



Two type quasi-contractions on quasi metric spaces and some fixed point results

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Abstract

In this paper, we introduce new concepts of quasi-contractions of type (A) and of type (B) in a quasi metric space and we present the differences between of them. Then we present some fixed point results. In the light of the theorems it is shown that, although the Hausdorffness condition of quasi metric space is needed for the mapping of quasi contraction of type (A), it is not necessary to guarantee the existence of fixed point for the mapping of quasi contraction of type (B). ©2017 All rights reserved.

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1. Introduction and preliminaries

Let X be any set. Then a function $d : X \times X \rightarrow [0, \infty)$ is said to be a quasi metric on X if it has the following properties for all $x, y, z \in X$;

- $d(x, x) = 0$;
- $d(x, y) \leq d(x, z) + d(z, y)$;
- $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.

If a quasi metric d satisfies the additional condition

- $d(x, y) = 0 \Rightarrow x = y$,

then d is said to be T_1 -quasi metric. It is clear that, every metric is a T_1 -quasi metric, every T_1 -quasi metric is a quasi metric. A quasi (resp. T_1 -quasi) metric space is a pair (X, d) such that X is a nonempty set and d is a quasi (resp. T_1 -quasi) metric. Given a quasi metric space (X, d) and any real number $\varepsilon > 0$, the open

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ball and closed ball, respectively, of radius ε and center $x_0 \in X$ is the set $B_d(x_0, \varepsilon) \subset X$ and $B_d[x_0, \varepsilon] \subset X$ defined by

$$B_d(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\},$$

and

$$B_d[x_0, \varepsilon] = \{y \in X : d(x_0, y) \leq \varepsilon\}.$$

Every quasi metric d on X generates a natural topology on X . This topology on X generated by the family of open balls $\{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$ as base is called the the topology τ_d (or, the topology induced by the quasi metric d). If (X, d) is a quasi metric space, then τ_d is a T_0 topology, and if (X, d) is a T_1 -quasi metric space, then τ_d is a T_1 topology. It is clear that if d is a quasi metric and τ_d is T_1 topology, then d is T_1 -quasi metric. However, there exist some examples of T_1 topological spaces which are not T_1 -quasi metrizable. Uncountable cofinite topological space, which is not first countable space, can be given an example to this situation.

Given a quasi metric d on X , the function d^{-1} defined by for all $x, y \in X$,

$$d^{-1}(x, y) = d(y, x)$$

is also a quasi metric on X and the functions d^s and d_+ defined by for all $x, y \in X$,

$$d^s(x, y) = \max \{d(x, y), d^{-1}(x, y)\}$$

and

$$d_+(x, y) = d(x, y) + d^{-1}(x, y)$$

are metrics on X . Moreover, metrics d^s and d_+ are equivalent. Let $\{x_n\}$ be a sequence in the quasi metric space (X, d) and $a \in X$. The sequence $\{x_n\}$ is said to be convergent to a with respect to τ_d if $d(a, x_n) \rightarrow 0$ as $n \rightarrow \infty$. This is called as d -convergence and denoted by the following

$$x_n \xrightarrow{d} a \Leftrightarrow d(a, x_n) \rightarrow 0.$$

Similarly, the sequence $\{x_n\}$ is said to be convergent to a with respect to $\tau_{d^{-1}}$ if $d(x_n, a) \rightarrow 0$ as $n \rightarrow \infty$. This is called as d^{-1} -convergence and denoted by the following

$$x_n \xrightarrow{d^{-1}} a \Leftrightarrow d(x_n, a) \rightarrow 0.$$

Finally, the sequence $\{x_n\}$ is said to be convergent to a with respect to τ_{d^s} if $d^s(a, x_n) \rightarrow 0$ as $n \rightarrow \infty$. This is called as d^s -convergence and denoted by the following

$$x_n \xrightarrow{d^s} a \Leftrightarrow d^s(a, x_n) \rightarrow 0.$$

Thus, it is clear that $x_n \xrightarrow{d^s} a \Leftrightarrow x_n \xrightarrow{d} a$ and $x_n \xrightarrow{d^{-1}} a$. More detailed information for some important properties of quasi metric spaces and their topological structures can be seen in [4, 7, 8].

There exist several different notions of Cauchy sequence and of completeness in the quasi metric version of studies of fixed point theory. In fact, there are seven different definitions of Cauchy sequence on quasi metric spaces. Also, taking into account three different convergence of a sequence in quasi metric spaces, there obtain twenty one different notions of completeness. We can find more detailed information about some kind of convergence, Cauchyness, completeness and some important properties of quasi metric space in [1, 2, 9–11]. In this present paper, we will denote by \mathbb{R} the set of all real numbers and by \mathbb{N} the set of all natural numbers and also we shall use the following notions.

