixed point set $\mathrm{Fix}(T)$ of a continuous pseudocontractive mapping T in a Hilbert space. We show that under suitable conditions, the sequences generated by the proposed iterative methods converge strongly to a common element of $\Omega := GMEP(\Theta, \varphi, B) \cap VI(C, F) \cap \mathrm{Fix}(T)$, which is a solution of a certain variational inequality. As a direct consequence, we find the unique solution of the minimization norm problem

$$\|x^*\| = \min\{\|x\| : x \in \Omega\}.$$

The results in this paper develop, improve upon and complement of the recent results announced by several authors in this direction.

2. Preliminaries and lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

We recall ([1, 11]) that a mapping F of C into H is called

(i) Lipschitzian, if there exists a constant $\kappa \geqslant 0$ such that

$$\|Fx - Fy\| \leqslant \kappa\|x - y\| \quad \forall x, y \in C;$$

(ii) monotone, if $\langle x - y, Fx - Fy\rangle \geqslant 0, \quad \forall x,\ y \in C$;

(iii) α-inverse-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Fx - Fy\rangle \geqslant \alpha\|Fx - Fy\|^2, \quad \forall x,\ y \in C;$$

(iv) η-strongly monotone, if there exists a positive real number η such that

$$\langle x - y, Fx - Fy\rangle \geqslant \eta\|x - y\|^2, \quad \forall x,\ y \in C.$$

We note that if F is an α-inverse-strongly monotone mapping of C into H, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Fx - Fy\| \leqslant \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$. Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

We recall ([1]) that a mapping $T : C \to H$ is said to be pseudocontractive, if

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be k-strictly pseudocontractive, if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. The class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive, (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive.

For solving the GMEP (1.1), the MEP (1.2), and the EP (1.3) for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leqslant 0$ for all $x,\ y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} \Theta(tz + (1 - t)x, y) \leqslant \Theta(x, y);$$

(A4) for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

We can prove the following lemma by using the same method as in [18, 42], and so we omit its proof.

Lemma 2.1. *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction form $C \times C$ to $\mathbb{R}$ satisfies* (A1)-(A4) *and $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $B : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0, \quad \forall y \in C.$$

Define a mapping $K_r : H \to C$ as follows:

$$K_r x = \left\{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0,\ \forall y \in C\right\}$$

for all $x \in H$ and $r > 0$. Then, the following hold:

(1) *For each $x \in H$, $K_r x \neq \emptyset$;*
(2) *K_r is single-valued;*
(3) *K_r is firmly nonexpansive, that is, for any $x,\ y \in H$,*

$$\|K_r x - K_r y\|^2 \leqslant \langle K_r x - K_r y, x - y\rangle;$$

(4) *$F(K_r) = GMEP(\Theta, \varphi, B)$;*
(5) *$GMEP(\Theta, \varphi, B)$ is closed and convex.*

We need the following lemmas for the proof of our main results.

Lemma 2.2 ([41])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Fz, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $F_r : H \to C$ by

$$F_r x = \left\{z \in C : \langle Fz, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C\right\}.$$

Then the following hold:

(i) F_r *is single-valued;*
(ii) F_r *is firmly nonexpansive, that is,*

$$\|F_r x - F_r y\|^2 \leqslant \langle F_r x - F_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) $\mathrm{Fix}(F_r) = VI(C, F)$;
(iv) $V(I, F)$ *is a closed convex subset of* C.

Lemma 2.3 ([41])**.** *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* $T : C \to H$ *be a continuous pseudocontractive mapping. Then, for* $r > 0$ *and* $x \in H$*, there exists* $z \in C$ *such that*

$$\langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leqslant 0, \quad \forall y \in C.$$

For $r > 0$ *and* $x \in H$*, define* $T_r : H \to C$ *by*

$$T_r x = \left\{ z \in C : \langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leqslant 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) T_r *is single-valued;*
(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{Fix}(T)$;
(iv) $\mathrm{Fix}(T)$ *is a closed convex subset of* C.

Lemma 2.4 ([34])**.** *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leqslant (1 - \lambda_n)s_n + \beta_n, \quad \forall n \geqslant 1,$$

where $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1 - \lambda_n) = 0$;
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemma is easily proven by property of inner product.

Lemma 2.5. *In a Hilbert space, there holds the inequality*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.6 ([26])**.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\gamma_n\}$ *be a sequence in* $[0, 1]$ *which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leqslant \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)y_n$*, for all* $n \geqslant 1$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.7. *Let* $V : C \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geqslant 0$*, and* $G : C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping with constants* κ *and* $\eta > 0$*. Then for* $0 \leqslant \gamma l < \mu\eta$*,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geqslant (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu G - \gamma V$ *is strongly monotone with constant* $\mu\eta - \gamma l$*.*

Finally, we need the following lemma (see [35] for the proof).

Lemma 2.8. *Let* C *be a nonempty closed subspace of a Hilbert space* H*. Let* $G : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping with constants* $\kappa > 0$ *and* $\eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t < \rho \leqslant 1$*. Then* $S := \rho I - t\mu G : C \to C$ *is a contraction with contractive constant* $\rho - t\tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$*.*

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace of H;
- Θ is a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4);
- $B : C \to H$ is a continuous monotone mapping;
- $F : C \to H$ is a continuous monotone mapping;
- $VI(C, F)$ is the set of the variational inequality problem (1.1) for F;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $K_{r_t} : H \to C$ is a mapping defined by

$$K_{r_t}x = \left\{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r_t}\langle y - u, u - x\rangle \geqslant 0,\ \forall y \in C\right\}$$

  for all $x \in H$ and for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $F_{r_t} : H \to C$ is a mapping defined by

$$F_{r_t}x = \left\{z \in C : \langle y - z, Fz\rangle + \frac{1}{r_t}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C\right\}$$

  for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_t} : H \to C$ is a mapping defined by

$$T_{r_t}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \leqslant 0, \quad \forall y \in C\right\}$$

  for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $K_{r_n} : H \to C$ is a mapping defined by

$$K_{r_n}x = \left\{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r_n}\langle y - u, u - x\rangle \geqslant 0,\ \forall y \in C\right\}$$

  for all $x \in H$ and for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;

- $F_{r_n} : H \to C$ is a mapping defined by
$$F_{r_n} x = \left\{ z \in C : \langle y - z, Fz \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geqslant 0, \quad \forall y \in C \right\}$$
for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by
$$T_{r_n} x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n) z - x \rangle \leqslant 0, \quad \forall y \in C \right\}$$
for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $V : C \to C$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $G : C \to C$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leqslant \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $\mathrm{GMEP}(\Theta, \varphi, B)$ is the set of solutions of the GMEP (1.1);
- $\mathrm{VI}(C, F)$ is the set of the variational inequality problem (1.5) for F;
- $\Omega := \mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, F) \cap \mathrm{Fix}(T) \neq \emptyset$.

By Lemmas 2.1, 2.2 and 2.3, K_{r_t}, F_{r_t}, T_{r_t}, K_{r_n}, F_{r_n} and T_{r_n} are nonexpansive and $\mathrm{Fix}(K_{r_t}) = \mathrm{GMEP}(\Theta, \varphi, B) = \mathrm{Fix}(K_{r_n})$, $\mathrm{VI}(C, F) = \mathrm{Fix}(F_{r_t}) = \mathrm{Fix}(F_{r_n})$, and $\mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t}) = \mathrm{Fix}(T_{r_n})$.

In this section, first we introduce the following modified hybrid iterative method that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:
$$\begin{cases} \Theta(u_t, y) + \langle Bu_t, y - u_t \rangle + \varphi(y) - \varphi(u_t) + \frac{1}{r_t} \langle y - u_t, u_t - x_t \rangle \geqslant 0, \ \ \forall y \in C, \\ x_t = t\gamma V x_t + (I - t\mu G)(\theta_t x_t + (1 - \theta_t) T_{r_t} F_{r_t} K_{r_t} x_t), \quad \forall t \in (0, 1), \end{cases} \tag{3.1}$$
where $r_t > 0$ for $t \in (0, 1)$, $\liminf_{t\to0} r_t > 0$, $\theta_t \in (0, 1)$ for $t \in (0, 1)$, and $0 < \liminf_{t\to0} \theta_t \leqslant \limsup_{t\to0} \theta_t < 1$.

