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Existence and multiplicity of solutions for a class of quasilinear elliptic systems in Orlicz-Sobolev spaces

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Abstract

In this paper, we investigate the following nonlinear and non-homogeneous elliptic system

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) = F_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(\phi_2(|\nabla v|)\nabla v) = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geqslant 2)$ with smooth boundary $\partial\Omega$, functions $\phi_i(t)t$ $(i = 1,2)$ are increasing homeomorphisms from $\mathbb{R}^+$ onto $\mathbb{R}^+$. When F satisfies some (ϕ_1, ϕ_2)-superlinear and subcritical growth conditions at infinity, by using the mountain pass theorem we obtain that system has a nontrivial solution, and when F satisfies an additional symmetric condition, by using the symmetric mountain pass theorem, we obtain that system has infinitely many solutions. Some of our results extend and improve those corresponding results in Carvalho et al. [M. L. M. Carvalho, J. V. A. Goncalves, E. D. da Silva, J. Math. Anal. Appl., **426** (2015), 466–483].

Keywords: Orlicz-Sobolev spaces, mountain pass theorem, symmetric mountain theorem.
2010 MSC: 35J20, 35J50, 35J55.

1. Introduction

In this paper, we investigate the existence and multiplicity of solutions for the following nonlinear and non-homogeneous elliptic system in Orlicz-Sobolev spaces:

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) = F_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(\phi_2(|\nabla v|)\nabla v) = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geqslant 2)$ with smooth boundary $\partial\Omega$, and ϕ_i $(i = 1,2) : (0,+\infty) \to (0,+\infty)$ are two functions which satisfy:

(ϕ_1) $\phi_i \in C^1(0,+\infty)$, $t\phi_i(t) \to 0$ as $t \to 0$, $t\phi_i(t) \to +\infty$ as $t \to +\infty$;

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(ϕ_2) $t \to t\phi_i(t)$ are strictly increasing;

(ϕ_3) $1 < l_i := \inf_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} \leqslant \sup_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} =: m_i < N$, where $\Phi_i(t) := \int_0^{|t|} s\phi_i(s)ds,\ t \in \mathbb{R}$,

and F satisfies

(F_0) $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x, 0, 0) = 0,\ x \in \Omega$.

Set $\phi_2 = \phi_1 =: \phi$, $v = u$ and $F(x, u, v) = F(x, v, u)$. Then the system (1.1) reduces to the following quasilinear elliptic equation:

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Under assumptions (ϕ_1) and (ϕ_2), equation (1.2) may be allowed to possess more complicated nonlinear or non-homogeneous term ϕ, which can be used to model many phenomena (see [20]). When ϕ is not homogeneous, an Orlicz-Sobolev space setting may be applied for this type of equations (see Section 2). In recent years, equations like (1.2) have caused great interests among scholars. We refer readers to [4, 9, 11–15, 17–20, 24, 28, 33] and reference therein for more information. In those papers, the existence and multiplicity of solutions were obtained by various methods. Among them, variational methods have been used widely. In Clément et al. [14], the authors firstly obtained that (1.2) has a nontrivial solution by variational method when the nonlinear term f satisfies Ambrosetti-Rabinowitz type growth and subcritical Orlicz-Sobolev growth conditions. Motivated by this paper, many scholars studied the existence and multiplicity of solutions when Ambrosetti-Rabinowitz type growth condition is replaced by other superlinear Orlicz-Sobolev growth conditions (see [12, 13]). When nonlinear term f has a critical Orlicz-Sobolev growth, the existence of a nontrivial solution was proved in [18, 19] and some other results were obtained in [33]. In [28] and [17], the authors obtained that (1.2) has at least two nontrivial solutions and infinitely many solutions, respectively, when the nonlinear term f has a sublinear Orlicz-Sobolev growth. In [11], by using the three critical points theorem due to Ricceri, the authors obtained that (1.2) has at least three solutions. Particularly, when $\phi(t) = |t|^{p-2} (p > 1)$, equation (1.2) is the p-Laplacian equation which has been studied extensively.

Set $\phi_1(t) = |t|^{p-2}$, $\phi_2(t) = |t|^{q-2}$ $(p, q > 1)$. Then system (1.1) reduces to the following (p, q)-Laplacian system:

$$\begin{cases} -\Delta_p u = F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The existence and multiplicity of solutions for systems like (1.3) have also received a wide range of interests. Some methods are important to investigate systems like (1.3), such as variational method (see [7, 8, 16]), Nehari manifold and fibering method (see [2, 10, 35]), three critical points theorem (see [3, 27]), etc.