A sequence $\{x_n\}$ in a quasi metric space (X, d) is called

- left K -Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k, \quad n \geq k \geq n_0, \quad d(x_k, x_n) < \varepsilon;$$

- right K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k, \quad n \geq k \geq n_0, \quad d(x_n, x_k) < \varepsilon;$$

- d^s -Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k \geq n_0, \quad d(x_n, x_k) < \varepsilon.$$

It is obvious that if a sequence is left K-Cauchy with respect to d , then it is right K-Cauchy with respect to d^{-1} and $\{x_n\}$ is d^s -Cauchy if and only if it is both left K-Cauchy and right K-Cauchy. If a sequence $\{x_n\}$ in a quasi metric space (X, d) satisfies $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$, then it is left K-Cauchy sequence (see [4]). In a quasi metric space, d or d^{-1} -convergent sequences may not be Cauchy in the sense mentioned above, although every convergent sequence is Cauchy in a metric space (see [9]).

A quasi metric space (X, d) called

- bicomplete if every d^s -Cauchy sequence is d^s -convergent;
- left (right) \mathcal{K} -complete if every left (right) K-Cauchy sequence is d -convergent;
- left (right) \mathcal{M} -complete if every left (right) K-Cauchy sequence is d^{-1} -convergent;
- left (right) Smyth complete if every left (right) K-Cauchy sequence is d^s -convergent.

If (X, d) is both left \mathcal{K} -complete and left \mathcal{M} -complete, then it is bicomplete. The same implications hold for the notions of right completeness.

The aim of this paper is to present several new fixed point results for quasi contractive mappings on some kind of complete in quasi metric spaces. It can be found some interesting and nice fixed point results on quasi metric spaces in [5, 6, 12].

2. Fixed point results

Definition 2.1. Let (X, d) be a quasi metric space and $T : X \rightarrow X$ be a mapping. Then T is said to be

- quasi contraction of type (A), if there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda M_A(x, y)$$

for all $x, y \in X$, where

$$M_A(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\};$$

- quasi contraction of type (B), if there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda M_B(x, y)$$

for all $x, y \in X$, where

$$M_B(x, y) = \max\{d(y, x), d(x, Tx), d(y, Ty), d(Ty, x), d(y, Tx)\}.$$

Example 2.2. Let $X = \{0, 1, 2, \dots\}$ and $d : X \times X \rightarrow \mathbb{R}$ be a quasi metric on X given by

$$\begin{aligned} d(n, n) &= 0, & \forall n \in X, \\ d(0, n) &= d(n, 1) = 0, & \forall n \geq 1, \\ d(n, 0) &= 1, & \forall n \geq 1, \\ d(n, m) &= \frac{1}{2^{n+1}} + \frac{1}{2^{m+1}}, & \text{otherwise.} \end{aligned}$$

Now define $T : X \rightarrow X$ by $T(n) = n + 1$ for all $n \in X$. Then T is quasi contraction of type (B) but not of

type (A).

Proposition 2.3. *Let (X, d) be a quasi-metric space. If T is a quasi contraction of type (A) or of type (B) on (X, d) , then it is a quasi contraction on the metric space (X, d^s) .*

Proof. Suppose T is a quasi contraction of type (A) on (X, d) . Then, there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda M_A(x, y)$$

for all $x, y \in X$. Thus, given $x, y \in X$, we also have

$$d(Ty, Tx) \leq \lambda M_A(y, x).$$

So, we obtain

$$d^s(Tx, Ty) \leq \lambda \max\{d^s(x, y), d^s(x, Tx), d^s(y, Ty), d^s(x, Ty), d^s(y, Tx)\}.$$

That is, T is a quasi contraction on (X, d^s) . Similarly, we show that if T is a quasi contraction of type (B) on (X, d) , then it is a quasi contraction on (X, d^s) . \square

Proposition 2.4. *Let (X, d) be a quasi metric space. If T is a quasi contraction of type (A) or of type (B) on (X, d) , then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, is a d^s -Cauchy sequence on (X, d) .*

Proof. By Proposition 2.3, T is a quasi contraction on (X, d^s) . Then, from the proof of [3, Theorem 4.12] we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d^s) . Therefore it is d^s -Cauchy sequence on (X, d) . \square

Theorem 2.5. *Let (X, d) be a bicomplete quasi metric space and $T : X \rightarrow X$ be a quasi contraction of type (A) or of type (B). Then T has a unique fixed point.*

Proof. Since (X, d) is bicomplete quasi metric space, then (X, d^s) is complete metric space. By Proposition 2.3, T is a quasi contraction on (X, d^s) . Then, from the proof of Theorem 4.12 in [3] we conclude that T has a unique fixed point in X . \square

In the following, we provide some fixed point results considering more weak completeness instead of bicompleteness of (X, d) .