Consider the following mapping Q_t on C defined by
$$Q_t x = t\gamma V x + (I - t\mu G)(\theta_t x + (1 - \theta_t) T_{r_t} F_{r_t} K_{r_t} x).$$
Let $R_t x = \theta_t x + (1 - \theta_t) T_{r_t} F_{r_t} K_{r_t} x$. Since T_{r_t}, F_{r_t}, and K_{r_t} are nonexpansive, we have for $x, \ z \in C$,
$$\|R_t x - R_t z\| \leqslant \theta_t \|x - z\| + (1 - \theta_t)\|x - z\| = \|x - z\|.$$
So, from Lemma 2.8, we derive
$$\begin{aligned} \|Q_t x - Q_t z\| &\leqslant t\gamma \|Vx - Vz\| + \|(I - t\mu G)R_t x - (I - t\mu G)R_t z\| \\ &\leqslant t\gamma l \|x - z\| + (1 - t\tau)\|x - z\| \\ &= (1 - t(\tau - \gamma l))\|x - z\|. \end{aligned}$$
Since $0 < 1 - t(\tau - \gamma l) < 1$, Q_t is a contraction. Therefore, by the Banach contraction principle, Q_t has a unique fixed point $x_t \in C$, which uniquely solves the fixed point equation
$$x_t = t\gamma V x_t + (I - t\mu G)(\theta_t x_t + (1 - \theta_t) T_{r_t} F_{r_t} K_{r_t} x_t).$$
Now, we establish the strong convergence of the net $\{x_t\}$ generated by (3.1) and show the existence of the $q \in \Omega$, which solves the variational inequality (3.2) below.

Theorem 3.1. *The nets $\{x_t\}$ and $\{u_t\}$ defined via (3.1) converge strongly, as $t \to 0$, to a point $q \in \Omega$, which is the unique solution of the following variational inequality:*
$$\langle (\mu G - \gamma V)q, p - q \rangle \geqslant 0, \quad \forall p \in \Omega. \tag{3.2}$$

Proof. First, we can show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \leqslant \gamma l < \tau$ and $\mu\eta \geqslant \tau \Leftrightarrow \rho \geqslant \eta$, it follows from Lemma 2.7 that

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geqslant (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu G - \gamma V$ is strongly monotone for $0 \leqslant \gamma l < \tau \leqslant \mu\eta$. So the variational inequality (3.2) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the variational inequality (3.2).

By Lemma 2.1, we know that $u_t = K_{r_t}x_t$. From now, we put $z_t := F_{r_t}u_t$, $w_t := T_{r_t}z_t$ and $y_t := \theta_t x_t + (1-\theta_t)T_{r_t}F_{r_t}K_{r_t}x_t$ $(= \theta_t x_t + (1-\theta_t)T_t z_t)$ for $t \in (0,1)$.

We divide the proof into several steps.

Step 1. We show that $\{x_t\}$ is bounded. To this end, take $p \in \Omega$. Then, from Lemma 2.1 (4), Lemma 2.2 (iii) and Lemma 2.3 (iii), it follows that $T_{r_t}p = p$, $F_{r_t}p = p$ and $p = K_{r_t}p$. Since K_{r_t} is nonexpansive, we have

$$\|u_t - p\|^2 = \|K_{r_t}x_t - K_{r_t}p\|^2 \leqslant \|x_t - p\|^2,$$

that is, $\|u_t - p\| \leqslant \|x_t - p\|$. Also

$$\begin{aligned} \|z_t - p\| &\leqslant \|F_{r_t}u_t - F_{r_t}p\| \\ &\leqslant \|u_t - p\| \leqslant \|x_t - p\|. \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned} \|y_t - p\| &\leqslant \theta_t\|x_t - p\| + (1-\theta_t)\|T_{r_t}F_{r_t}K_{r_t}x_t - p\| \\ &\leqslant \theta_t\|x_t - p\| + (1-\theta_t)\|x_t - p\| \\ &\leqslant \|x_t - p\|. \end{aligned} \tag{3.4}$$

Therefore it follows from (3.4) and Lemma 2.8 that

$$\begin{aligned} \|x_t - p\| &= \|t\gamma Vx_t + (I - t\mu G)y_t - p\| \\ &= \|t(\gamma Vx_t - \gamma Vp) + (I - t\mu G)y_t - (I - t\mu G)p + t(\gamma Vp - \mu Gp)\| \\ &\leqslant t\gamma l\|x_t - p\| + (1 - t\tau)\|y_t - p\| + t\|\gamma Vp - \mu Gp\| \\ &\leqslant t\gamma l\|x_t - p\| + (1 - t\tau)\|x_t - p\| + t(\gamma\|Vp\| + \mu\|Gp\|). \end{aligned}$$

So, we derive

$$\|x_t - p\| \leqslant \frac{\gamma\|Vp\| + \mu\|Gp\|}{\tau - \gamma l}.$$

Thus, $\{x_t\}$ is bounded, and $\{u_t\}$, $\{y_t\}$, $\{Gy_t\}$, $\{z_t\}$, $\{Vx_t\}$ and $\{Fu_t\}$ are also bounded.

Step 2. We show that $\lim_{t\to 0}\|x_t - w_t\| = \lim_{t\to 0}\|x_t - T_{r_t}z_t\| = 0$. In fact, observing

$$\begin{aligned} \|x_t - T_{r_t}z_t\| &= \|t\gamma Vx_t + (I - t\mu G)y_t - T_{r_t}z_t\| \\ &\leqslant t\|\gamma Vx_t - \mu Gy_t\| + \|y_t - T_{r_t}z_t\| \\ &= t\|\gamma Vx_t - \mu Gy_t\| + \|\theta_t x_t + (1-\theta_t)T_{r_t}z_t - T_{r_t}z_t\| \\ &= t\|\gamma Vx_t - \mu Gy_t\| + \theta_t\|x_t - T_{r_t}z_t\|, \end{aligned}$$

we have

$$\|x_t - T_{r_t}z_t\| \leqslant \frac{t}{1-\theta_t}\|\gamma Vx_t - \mu Gy_t\| \to 0 \ \text{ as } t \to 0.$$

Step 3. We show that $\lim_{t\to 0}\|x_t - u_t\| = 0$. To this end, let $p \in \Omega$. Since K_{r_t} is firmly nonexpansive and $u_t = K_{r_t}x_t$, we have

$$\|u_t - p\|^2 = \|K_{r_t}x_t - K_{r_t}p\|^2$$

$$\begin{aligned}
&\leqslant \langle K_{r_t}x_t - K_{r_t}p, x_t - x_t - p\rangle\\
&= \frac{1}{2}(\|u_t - p\|^2 + \|x_t - p\|^2) - \frac{1}{2}(\|x_t - p - (u_t - p)\|^2)\\
&= \frac{1}{2}(\|u_t - p\|^2 + \|x_t - p\|^2 - \|x_t - u_t\|^2),
\end{aligned}$$

and hence

$$\|u_t - p\|^2 \leqslant \|x_t - p\|^2 - \|x_t - u_t\|^2. \tag{3.5}$$

Moreover, from $z_t = F_{r_t}u_t$, we get

$$\|z_t - p\|^2 = \|F_{r_t}u_t - F_{r_t}p\|^2 \leqslant \|u_t - p\|^2. \tag{3.6}$$