To the best of our knowledge, there are few papers considering the existence and multiplicity of solutions for systems like (1.1) except for [23, 34, 36]. In [23], Huentutripay-Manásevich studied an eigenvalue problem to the following system:

$$\begin{cases} -\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\operatorname{div}(\phi_2(|\nabla v|)\nabla v) = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function F has the form

$$F(x, u, v) = A_1(x, u) + b(x)\Gamma_1(u)\Gamma_2(v) + A_2(x, v).$$

For a certain λ, the authors translated the existence of solution into a suitable minimizing problem and proved the existence of solution under some reasonable restriction. In [23], the Orlicz-Sobolev spaces are not necessary to be reflexive.

In [36], Xia and Wang considered a class of quasilinear elliptic system like (1.1). When Φ_i are p_i-uniformly convex and have a q_i-asymptotical growth at infinity, $i = 1, 2$, respectively (see (4) and (6) in [36]), and F satisfies the following Ambrosetti-Rabinowitz type condition (see (5) in [36]):

$$0 < \theta F(x, u, v) \leqslant uF_u(x, u, v) + vF_v(x, u, v), \quad \text{for all } x \in \overline{\Omega}, \quad |u| + |v| > T,$$

where $\theta > \max\{q_1, q_2\}$ and $T > 0$, by using the mountain pass theorem, Xia and Wang proved the system has a nontrivial solution.

In [34], we investigated the following system:

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) = \lambda_1 F_u(x, u, v) - \lambda_2 G_u(x, u, v) - \lambda_3 H_u(x, u, v) & \text{in } \Omega, \\ -\mathrm{div}(\phi_2(|\nabla v|)\nabla v) = \lambda_1 F_v(x, u, v) - \lambda_2 G_v(x, u, v) - \lambda_3 H_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_1, λ_2, λ_3 are three parameters, functions F, G, H are of class $C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ and satisfy some reasonable growth conditions. By using a three critical points theorem due to B. Ricceri [32], we proved that system has at least three solutions. With some additional conditions, by using a four critical points theorem due to Anello [5], we proved that system has at least four solutions.

In Carvalho et al. [12], by using the mountain pass theorem, authors obtained that equation (1.2) has at least one or two nontrivial solutions. To be precise, they obtained the following result:

Theorem 1.1 ([12, Theorem 1.1]). *Assume that ϕ and f satisfy*

$(\phi_1)'$ $\phi \in C^1(0, +\infty)$, $t\phi(t) \to 0$ *as* $t \to 0$, $t\phi(t) \to +\infty$ *as* $t \to +\infty$*;*

$(\phi_2)'$ $t \to t\phi(t)$ *is strictly increasing;*

$(\phi_3)'$

$$1 < l := \inf_{t>0} \frac{t^2\phi(t)}{\Phi(t)} \leqslant \sup_{t>0} \frac{t^2\phi(t)}{\Phi(t)} =: m < N,$$

where

$$\Phi(t) := \int_0^{|t|} s\phi(s)ds, \quad t \in \mathbb{R};$$

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $f(x, 0) = 0$, $x \in \Omega$*;*

$(f_1)'$ *a constant* $C > 0$ *and an N-function (see Section 2) exists*

$$\Psi(t) := \int_0^{|t|} \psi(s)ds, \quad t \in \mathbb{R},$$

where $\psi : [0, +\infty) \to \mathbb{R}$ *is continuous and it satisfies*

$$1 < l \leqslant m < l_\Psi := \inf_{t>0} \frac{t\psi(t)}{\Psi(t)} \leqslant \sup_{t>0} \frac{t\psi(t)}{\Psi(t)} =: m_\Psi < l^* := \frac{lN}{N - l},$$

such that

$$|f(x, t)| \leqslant C(1 + \psi(|t|)), \quad (x, t) \in \Omega \times \mathbb{R};$$

$(f_2)'$

$$\limsup_{t\to 0} \frac{|f(x, t)|}{|t|\phi(t)} = \lambda < \lambda_1, \ \textit{uniformly in } \ x \in \Omega,$$

where $\lambda_1 > 0$ *satisfies the Poincaré inequality in* [1] *given by*

$$\lambda_1 \int_\Omega \Phi(u)dx \leqslant \int_\Omega \Phi(|\nabla(u)|)dx, \quad \forall u \in W_0^{1,\Phi}(\Omega);$$

$(f_3)'$

$$\lim_{t\to\infty}\frac{f(x,t)}{|t|^{m-2}t}=+\infty,\quad \textit{uniformly in}\ \ x\in\Omega;$$