Proposition 2.6. *Let (X, d) be a T_1 -quasi metric space. If T is a quasi contraction of type (A) or of type (B) on (X, d) , then T has at most one fixed point.*

Proof. Let T be the quasi contraction of type (A) and let T has two different fixed points, say $z, w \in X$. Since d is a T_1 -quasi metric $\min\{d(z, w), d(w, z)\} > 0$. Now if $d(w, z) \leq d(z, w)$, then we get

$$\begin{aligned} 0 < d(z, w) &= d(Tz, Tw) \\ &\leq \lambda M_A(z, w) \\ &= \lambda \max\{d(z, w), d(z, Tz), d(w, Tw), d(z, Tw), d(w, Tz)\} \\ &= \lambda d(z, w), \end{aligned}$$

which is a contradiction. If, $d(w, z) > d(z, w)$, then we get

$$\begin{aligned} 0 < d(w, z) &= d(Tw, Tz) \\ &\leq \lambda M_A(w, z) \\ &= \lambda \max\{d(w, z), d(w, Tw), d(z, Tz), d(w, Tz), d(z, Tw)\} \\ &= \lambda d(w, z), \end{aligned}$$

which is a contradiction. Therefore T has at most one fixed point. Similar result can be obtained when T is a quasi contraction of type (B). \square

We will consider the sequential continuity of a mapping T in our theorems.

Definition 2.7. Let X be a nonempty set, d and ρ be two quasi metrics on X and $T : X \rightarrow X$ be a mapping. Then T is said to be sequentially d - ρ -continuous at $x \in X$, if for all sequence $\{x_n\}$ in X such that $d(x, x_n) \rightarrow 0$ implies $\rho(Tx, Tx_n) \rightarrow 0$. If T is sequentially d - ρ -continuous at all points of X , then T is said to be sequentially d - ρ -continuous on X .

Our first main results is as follows:

Theorem 2.8. Let (X, d) be a left \mathcal{K} -complete T_1 -quasi metric space and $T : X \rightarrow X$ be a quasi contraction of type (A). Then, T has a unique fixed point in X provided that one of the following conditions holds:

- (i) (X, τ_d) is Hausdorff space and T is sequentially d - d -continuous;
- (ii) T is sequentially d - d^{-1} -continuous;
- (iii) T has the following property:

$$\inf\{d(y, x) + d(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. By Proposition 2.4, the sequence $\{x_n\}$ is a left K -Cauchy sequence in the quasi metric space (X, d) . Then, since (X, d) is left \mathcal{K} -complete, there exists $z \in X$ such that $\{x_n\}$ is d -converges to z , that is, $d(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Now, we will consider the following three cases:

- (i) Let (X, τ_d) be Hausdorff space and T be sequentially d - d -continuous mapping. Then

$$d(Tz, x_{n+1}) = d(Tz, Tx_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since (X, d) is Hausdorff space, we get $z = Tz$.

- (ii) Let T be sequentially d - d^{-1} -continuous. Then, since $d(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$, we get

$$d(Tx_n, Tz) = d^{-1}(Tz, Tx_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, we obtain

$$d(z, Tz) \leq d(z, x_{n+1}) + d(Tx_n, Tz) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since d is T_1 -quasi metric, we get $z = Tz$.

- (iii) Let T has a property that $\inf\{d(y, x) + d(x, Tx) : x \in X\} > 0$ for every $y \in X$ with $y \neq Ty$. Now, suppose that $z \neq Tz$, then by our assumption, we obtain (by taking on account the left K -Cauchyness of $\{x_n\}$)

$$\begin{aligned} 0 &< \inf\{d(z, x) + d(x, Tx) : x \in X\} \\ &\leq \inf\{d(z, x_n) + d(x_n, Tx_n) : n \in \mathbb{N}\} \\ &= \inf\{d(z, x_n) + d(x_n, x_{n+1}) : n \in \mathbb{N}\} \\ &= 0, \end{aligned}$$

which is a contradiction. Hence $z = Tz$. By Proposition 2.6, the fixed point is unique.

□

The following example shows that the Hausdorffness condition can not be removed in Theorem 2.8.