By (3.5) and (3.6), we obtain

$$\begin{aligned}
\|x_t - p\|^2 &= \|t(\gamma V x_t - \mu G y_t) + (y_t - p)\|^2\\
&= \|t(\gamma V x_t - \mu G y_t) + \theta_t(x_t - T_{r_t}z_t) + (T_{r_t}z_t - p)\|^2\\
&\leqslant [(\|t(\gamma V x_t - \mu G y_t)\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t}z_t\|]^2\\
&= t^2\|\gamma V x_t - \mu G y_t\|^2 + 2t\|\gamma V x_t - \mu G y_t\|\|z_t - p\| + \|z_t - p\|^2\\
&\quad + \theta_t\|x_t - T_{r_t}z_t\|[2(t\|\gamma V x_t - \mu G y_t\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t}z_t\|]\\
&\leqslant t\|\gamma V x_t - \mu G y_t\|^2 + \|u_t - p\|^2 + M_t\\
&\leqslant t\|\gamma V x_t - \mu G y_t\|^2 + (\|x_t - p\|^2 - \|x_t - u_t\|^2) + M_t,
\end{aligned} \tag{3.7}$$

where

$$M_t = \theta_t\|x_t - T_{r_t}z_t\|[2(t\|\gamma V x_t - \mu G y_t\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t}z_t\|] + 2t\|\gamma V x_t - \mu G y_t\|\|z_t - p\|. \tag{3.8}$$

Now, from (3.7), we derive

$$\|x_t - u_t\|^2 \leqslant t\|\gamma V x_t - \mu G y_t\|^2 + M_t.$$

Since $M_t \to 0$ by Step 2, we have

$$\lim_{t\to 0}\|x_t - u_t\| = 0.$$

Step 4. We show that $\lim_{t\to 0}\|u_t - z_t\| = 0$. To this end, let $p \in \Omega$. Using $z_t = F_{r_t}u_t$ and $p = F_{r_t}p$, we obtain

$$\begin{aligned}
\|z_t - p\|^2 &= \|F_{r_t}u_t - F_{r_t}p\|^2\\
&\leqslant \langle F_{r_t}u_t - F_{r_t}p, u_t - p\rangle\\
&= \langle z_t - p, u_t - p\rangle\\
&\leqslant \frac{1}{2}[\|z_t - p\|^2 + \|u_t - p\|^2 - \|u_t - z_t\|^2],
\end{aligned}$$

that is,

$$\begin{aligned}
\|z_t - p\|^2 &\leqslant \|u_t - p\|^2 - \|u_t - z_t\|^2\\
&\leqslant \|x_t - p\|^2 - \|u_t - z_t\|^2.
\end{aligned} \tag{3.9}$$

Thus, from (3.7) and (3.9), we deduce

$$\begin{aligned}
\|x_t - p\|^2 &\leqslant t\|\gamma V x_t - \mu G y_t\|^2 + \|z_t - p\|^2 + M_t\\
&\leqslant t\|\gamma V x_t - \mu G y_t\|^2 + \|x_t - p\|^2 - \|u_t - z_t\|^2 + M_t,
\end{aligned}$$

which implies that

$$\|u_t - z_t\|^2 \leqslant t\|\gamma V x_t - \mu G y_t\|^2 + M_t,$$

where M_t is of in (3.8). From $\lim_{t\to 0} M_t = 0$, it follows that

$$\lim_{t\to 0}\|u_t - z_t\| = 0.$$

Step 5. We show that $\lim_{t\to 0}\|z_t - w_t\| = \lim_{t\to 0}\|z_t - T_{r_t}z_t\| = 0$. In fact, since $\|w_t - z_t\| = \|T_{r_t}z_t - z_t\| \leqslant \|T_{r_t}z_t - x_t\| + \|x_t - u_t\| + \|u_t - z_t\|$, by Step 2, Step 3 and Step 4, we conclude that

$$\lim_{t\to 0}\|w_t - z_t\| = 0.$$

Step 6. We show that $\lim_{t\to 0}\|x_t - z_t\| = 0$. In fact, from Step 2 and Step 4, it follows that

$$\|x_t - z_t\| \leqslant \|x_t - T_{r_t}z_t\| + \|T_{r_t}z_t - z_t\| \to 0 \ \text{ as } t \to 0.$$

Step 7. We show that $\{x_t\}$ is relatively norm compact as $t \to 0$. To this end, let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $u_n := u_{t_n}$, $z_n := z_{t_n}$, $w_n := w_{t_n}$ and $r_n := r_{t_n}$. First of all, by (3.1), we deduce

$$\begin{aligned}
\|x_t - p\|^2 =& \|t\gamma V x_t + (I - t\mu G)y_t - p\|^2\\
=& \|(I - t\mu G)y_t - (I - t\mu G)p - t(\mu G - \gamma V)p + t\gamma(Vx_t - Vp)\|^2\\
=& \|(I - t\mu G)y_t - (I - t\mu G)p\|^2 - 2t[\langle(\mu G - \gamma V)p, y_t - p\rangle - t\langle(\mu G - \gamma V)p, \mu G y_t - \mu G p\rangle]\\
&+ 2t\gamma[\langle Vx_t - Vp, y_t - p\rangle - t\langle Vx_t - Vp, \mu G y_t - \mu G p\rangle]\\
&- 2t^2\gamma\langle(\mu G - \gamma V)p, Vx_t - Vp\rangle + t^2\|(\mu G - \gamma V)p\|^2 + t^2\gamma^2\|Vx_t - Vp\|^2\\
\leqslant& (1 - t\tau)^2\|y_t - p\|^2 - 2t\langle(\mu G - \gamma V)p, y_t - p\rangle + 2t\gamma l\|x_t - p\|\|y_t - p\|\\
&+ 2t^2\|(\mu G - \gamma V)p\|(\|\mu G y_t\| + \|\mu G p\|)\\
&+ 2t^2\gamma l\|x_t - p\|(\|\mu G y_t\| + \|\mu G p\|) + 2t^2 l\|(\mu G - \gamma V)p\|\|x_t - p\|\\
&+ t^2(\|(\mu G - V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2)\\
=& (1 - 2t\tau + t^2\tau^2)\|y_t - p\|^2 - 2t\langle(\mu G - \gamma V)p, y_t - p\rangle\\
&+ 2t\gamma l\|x_t - p\|\|y_t - p\| + 2t^2\|(\mu G - \gamma V)p\|(\|(\mu G)y_t\| + \|\mu G p\|)\\
&+ 2t^2\gamma l\|x_t - p\|(\|\mu G y_t\| + \|\mu G p\|) + 2t^2 l\|(\mu G - \gamma V)p\|\|x_t - p\|\\
&+ t^2(\|(\mu G - V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2)\\
\leqslant& (1 - 2t\tau)\|y_t - p\|^2 + 2t\langle(\mu G - \gamma V)p, p - y_t\rangle + t\gamma l(\|x_t - p\|^2 + \|y_t - p\|^2) + t^2 M,
\end{aligned}$$

where

$$\begin{aligned}
M = \ & \sup\{\tau^2\|y_t - p\|^2 + 2(\|(\mu G - \gamma V)p\| + \gamma l\mu\|x_t - p\|)(\|Gy_t\| + \|Gp\|)\\
&+ 2l\|(\mu G - \gamma V)p\|\|x_t - p\| + \|(\mu G - \gamma V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2\}.
\end{aligned}$$

Hence, for small enough t, by (3.4), we obtain

$$\begin{aligned}
\|x_t - p\|^2 &\leqslant \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|y_t - p\|^2 + \frac{2t}{1 - t\gamma l}\langle(\mu G - \gamma V)p, p - y_t\rangle + \frac{t^2}{1 - t\gamma l}M\\
&\leqslant \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|x_t - p\|^2 + \frac{2t}{1 - t\gamma l}\langle(\mu G - \gamma V)p, p - y_t\rangle + \frac{t^2}{1 - t\gamma l}M.
\end{aligned} \tag{3.10}$$