$(f_4)'$ *an N-function exists*

$$\Gamma(t):=\int_0^{|t|}\gamma(s)ds,\quad t\in\mathbb{R},$$

where $\gamma:[0,+\infty)\to\mathbb{R}$ *is continuous and it satisfies*

$$\frac{N}{l}<l_\Gamma:=\inf_{t>0}\frac{t\gamma(t)}{\Gamma(t)}\leqslant\sup_{t>0}\frac{t\gamma(t)}{\Gamma(t)}=:m_\Gamma<\infty,$$

such that

$$\Gamma\left(\frac{F(x,t)}{|t|^l}\right)\leqslant C\overline{F}(x,t),\quad x\in\Omega,|t|\geqslant R,$$

where C, R *are positive constants and*

$$\overline{F}(x,t)=tf(x,t)-mF(x,t),\quad (x,t)\in\Omega\times\mathbb{R}.$$

Then (1.2) *admits*

(i) *one nonzero solution;*

(ii) *two nonzero solutions, say* $u,v\in C^{1,\alpha}(\overline{\Omega})$ *with* $0<\alpha<1$ *such that*

$$u>0\quad \textit{and}\quad v<0\quad \textit{in}\ \Omega,$$

provided that $(\phi_3)'$ *is replaced by a stronger condition*

$(\phi_4)'$

$$0<l-1:=\inf_{t>0}\frac{(\phi(t)t)'}{\phi(t)}\leqslant\sup_{t>0}\frac{(\phi(t)t)'}{\phi(t)}=:m-1<N-1.$$

Motivated by [12], in this paper, by using the mountain pass theorem, we obtain that system (1.1) has a nontrivial solution (see Theorem 3.1 in Section 3) and the result extends the result (i) of Theorem 1.1 to system case. It is remarkable that, when system (1.1) reduces to (1.2), our corresponding result still improves the result (i) of Theorem 1.1 and one can see the details in Section 5 where we also offer an example that satisfies our results but does not satisfy Theorem 1.1. Besides, by using the symmetric mountain pass theorem, we obtain that system solutions. Since the system case is different from the scalar case, we will come across some additional difficulties, for example, the direct sum decomposition of working space. Of course more computing skills are needed in the process of our proofs.

This paper is organized as follows. In Section 2, we recall some preliminary knowledge on Orlicz and Orlicz-Sobolev spaces. In Section 3, we present the existence result (Theorem 3.1 below) and complete the proofs by using mountain pass theorem. In Section 4, we present the multiplicity result (Theorem 4.1 below) and complete the proofs by using symmetric mountain pass theorem. In Section 5, we present the results for (1.2), which correspond to Theorem 3.1 and Theorem 4.1, and compare the results with Theorem 1.1. In Section 6, we present some examples to illustrate our Theorem 3.1 and Theorem 4.1.

2. Preliminaries

In this paper, we study system (1.1) where ϕ_i may be nonlinear and non-homogeneous. To deal with such problem, we need to introduce Orlicz and Orlicz-Sobolev spaces. In this section, we present some fundamental notions and important properties about Orlicz and Orlicz-Sobolev spaces. We refer readers for more details to the books [1, 31] and the references quoted in them.

Definition 2.1 (see [1]). Let $\phi : [0, +\infty) \to [0, +\infty)$ be a right continuous, monotone increasing function with

(1) $\phi(0) = 0$;

(2) $\lim_{t\to+\infty} \phi(t) = +\infty$;

(3) $\phi(t) > 0$ whenever $t > 0$.

Then the function defined on $\mathbb{R}$ by $\Phi(t) = \int_0^{|t|} \phi(s)ds$ is called an N-function.

By the definition of N-function Φ, it is obvious that $\Phi(0) = 0$ and Φ is strictly convex. We recall that an N-function Φ satisfies a Δ_2-condition globally (or near infinity) if

$$\sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} < +\infty \quad (\text{ or } \limsup_{t\to\infty} \frac{\Phi(2t)}{\Phi(t)} < +\infty),$$

which implies that there exists a constant $K > 0$, such that $\Phi(2t) \leqslant K\Phi(t)$ for all $t \geqslant 0$ (or $t \geqslant t_0 > 0$). We also state the equivalent form that Φ satisfies a Δ_2-condition globally (or near infinity) if and only if for any $c \geqslant 1$, there exists a constant $K_c > 0$ such that $\Phi(ct) \leqslant K_c\Phi(t)$ for all $t \geqslant 0$ (or $t \geqslant t_0 > 0$).

Definition 2.2 (see [1]). For an N-function Φ, we define

$$\widetilde{\Phi}(t) = \int_0^{|t|} \phi^{-1}(s)ds, \quad t \in \mathbb{R},$$

where ϕ^{-1} is the right inverse of the right derivative ϕ of Φ. Then $\widetilde{\Phi}$ is an N-function called the complement of Φ.