Example 2.9. Let $X = \{1, 2, 3, \dots\}$ and d be a quasi metric on X given by $d(n, n) = 0$ for all $n \in X$, and $d(n, m) = \frac{1}{m}$ otherwise. Then d is a T_1 -quasi metric on X however (X, τ_d) is not Hausdorff. Since every left K -Cauchy sequence d -converges to 1, then (X, d) is left \mathcal{K} -complete quasi metric space. Now we define $T : X \rightarrow X$ as $Tn = 2n$ for all $n \in X$, then T is a quasi contraction of type (A) with $\lambda = \frac{1}{2}$. Further T is sequentially d - d -continuous. However T has no fixed point. (Note that both (ii) and (iii) conditions are not satisfied)

If we consider the quasi contraction of type (B) instead of type (A), we do not need the Hausdorffness condition of (X, d) .

Theorem 2.10. Let (X, d) be a left \mathcal{K} -complete T_1 -quasi metric space and $T : X \rightarrow X$ be a quasi contraction of type (B). Then, T has a unique fixed point in X provided that one of the following conditions holds:

- (i) T is sequentially d - d -continuous;
- (ii) T is sequentially d - d^{-1} -continuous;
- (iii) T has the following property:

$$\inf\{d(y, x) + d(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. By Proposition 2.4, the sequence $\{x_n\}$ is a left K -Cauchy sequence in the quasi metric space (X, d) . Then, since (X, d) is left \mathcal{K} -complete, there exists $z \in X$ such that $\{x_n\}$ d -converges to z , that is, $d(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Now, assume (i) holds. Since T is sequentially d - d -continuous mapping, then

$$d(Tz, x_n) = d(Tz, Tx_{n-1}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Suppose $z \neq Tz$, then

$$\begin{aligned} 0 < d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + \lambda M_B(x_n, z) \\ &= d(z, x_{n+1}) + \lambda \max\{d(z, x_n), d(x_n, Tx_n), d(z, Tz), d(Tz, x_n), d(z, Tx_n)\} \\ &= d(z, x_{n+1}) + \lambda \max\{d(z, x_n), d(x_n, x_{n+1}), d(z, Tz), d(Tz, Tx_{n-1}), d(z, x_{n+1})\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we get $0 < d(z, Tz) \leq \lambda d(z, Tz)$, which is a contradiction. Therefore we have $z = Tz$. By Proposition 2.6, the fixed point is unique. The proof is similar to proof of Theorem 2.8 when the other cases (ii) or (iii) hold. □

Now we present similar results when the quasi metric space is left \mathcal{M} -complete. However, we need the Hausdorffness condition in the case (i) in the following theorem.

Theorem 2.11. Let (X, d) be a left \mathcal{M} -complete T_1 -quasi metric space and $T : X \rightarrow X$ be a quasi contraction of type (A) or of type (B). Then, T has a unique fixed point in X provided that one of the following conditions holds:

- (i) $(X, \tau_{d^{-1}})$ is Hausdorff space T is sequentially d^{-1} - d^{-1} -continuous;
- (ii) T is sequentially d^{-1} - d -continuous;
- (iii) T has the following property:

$$\inf\{d(x, y) + d(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. By Proposition 2.4, the sequence $\{x_n\}$ is a left K-Cauchy sequence in the quasi metric space. Since (X, d) is left \mathcal{M} -complete, there exists $z \in X$ such that $\{x_n\}$ d^{-1} -converges to z , that is, $d^{-1}(z, x_n) = d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

Now, we will consider the following three cases:

(i) Let $(X, \tau_{d^{-1}})$ be Hausdorff space and T be sequentially d^{-1} - d^{-1} -continuous mapping. Then

$$d^{-1}(Tz, x_{n+1}) = d^{-1}(Tz, Tx_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $(X, \tau_{d^{-1}})$ is Hausdorff space, we get $z = Tz$.

(ii) Let T be sequentially d^{-1} - d -continuous. Then, since $d(x_n, z) = d^{-1}(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$, we get $d(Tz, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. So, we have

$$d(Tz, z) \leq d(Tz, Tx_n) + d(x_{n+1}, z) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since d is T_1 -quasi metric, we get $z = Tz$.

(iii) Let T has a property that $\inf\{d(x, y) + d(x, Tx) : x \in X\} > 0$ for every $y \in X$ with $y \neq Ty$. Now, assume that $z \neq Tz$, then by our assumption, we obtain (by taking on account the left K-Cauchyness of $\{x_n\}$)

$$\begin{aligned} 0 &< \inf\{d(x, z) + d(x, Tx) : x \in X\} \\ &\leq \inf\{d(x_n, z) + d(x_n, Tx_n) : n \in \mathbb{N}\} \\ &= \inf\{d(x_n, z) + d(x_n, x_{n+1}) : n \in \mathbb{N}\} \\ &= 0, \end{aligned}$$

which is a contradiction. Hence $z = Tz$. □

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