Observe that

$$\begin{aligned}
\langle(\mu G - \gamma V)p, p - y_t\rangle &= \langle(\mu G - \gamma V)p, p - (\theta_t x_t + (1 - \theta_t)T_{r_t}z_t)\rangle\\
&= \langle(\mu G - \gamma V)p, p - T_{r_t}z_t\rangle + \theta_t\langle(\mu G - \gamma V)p, T_{r_t}z_t - x_t\rangle\\
&= \langle(\mu G - \gamma V)p, p - z_t\rangle + \langle(\mu G - \gamma V)p, z_t - T_{r_t}z_t\rangle\\
&\quad + \theta_t\langle(\mu G - \gamma V)p, T_{r_t}z_t - x_t\rangle\\
&= \langle(\mu G - \gamma V)p, p - z_t\rangle + \langle(\mu G - \gamma V)p, z_t - T_{r_t}z_t\rangle\\
&\quad + \theta_t\langle(\mu G - \gamma V)p, T_{r_t}z_t - x_t\rangle\\
&\leqslant \langle(\mu G - \gamma V)p, p - z_t\rangle + \|(\mu G - \gamma V)p\|\|z_t - T_{r_t}z_t\|\\
&\quad + \theta_t\|(\mu G - \gamma V)p\|\|T_{r_t}z_t - x_t\|\\
&\leqslant \langle(\mu G - \gamma V)p, p - z_t\rangle + L_t,
\end{aligned} \tag{3.11}$$

where $L_t = \sup\{\|(\mu G-\gamma V)p\|\|z_t - T_{r_t}z_t\| + \theta_t\|(\mu G-\gamma V)p\|\|T_{r_t}z_t - x_t\|\}$. Then, from (3.10) and (3.11), we derive that

$$\|x_t - p\|^2 \leqslant \frac{1}{\tau-\gamma l}\langle(\mu G-\gamma V)p, p-z_t\rangle + \frac{tM}{2(\tau-\gamma l)} + \frac{L_t}{\tau-\gamma l}.$$

In particular,

$$\|x_n - p\|^2 \leqslant \frac{1}{\tau-\gamma l}\langle(\mu G-\gamma V)p, p-z_n\rangle + \frac{t_n M}{2(\tau-\gamma l)} + \frac{L_{t_n}}{\tau-\gamma l}. \tag{3.12}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $q \in C$. Then, by the same argument as in Step 6 in the proof of [15, Theorem 3.1], we can show that $q \in \Omega$. For the sake of completeness, we include its proof.

First, we show that $q \in GMEP(\Theta, \varphi, B)$. Indeed, by $u_n = K_{r_n}x_n$, we know that

$$\Theta(u_n, y) + \langle Bu_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle Bu_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geqslant \Theta(y, u_n), \quad \forall y \in C.$$

Hence

$$\langle Bu_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geqslant \Theta(y, u_n), \quad \forall y \in C. \tag{3.13}$$

For t with $0 < t \leqslant 1$ and $w \in C$, let $v_t = tv + (1-t)q$. Since $v \in C$ and $q \in C$, we have $w_t \in C$. So, from (3.13), we have

$$\begin{aligned}\langle Bv_t, v_t - u_n\rangle &\geqslant \langle Bv_t, v_t-u_n\rangle - \varphi(v_t) + \varphi(u_n) - \langle Bu_n, v_t - u_n\rangle - \langle v_t - u_n, \frac{u_n - x_n}{r_n}\rangle + \Theta(v_t, u_n)\\ &= \langle Bv_t - Bu_n, v_t - u_n\rangle - \varphi(v_t) + \varphi(u_n) - \langle v_t - u_n, \frac{u_n-x_n}{r_n}\rangle + \Theta(v_t, u_n).\end{aligned}$$

Since $\|u_n - x_n\| \to 0$ by Step 3, $\frac{u_n-x_n}{r_n} \to 0$ and $u_n \rightharpoonup q$. Moreover, from the monotonicity of B, we have $\langle Bv_t - Bu_n, y_t - u_n\rangle \geqslant 0$. So, from (A4) and the weak lower semicontinuity of φ, if follows that

$$\langle Bv_t, v_t - q\rangle \geqslant -\varphi(v_t) + \varphi(q) + \Theta(v_t, q) \ \text{ as } i \to \infty. \tag{3.14}$$

By (A1), (A4) and (3.14), we also obtain

$$\begin{aligned}0 &= \Theta(v_t, v_t) + \varphi(v_t) - \varphi(v_t)\\ &\leqslant t\Theta(v_t, v) + (1-t)\Theta(v_t, q) + t\varphi(v) + (1-t)\varphi(q) - \varphi(v_t)\\ &\leqslant t[\Theta(v_t, v) + \varphi(v) - \varphi(v_t)] + (1-t)\langle Bv_t, v_t - q\rangle\\ &= t[\Theta(v_t, v) + \varphi(v) - \varphi(v_t)] + (1-t)t\langle Bv_t, v - q\rangle,\end{aligned}$$

and hence

$$0 \leqslant \Theta(v_t, v) + \varphi(v) - \varphi(v_t) + (1-t)\langle Bv_t, v-q\rangle. \tag{3.15}$$

Letting $t \to 0$ in (3.15), we have for each $v \in C$

$$\Theta(q, v) + \langle Bq, v-q\rangle + \varphi(v) - \varphi(q) \geqslant 0.$$

This implies that $q \in GMEP(\Theta, \varphi, B)$.

Second, we show that $q \in VI(C, F)$. In fact, from the definition of $z_n = F_{r_n}u_n$, we have

$$\langle y - z_n, Fz_n\rangle + \langle y - z_n, \frac{z_n - u_n}{r_n}\rangle \geqslant 0, \quad \forall y \in C. \tag{3.16}$$

Set $v_t = tv + (1-t)q$, for all $t \in (0,1]$ and $v \in C$. Then, $v_t \in C$. From (3.16), it follows that

$$\begin{aligned} \langle v_t - z_n, Fv_t\rangle &\geqslant \langle v_t - z_n, Fv_t\rangle - \langle v_t - z_n, Fz_n\rangle - \langle v_t - z_n, \frac{z_n - u_n}{r_n}\rangle \\ &= \langle v_t - z_n, Fv_t - Fz_n\rangle - \langle v_t - z_n, \frac{z_n - u_n}{r_n}\rangle. \end{aligned} \tag{3.17}$$

By Step 4, we have $\frac{z_n - u_n}{r_n} \to 0$ as $n \to \infty$. Moreover, since $x_n \rightharpoonup q$, by Step 6, we have $z_n \rightharpoonup q$ as $n \to \infty$. Since F is monotone, we also have that $\langle v_t - z_n, Fv_t - Fz_n\rangle \geqslant 0$. Thus, from (3.17), it follows that

$$0 \leqslant \lim_{n\to\infty} \langle v_t - z_n, Fv_t\rangle = \langle v_t - q, Fv_t\rangle,$$

and hence

$$\langle v - q, Fv_t\rangle \geqslant 0, \quad \forall v \in C.$$

If $t \to 0$, the continuity of F yields that

$$\langle v - q, Fq\rangle \geqslant 0, \quad \forall v \in C.$$

This implies that $q \in VI(C, F)$.

Third, we show that $q \in Fix(T)$. In fact, from the definition of $w_n = T_{r_n} z_n$, we have

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1+r_n)w_n - z_n\rangle \leqslant 0, \quad \forall y \in C. \tag{3.18}$$

Put $v_t = tv + (1-t)q$ for all $t \in (0,1]$ and $v \in C$. Then $v_t \in C$ and from (3.18) and pseudocontractivity of T, it follows that

$$\begin{aligned} \langle w_n - v_t, Tv_t\rangle &\geqslant \langle w_n - v_t, Tv_t\rangle + \langle v_t - w_n, Tw_n\rangle - \frac{1}{r_n}\langle v_t - w_n, (1+r_n)w_n - z_n\rangle \\ &= -\langle v_t - w_n, Tv_t - Tw_n\rangle - \frac{1}{r_n}\langle v_t - u_n, w_n - z_n\rangle - \langle v_t - w_n, w_n\rangle \\ &\geqslant -\|v_t - w_n\|^2 - \frac{1}{r_n}\langle v_t - w_n, w_n - z_n\rangle - \langle v_t - w_n, w_n\rangle \\ &= -\langle v_t - w_n, v_t\rangle - \langle v_t - w_n, \frac{w_n - z_n}{r_n}\rangle. \end{aligned} \tag{3.19}$$

By Step 5, we get $\frac{w_n - z_n}{r_n} \to 0$ as $n \to \infty$. Moreover, since $x_n \rightharpoonup q$, by Step 2, we have $w_n \rightharpoonup q$ as $n \to \infty$. Therefore, from (3.19), as $n \to \infty$, it follows that

$$\langle q - v_t, Tv_t\rangle \geqslant \langle q - v_t, v_t\rangle,$$

and hence

$$-\langle v - q, Tv_t\rangle \geqslant -\langle v - q, v_t\rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is continuous, we get

$$-\langle v - q, Tq\rangle \geqslant -\langle v - q, q\rangle, \quad \forall v \in C.$$

Now, let $v = Tq$. Then we obtain $q = Tq$ and hence $q \in Fix(T)$. Therefore, $q \in \Omega$.

Now, we substitute q for p in (3.12) to obtain

$$\|x_n - q\|^2 \leqslant \frac{1}{\tau - \gamma l}\langle(\mu G - \gamma V)q, q - z_n\rangle + \frac{t_n M}{2(\tau - \gamma l)} + \frac{L_{t_n}}{\tau - \gamma l}. \tag{3.20}$$

Note that $z_n \rightharpoonup q$ by Step 5 and $\lim_{t\to 0} L_t = 0$ by Step 2 and Step 4. This fact and the inequality (3.20) imply that $x_n \to q$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Step 8. We show that q solves the variational inequality (3.2). In fact, taking the limit in (3.12) as $n \to \infty$, we get

$$\|q-p\|^2 \leqslant \frac{1}{\tau-\gamma l}\langle(\mu G-\gamma V)p, p-q\rangle, \quad \forall p \in \Omega.$$

In particular, q solves the following variational inequality

$$q \in \Omega, \quad \langle(\mu G-\gamma V)p, p-q\rangle \geqslant 0, \quad p \in \Omega,$$

or the equivalent dual variational inequality (see [20])

$$q \in \Omega, \quad \langle(\mu G-\gamma V)q, p-q\rangle \geqslant 0, \quad p \in \Omega. \tag{3.21}$$

Step 9. We show that the entire net $\{x_t\}$ converges strongly to q. To this end, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ and assume $x_{n_k} \to \hat{q}$. By the same as the proof above, we have $\hat{q} \in \Omega$. Moreover, it follows from (3.21) that

$$\langle(\mu G-\gamma V)q, \hat{q}-q\rangle \geqslant 0. \tag{3.22}$$

Interchanging q and $\hat{q}$, we obtain

$$\langle(\mu G-\gamma V)\hat{q}, q-\hat{q}\rangle \geqslant 0. \tag{3.23}$$

Lemma 2.7 and adding these two inequalities (3.22) and (3.23) yields

$$(\mu\eta-\gamma l)\|q-\hat{q}\|^2 \leqslant \langle(\mu G-\gamma V)q-(\mu G-\gamma V)\hat{q}, q-\hat{q}\rangle \leqslant 0.$$

Hence $q=\hat{q}$. Therefore we conclude that $x_t \to q$ as $t \to 0$. Moreover, by Step 3, we obtain that $u_t \to q$ as $t \to 0$. □

From Theorem 3.1, we can deduce the following result.

Corollary 3.2. *Let $\{x_t\}$ and $\{u_t\}$ be nets generated by*

$$\begin{cases} \Theta(u_t, y)+\langle Bu_t, y-u_t\rangle+\varphi(y)-\varphi(u_t)+\frac{1}{r_t}\langle y-u_t, u_t-x_t\rangle \geqslant 0, & \forall y \in C, \\ x_t=(1-t)(\theta_t x_t+(1-\theta_t)T_{r_t}F_{r_t}K_{r_t}x_t), & \forall t \in (0,1). \end{cases}$$

Then $\{x_t\}$ and $\{u_t\}$ converge strongly, as $t \to 0$, to a point $q \in \Omega$, which solves the following minimum norm problem: find $x^* \in \Omega$ such that

$$\|x^*\| = \min_{x \in \Omega}\|x\|. \tag{3.24}$$

Proof. In (3.12) with $G=I$, $\mu=1$, $\tau=1$, $V=0$, and $l=0$, letting $t \to 0$ yields

$$\|q-p\|^2 \leqslant \langle p, p-q\rangle, \quad \forall p \in \Omega.$$

Equivalently,

$$\langle q, p-q\rangle \geqslant 0, \quad \forall p \in \Omega.$$

This obviously implies that

$$\|q\|^2 \leqslant \langle p, q\rangle \leqslant \|p\|\|q\|, \quad \forall p \in \Omega.$$

It turns out that $\|q\| \leqslant \|p\|$ for all $p \in \Omega$. Therefore, q is the minimum-norm point of Ω. □

Now, we propose the following modified hybrid iterative method which generates a sequence in an explicit way:

$$\begin{cases} \Theta(u_n, y)+\langle Bu_n, y-u_n\rangle+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geqslant 0, & \forall y \in C, \\ y_n = \beta_n x_n+(1-\beta_n)T_{r_n}F_{r_n}K_{r_n}x_n, & \\ x_{n+1} = \alpha_n\gamma Vx_n+(I-\alpha_n\mu G)y_n, & \forall n \geqslant 1, \end{cases} \tag{3.25}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$; $\{r_n\} \subset (0,\infty)$; and $x_1 \in C$ is an arbitrary initial guess.

Theorem 3.3. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the explicit method (3.25). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(C4) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \Omega$, which is the unique solution of the variational inequality (3.2).

Proof. Note that from the condition (C1), without loss of generality, we assume that $\alpha_n(\tau - \gamma l) < 1$ for $n \geqslant 1$. From now, we put $u_n = K_{r_n}x_n$, $z_n = F_{r_n}u_n$ and $w_n = T_{r_n}z_n$, for $n \geqslant 1$.

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega$. Then, by Lemma 2.1 (iv), Lemma 2.2 (iii) and Lemma 2.3 (iii), $p = K_{r_n}p$, $p = F_{r_n}p$ and $p = T_{r_n}p$. From $z_n = F_{r_n}u_n$ and the fact that F_{r_n} is nonexpansive, it follows that

$$\|z_n - p\| \leqslant \|F_{r_n}u_n - F_{r_n}p\| \leqslant \|u_n - p\|.$$

Also, by $u_n = K_{r_n}x_n$,

$$\|u_n - p\| = \|K_{r_n}x_n - K_{r_n}p\| \leqslant \|x_n - p\|,$$

and so

$$\|z_n - p\| \leqslant \|x_n - p\|. \tag{3.26}$$

Now, by (3.26), we obtain that

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (I - \beta_n)T_{r_n}z_n - p\| \\ &\leqslant \beta_n\|x_n - p\| + \|(I - \beta_n)T_{r_n}z_n - (I - \beta_n)T_{r_n}p\| \\ &\leqslant \beta_n\|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leqslant \beta_n\|x_n - p\| + (1 - \beta_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.27}$$

Thus, noting Lemma 2.8 and (3.27), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \alpha_n\|(\gamma V(x_n) - \gamma V(p)\| + \|(I - \alpha_n\mu G)y_n - (I - \alpha_n\mu G)p\| + \alpha_n\|\gamma V(p) - \mu Gp\| \\ &\leqslant \alpha_n\gamma l\|x_n - p\| + (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n\|\gamma V(p) - \mu Gp\| \\ &\leqslant \alpha_n\gamma l\|x_n - p\| + (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\|\gamma V(p) - \mu Gp\| \\ &= (1 - (\tau - \gamma l)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma V(p) - \mu Gp\|. \end{aligned} \tag{3.28}$$

By induction, it follows from (3.28) that

$$\|x_n - p\| \leqslant \max\left\{\|x_1 - p\|, \frac{\|\gamma V(p) - \mu Gp\|}{\tau - \gamma l}\right\}, \quad \forall n \geqslant 1.$$

Therefore $\{x_n\}$ is bounded, and so $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{V(x_n)\}$, $\{Fu_n\}$, $\{Gy_n\}$, and $\{GT_{r_n}z_n\}$ are bounded. Moreover, since $\|T_n z_n - p\| \leqslant \|x_n - p\|$, $\{T_{r_n}z_n\}$ is also bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Indeed, since $z_n = F_{r_n}u_n$, and $z_{n-1} = F_{r_{n-1}}u_{n-1}$, we get

$$\langle y - z_n, Fz_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - u_n\rangle \geqslant 0, \quad \forall y \in C, \tag{3.29}$$

and

$$\langle y - z_{n-1}, Fz_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - z_{n-1}, z_{n-1} - u_{n-1}\rangle \geqslant 0, \quad \forall y \in C. \tag{3.30}$$

Putting $y = z_{n-1}$ in (3.29) and $y = z_n$ in (3.30), we obtain

$$\langle z_{n-1} - z_n, Fz_n\rangle + \frac{1}{r_n}\langle z_{n-1} - z_n, z_n - u_n\rangle \geqslant 0, \tag{3.31}$$

and

$$\langle z_n - z_{n-1}, Fz_{n-1}\rangle + \frac{1}{r_{n-1}}\langle z_n - z_{n-1}, z_{n-1} - u_{n-1}\rangle \geqslant 0. \tag{3.32}$$

Adding up (3.31) and (3.32), we derive

$$-\langle z_n - z_{n-1}, Fz_n - Fz_{n-1}\rangle + \langle z_{n-1} - z_n, \frac{z_n - u_n}{r_n} - \frac{z_{n-1} - u_{n-1}}{r_{n-1}}\rangle \geqslant 0.$$

Since F is monotone, we have

$$\langle z_{n-1} - z_n, \frac{z_n - u_n}{r_n} - \frac{z_{n-1} - u_{n-1}}{r_{n-1}}\rangle \geqslant 0,$$

and hence

$$\langle z_n - z_{n-1}, z_{n-1} - z_n + z_n - u_{n-1} - \frac{r_{n-1}}{r_n}(z_n - u_n)\rangle \geqslant 0. \tag{3.33}$$

Without loss of generality, let us assume that there exists a real number $r_n > b > 0$ for $n \geqslant 1$. Then, from (3.33), we get

$$\begin{aligned}
\|z_n - z_{n-1}\|^2 &\leqslant \langle z_n - z_{n-1}, z_n - u_n + u_n - u_{n-1} - \frac{r_{n-1}}{r_n}(z_n - u_n)\rangle \\
&= \langle z_n - z_{n-1}, u_n - u_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(z_n - u_n)\rangle \\
&\leqslant \|z_n - z_{n-1}\| \left[\|u_n - u_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|z_n - u_n\|\right].
\end{aligned}$$

This implies that

$$\|z_n - z_{n-1}\| \leqslant \|u_n - u_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_1, \tag{3.34}$$

where $M_1 = \sup\{\|z_n - u_n\| : n \geqslant 1\}$.

On the other hand, from $u_{n-1} = K_{r_{n-1}}x_{n-1}$ and $u_n = K_{r_n}x_n$, it follows that

$$\Theta(u_{n-1}, y) + \langle Bu_{n-1}, y - u_{n-1}\rangle + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \tag{3.35}$$

and

$$\Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C. \tag{3.36}$$

Substituting $y = u_n$ into (3.35) and $y = u_{n-1}$ into (3.36), we obtain

$$\Theta(u_{n-1}, u_n) + \langle Bu_{n-1}, u_n - u_{n-1}\rangle + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geqslant 0,$$

and

$$\Theta(u_n, u_{n-1}) + \langle Bu_n, u_{n-1} - u_n\rangle + \varphi(u_{n-1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geqslant 0.$$

By (A2), we have

$$\langle u_n - u_{n-1}, Bu_{n-1} - Bu_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\rangle \geqslant 0,$$

and then

$$\langle u_n - u_{n-1}, r_{n-1}(Bu_{n-1} - Bu_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geqslant 0.$$

So, it follows that

$$\begin{aligned}&\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + r_{n-1}\langle u_n - u_{n-1}, Bu_{n-1} - Bu_n\rangle \\ &+ \langle u_n - u_{n-1}, x_n - x_{n-1}\rangle + \left(1 - \frac{r_{n-1}}{r_n}\right)\langle u_n - u_{n-1}, u_n - x_n\rangle \geqslant 0.\end{aligned} \tag{3.37}$$

Then, from (3.37), $r_n > b > 0$ for $n \geqslant 1$, and the fact that $\langle u_n - u_{n-1}, Bu_{n-1} - Bu_n\rangle \leqslant 0$, we have

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &\leqslant \langle u_n - u_{n-1}, x_n - x_{n-1}\rangle + \left(1 - \frac{r_{n-1}}{r_n}\right)\langle u_n - u_{n-1}, u_n - x_n\rangle \\ &\leqslant \|u_n - u_{n-1}\|\left[\|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - x_n\|\right],\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - u_{n-1}\| &\leqslant \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\leqslant \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_2,\end{aligned} \tag{3.38}$$

where $M_2 = \sup\{\|u_n - x_n\| : n \geqslant 1\}$. Substituting (3.38) into (3.34), we have

$$\|z_n - z_{n-1}\| \leqslant \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2). \tag{3.39}$$

On another hand, let $w_n = T_{r_n}z_n$ and $w_{n-1} = T_{r_{n-1}}z_{n-1}$. Then we get

$$\langle y - w_{n-1}, Tw_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - w_{n-1}, (1 + r_{n-1})w_{n-1} - z_{n-1}\rangle \leqslant 0, \quad \forall y \in C, \tag{3.40}$$

and

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1 + r_n)w_n - z_n\rangle \leqslant 0, \quad \forall y \in C. \tag{3.41}$$

Putting $y = w_n$ in (3.40) and $y = w_{n-1}$ in (3.41), we obtain

$$\langle w_n - w_{n-1}, Tw_{n-1}\rangle - \frac{1}{r_{n-1}}\langle w_n - w_{n-1}, (1 + r_{n-1})w_{n-1} - z_{n-1}\rangle \leqslant 0, \tag{3.42}$$

and

$$\langle w_{n-1} - w_n, Tw_n\rangle - \frac{1}{r_n}\langle w_{n-1} - w_n, (1 + r_n)w_n - z_n\rangle \leqslant 0. \tag{3.43}$$

Adding up (3.42) and (3.43), we have

$$\langle w_n - w_{n-1}, Tw_{n-1} - Tw_n\rangle - \langle w_n - w_{n-1}, \frac{(1 + r_{n-1})w_{n-1} - z_{n-1}}{r_{n-1}} - \frac{(1 + r_n)w_n - z_n}{r_n}\rangle \leqslant 0,$$

which implies that

$$\langle w_n - w_{n-1}, (w_n - Tw_n) - (w_{n-1} - Tw_{n-1})\rangle - \langle w_n - w_{n-1}, \frac{w_{n-1} - z_{n-1}}{r_{n-1}} - \frac{w_n - z_n}{r_n}\rangle \leqslant 0.$$

Now, using the fact that T is pseudocontractive, we induce

$$\langle w_n - w_{n-1}, \frac{w_{n-1} - z_{n-1}}{r_{n-1}} - \frac{w_n - z_n}{r_n}\rangle \geqslant 0,$$

and hence

$$\langle w_n - w_{n-1}, w_{n-1} - w_n + w_n - z_{n-1} - \frac{r_{n-1}}{r_n}(w_n - z_n)\rangle \geqslant 0. \tag{3.44}$$

Since $r_n > b > 0$ for $n \geqslant 1$, by (3.44), we have

$$\begin{aligned}\|w_n - w_{n-1}\|^2 &\leqslant \langle w_n - w_{n-1}, z_n - z_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(w_n - z_n)\rangle \\ &\leqslant \|w_n - w_{n-1}\|\left[\|z_n - z_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|w_n - z_n\|\right],\end{aligned}$$

which implies

$$\|w_n - w_{n-1}\| \leqslant \|z_n - z_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_3, \tag{3.45}$$

where $M_3 = \sup\{\|w_n - z_n\| : n \geqslant 1\}$. From (3.39) and (3.45), it follows that

$$\|T_{r_n} z_n - T_{r_{n-1}} z_{n-1}\| \leqslant \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2 + M_3). \tag{3.46}$$

Now, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n)k_n, \quad \forall n \geqslant 1.$$

Then, from the definition of k_n, we obtain

$$\begin{aligned}k_{n+1} - k_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma V(x_{n+1}) + (I - \alpha_{n+1}\mu G)y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\gamma V(x_n) + (I - \alpha_n G)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma V(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n\gamma V(x_n)}{1-\beta_n} - \frac{(I - \alpha_n\mu G)(\beta_n x_n + (1-\beta_n)T_{r_n}z_n) - \beta_n x_n}{1-\beta_n x_n} \\ &\quad + \frac{(I - \alpha_{n+1}\mu G)(\beta_{n+1}x_{n+1} + (1-\beta_{n+1})T_{r_{n+1}}z_{n+1}) - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}x_{n+1}} \\ &= \frac{\alpha_{n+1}\gamma V(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n\gamma V(x_n)}{1-\beta_n} - \frac{\beta_n x_n + (1-\beta_n)T_{r_n}z_n - \beta_n x_n}{1-\beta_n} + \frac{\alpha_n\mu G y_n}{1-\beta_n} \\ &\quad + \frac{\beta_{n+1}x_{n+1} + (1-\beta_{n+1})T_{r_{n+1}}x_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}\mu G)y_{n+1}}{1-\beta_{n+1}} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma V(x_{n+1}) - \mu G y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(\gamma V(x_n) - \mu G y_n) + T_{r_{n+1}}z_{n+1} - T_{r_n}z_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma V(x_{n+1}) - \mu G y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(\gamma V(x_n) - \mu G y_n) \\ &\quad + T_{r_{n+1}}z_{n+1} - T_{r_{n+1}}z_n + (T_{r_{n+1}}z_n - T_{r_n}z_n).\end{aligned}$$

So, it follows from (3.46) that

$$\begin{aligned}&\|k_{n+1} - k_n\| - \|x_{n+1} - x_n\| \\ &\quad\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|V(x_{n+1})\| + \mu\|Gy_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\gamma\|V(x_n)\| + \mu\|Gy_n\|) \\ &\qquad + \|T_{n+1}z_{n+1} - T_{n+1}z_n\| + \|T_{r_{n+1}}z_n - T_{r_n}z_n\| \\ &\quad\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|V(x_{n+1})\| + \mu\|Gy_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\gamma\|V(x_n)\| + \mu\|Gy_n\|) \\ &\qquad + \|z_{n+1} - z_n\| + \frac{1}{b}|r_n - r_{n-1}|M_3 \\ &\quad\leqslant \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_4 + \|z_{n+1} - z_n\| + \frac{1}{b}|r_n - r_{n-1}|M_3 \\ &\quad\leqslant \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_4 + \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2 + M_3),\end{aligned} \tag{3.47}$$

where $M_4 = \sup\{\gamma\|V(x_n)\| + \mu\|Gy_n\| : n \geqslant 1\}$. Thus, by conditions (C1), (C3) and (C4), from (3.47) we have

$$\limsup_{n\to\infty}(\|k_{n+1} - k_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

Hence, by Lemma 2.5,

$$\lim_{n\to\infty} \|k_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|k_n - x_n\| = 0.$$

Also from (3.38) and (3.39), it follows that

$$\lim_{ntoo} \|u_{n+1} - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|z_{n+1} - z_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|x_n - T_{r_n}z_n\| = 0$. Noting that $x_{n+1} = \alpha_n\gamma V(x_n) + (I - \alpha_n\mu G)y_n$, we have

$$\begin{aligned}
\|x_n - T_{r_n}z_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n}z_n\| \\
&\leqslant \|x_n - x_{n+1}\| + \alpha_n\|\gamma V(x_n) - \mu Gy_n\| + \|y_n - T_{r_n}z_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma V(x_n) - \mu Gy_n\| + \|\beta_n x_n + (1 - \beta_n)T_{r_n}z_n - T_{r_n}z_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma V(x_n) - \mu Gy_n\| + \beta_n\|x_n - T_{r_n}z_n\|,
\end{aligned}$$

that is,

$$\|x_n - T_{r_n}z_n\| \leqslant \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma V(x_n) - \mu Gy_n\|.$$

From the conditions (C1), (C3) and Step 2, it follows that

$$\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|x_n - T_{r_n}z_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. To this end, let $p \in \Omega$. Since K_{r_n} is firmly nonexpansive and $u_n = K_{r_n}x_n$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|K_{r_n}x_n - K_{r_n}p\|^2 \\
&\leqslant \langle K_{r_n}x_n - K_{r_n}p, x_n - x_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2) - \frac{1}{2}(\|x_n - p - (u_n - p)\|^2) \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \leqslant \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.48}$$

From $z_n = F_{r_n}u_n$, we also get

$$\|z_n - p\|^2 = \|F_{r_n}u_n - F_{r_n}p\|^2 \leqslant \|u_n - p\|^2. \tag{3.49}$$

By (3.48) and (3.49), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma V(x_n) - \mu Gy_n) + (y_n - p)\|^2 \\
&= \|\alpha_n(\gamma V(x_n) - \mu Gy_n) + \beta_n(x_n - T_{r_n}z_n) + (T_{r_n}z_n - p)\|^2 \\
&\leqslant [(\|\alpha_n(\gamma V(x_n) - \mu Gy_n)\| + \|z_n - p\|) + \beta_n\|x_n - T_{r_n}z_n\|]^2 \\
&= \alpha_n^2\|\gamma V(x_n) - \mu Gy_n\|^2 + 2\alpha_n\|\gamma V(x_n) - \mu Gy_n\|\|z_n - p\| + \|z_n - p\|^2 \\
&\quad + \beta_n\|x_n - T_nz_n\|[2(\alpha_n\|\gamma V(x_n) - \mu Gy_n\| + \|z_n - p\|) + \beta_n\|x_n - T_{r_n}z_n\|] \\
&\leqslant \alpha_n\|\gamma V(x_n) - \mu Gy_n\|^2 + \|u_n - p\|^2 + M_n \\
&\leqslant \alpha_n\|\gamma V(x_n) - \mu Gy_n\|^2 + (\|x_n - p\|^2 - \|x_n - u_n\|^2) + M_n,
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned} M_n = {} & \beta_n\|x_n - w_n\|[2(\alpha_n\|\gamma V(x_n) - \mu G y_n\| + \|z_n - p\|) + \beta_n\|x_n - w_n\|] \\ & + 2\alpha_n\|\gamma V(x_n) - \mu G y_n\|\|z_n - p\|. \end{aligned} \tag{3.51}$$

Now, from (3.50), we derive

$$\begin{aligned} \|x_n - u_n\|^2 & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + M_n + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + M_n + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned}$$

Since $\alpha_n \to 0$ by the condition (C1), $\|x_{n+1} - x_n\| \to 0$ by Step 2 and $M_n \to 0$ by Step 3 and the condition (C1), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Step 5. We show that $\lim_{n\to\infty} \|u_n - z_n\| = 0$. To this end, let $p \in \Omega$. Using $z_n = F_{r_n} u_n$ and $p = F_{r_n} p$, and firmly nonexpansivity of F_{r_n}, we observe that

$$\begin{aligned} \|z_n - p\|^2 & = \|F_{r_n} u_n - F_{r_n} p\|^2 \\ & \leqslant \langle F_{r_n} u_n - F_{r_n} p, u_n - p\rangle \\ & = \langle z_n - p, u_n - p\rangle \\ & \leqslant \frac{1}{2}(\|z_n - p\|^2 + \|u_n - p\|^2 - \|u_n - z_n\|^2), \end{aligned}$$

that is,

$$\|z_n - p\|^2 \leqslant \|u_n - p\|^2 - \|u_n - z_n\|^2 \leqslant \|x_n - p\|^2 - \|u_n - z_n\|^2. \tag{3.52}$$

Now, from (3.50) and (3.52), we compute

$$\begin{aligned} \|x_{n+1} - p\|^2 & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + \|z_n - p\|^2 + M_n \\ & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + \|x_n - p\|^2 - \|u_n - z_n\|^2 + M_n, \end{aligned}$$

where M_n is of in (3.51). So, we get

$$\begin{aligned} \|u_n - z_n\|^2 & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + M_n + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ & \leqslant \alpha_n\|\gamma V(x_n) - \mu G y_n\|^2 + M_n + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned}$$

From the condition (C1), Step 2 and $\lim_{n\to\infty} M_n = 0$, it follows that

$$\lim_{n\to\infty} \|u_n - z_n\| = 0.$$

Step 6. We show that $\lim_{n\to\infty} \|z_n - w_n\| = \lim_{n\to\infty} \|z_n - T_{r_n} z_n\| = 0$. Since $\|w_n - z_n\| \leqslant \|w_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\|$, by Step 3, Step 4 and Step 5, we conclude that

$$\lim_{n\to\infty} \|w_n - z_n\| = \lim_{n\to\infty} \|T_n z_n - z_n\| = 0.$$

Step 7. We show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Indeed, from Step 3 and Step 6, we have

$$\|x_n - z_n\| \leqslant \|x_n - w_n\| + \|w_n - z_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 8. We show that $\limsup_{n\to\infty}\langle(\gamma V - \mu G)q, x_n - q\rangle \leqslant 0$, where q is a solution of the variational inequality (3.2). To this end, first we prove that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu G)q, w_n - q\rangle = \limsup_{n\to\infty}\langle(\gamma V - \mu G)q, T_{r_n} z_n - q\rangle \leqslant 0.$$

Since $\{z_n\}$ is bounded, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle = \lim_{i\to\infty}\langle(\gamma V-\mu G)q, w_{n_i}-q\rangle. \tag{3.53}$$

Without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z \in C$. From $\|w_n - z_n\| \to 0$ by Step 6, it follows that $w_{n_i} \rightharpoonup z$. Moreover, from Step 3 and Step 4, it follows that $x_{n_i} \rightharpoonup z$ and $u_{n_i} \rightharpoonup z$. Thus, by the same argument as in Step 7 of the proof of Theorem 3.1 together with Step 4, Step 5 and Step 6, we obtain $z \in \Omega$. So, from (3.53), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle &= \lim_{i\to\infty}\langle(\gamma V-\mu G)q, w_{n_i}-q\rangle\\ &= \langle(\gamma V-\mu G)q, z-q\rangle \leqslant 0.\end{aligned} \tag{3.54}$$

Since $\lim_{n\to\infty}\|x_n - w_n\| = 0$ by Step 3, from (3.54), we conclude that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, x_n-q\rangle\\ &\qquad\leqslant \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, x_n-w_n\rangle + \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle\\ &\qquad\leqslant \limsup_{n\to\infty}\|(\gamma V-\mu G)q\|\|x_n-w_n\| + \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle \leqslant 0.\end{aligned}$$

Step 9. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$ and $\lim_{n\to\infty}\|u_n - q\| = 0$, where q is a solution of the variational inequality (3.2). Indeed, from (3.3), Lemma 2.5 and Lemma 2.8, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n\gamma V(x_n) + (I-\alpha_n\mu G)y_n - p\|^2\\ &= \|\alpha_n(\gamma V(x_n)-\gamma V(q)) + (I-\alpha_n\mu G)y_n - (I-\alpha_n\mu G)q + \alpha_n(\gamma V(q)-\mu Gq)\|^2\\ &\leqslant [\alpha_n\gamma l\|x_n-q\| + (1-\alpha_n\tau)\|y_n-q\|]^2 + 2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ &\leqslant [\alpha_n\gamma l\|x_n-q\| + (1-\alpha_n\tau)(\beta_n\|x_n-q\| + (1-\beta_n)\|T_{r_n}z_n-q\|)]^2\\ &\quad + 2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ &\leqslant [\alpha_n\gamma l\|x_n-q\| + (1-\alpha_n\tau)(\beta_n\|x_n-q\| + (1-\beta_n)\|z_n-q\|)]^2\\ &\quad + 2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ &\leqslant [\alpha_n\gamma l\|x_n-q\| + (1-\alpha_n\tau)\|x_n-q\|]^2 + 2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ &\leqslant (1-(\tau-\gamma l)\alpha_n)\|x_n-q\|^2 + 2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ &= (1-\overline{\alpha_n})\|x_n-q\|^2 + \overline{\beta_n},\end{aligned} \tag{3.55}$$

where $\overline{\alpha_n} = (\tau-\gamma l)\alpha_n$ and $\overline{\beta_n} = 2(\tau-\gamma l)\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle$. From the conditions (C1) and (C2), and Step 8, it is easily seen that $\overline{\alpha}_n \to 0$, $\sum_{n=1}^{\infty}\overline{\alpha}_n = \infty$, and $\limsup_{n\to\infty}\frac{\overline{\beta}_n}{\overline{\alpha}_n} \leqslant 0$. Hence, by applying Lemma 2.4 to (3.55), we conclude $x_n \to q$ as $n \to \infty$. Moreover, by Step 4, we obtain that $u_n \to q$ as $n \to \infty$. This completes the proof. □

From Theorem 3.3, we deduce immediately the following result.

Corollary 3.4. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases}\Theta(u_n, y) + \langle Bu_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geqslant 0, \quad \forall y \in C,\\ y_n = \beta_n x_n + (1-\beta_n)T_{r_n}F_{r_n}K_{r_n}x_n,\\ x_{n+1} = (1-\alpha_n)y_n, \quad \forall n \geqslant 1.\end{cases}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ be sequences satisfying conditions (C1), (C2), (C3), *and* (C4) *in Theorem* 3.3. *Then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $q \in \Omega$, which solves the minimum norm problem* (3.24).

Proof. Take $G = I$, $\mu = 1$ $\tau = 1$, $V = 0$, and $l = 0$ in Theorem 3.3. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, p - q \rangle \geqslant 0, \quad \forall p \in \Omega.$$

This is equivalent to $\|q\|^2 \leqslant \langle p, q \rangle \leqslant \|p\|\|q\|$ for all $p \in \Omega$. It turns out that $\|q\| \leqslant \|p\|$ for all $p \in \Omega$ and q is the minimum-norm point of Ω. □

Remark 3.5.

(1) For finding a common element of $\mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$, where B is a continuous monotone mapping, F is a continuous monotone mapping, and T is a continuous pseudocontractive mapping, Theorem 3.1 and Theorem 3.3 are new ones different from previous those introduced by several authors. Consequently, in the sense that our convergence is for the more general class of continuous monotone mappings and the more general class of continuous pseudocontractive mappings, our results improve, develop and complement the corresponding results, which were obtained recently by several authors in references; for example, see [5, 6, 8, 12, 13, 15, 16, 19, 24, 25, 27–29, 32, 36–39] and the references therein.

(2) We point out that Corollary 3.2 and Corollary 3.4 for finding the minimum-norm point of $\mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$ are also new ones different from previous those introduced by several authors.

(3) We recall some special cases of the generalized mixed equilibrium problem (1.1) as follows:

   (i) If $\Theta(x, y) = 0$ for all $x, y \in C$, the GMEP (1.1) reduces the following generalized variational inequality problem (for short, GVI) of finding $x \in C$ such that

$$\langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{3.56}$$

   (ii) If $B = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, the GMEP (1.1) reduces the following minimization problem (for short, MP) finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{3.57}$$

Applying Theorem 3.1, Theorem 3.3, Corollary 3.2, and Corollary 3.4, we can also establish the new corresponding results for the GEP (1.2), the MEP (1.3), the EP (1.4), the GVI (3.56) and the MP (3.57).

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