It holds that Young's inequality (see [1, 31])

$$st \leqslant \Phi(s) + \widetilde{\Phi}(t) \quad s, t \geqslant 0, \tag{2.1}$$

and inequality (see [18, Lemma A.2])

$$\widetilde{\Phi}(\phi(t)) \leqslant \Phi(2t), \quad t \geqslant 0.$$

Now, we recall the Orlicz space $L^\Phi(\Omega)$ associated with Φ. When Φ satisfies Δ_2-condition globally, the Orlicz space $L^\Phi(\Omega)$ is the vectorial space of the measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi(|u|)dx < +\infty,$$

where $\Omega \subset \mathbb{R}^N$ is an open set. $L^\Phi(\Omega)$ is a Banach space endowed with Luxemburg norm

$$\|u\|_\Phi := \inf\{\lambda > 0 : \int_\Omega \Phi\left(\frac{u}{\lambda}\right) dx \leqslant 1\}, \quad \text{for } u \in L^\Phi(\Omega).$$

Particularly, when $\Phi(t) = |t|^p (p > 1)$, the corresponding Orlicz space $L^\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ and the corresponding Luxemburg norm $\|u\|_\Phi$ is equal to the classical $L^p(\Omega)$ norm, that is,

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, \quad \text{for } u \in L^p(\Omega).$$

The fact that Φ satisfies Δ_2-condition globally implies that

$$u_n \to u \text{ in } L^\Phi(\Omega) \Longleftrightarrow \int_\Omega \Phi(u_n - u)dx \to 0. \tag{2.2}$$

Moreover, a generalized type of Hölder's inequality (see [1, 31])

$$\left|\int_\Omega uv\,dx\right| \leqslant 2\|u\|_\Phi \|v\|_{\widetilde{\Phi}}, \quad \text{for all } u \in L^\Phi(\Omega),\ \ v \in L^{\widetilde{\Phi}}(\Omega),$$

can be gained by applying Young's inequality (2.1).

The corresponding Orlicz-Sobolev space (see [1, 31]) is defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega), i = 1, \cdots, N\right\},$$

with the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi.$$

When Ω is bounded, $W_0^{1,\Phi}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$ has an equivalent norm

$$\|u\|_{0,\Phi} := \|\nabla u\|_\Phi,$$

which can be obtained by using the Poincaré inequality in [22] given as

$$\int_\Omega \Phi(u)dx \leqslant \int_\Omega \Phi(2d|\nabla(u)|)dx, \quad \forall u \in W_0^{1,\Phi}(\Omega),$$

where $d=\mathrm{diam}(\Omega)$.

Next, we give some inequalities which will be used in our proofs. For more details, we refer the reader to the papers [1, 18].

Lemma 2.3 ([1, 18])**.** *If Φ is an N-function, then the following conditions are equivalent:*

(1)

$$1 \leqslant l = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \leqslant \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} = m < +\infty; \tag{2.3}$$

(2) *let* $\zeta_0(t) = \min\{t^l, t^m\}$, $\zeta_1(t) = \max\{t^l, t^m\}$, $t \geqslant 0$. Φ *satisfies*

$$\zeta_0(t)\Phi(\rho) \leqslant \Phi(\rho t) \leqslant \zeta_1(t)\Phi(\rho), \quad \forall \rho, t \geqslant 0;$$

(3) Φ *satisfies a* Δ_2*-condition globally.*

Lemma 2.4. *If Φ is an N-function and (2.3) holds, then Φ satisfies*

$$\zeta_0(\|u\|_\Phi) \leqslant \int_\Omega \Phi(u)dx \leqslant \zeta_1(\|u\|_\Phi), \quad \forall u \in L^\Phi(\Omega).$$

Lemma 2.5. *If Φ is an N-function and (2.3) holds with $l > 1$. Let $\widetilde{\Phi}$ be the complement of Φ and $\zeta_2(t) = \min\{t^{\widetilde{l}}, t^{\widetilde{m}}\}$, $\zeta_3(t) = \max\{t^{\widetilde{l}}, t^{\widetilde{m}}\}$, for $t \geqslant 0$, where $\widetilde{l} := \frac{l}{l-1}$ and $\widetilde{m} := \frac{m}{m-1}$. Then $\widetilde{\Phi}$ satisfies*

(1)

$$\widetilde{m} = \inf_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} \leqslant \sup_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} = \widetilde{l};$$

(2)

$$\zeta_2(t)\widetilde{\Phi}(\rho) \leqslant \widetilde{\Phi}(\rho t) \leqslant \zeta_3(t)\widetilde{\Phi}(\rho), \quad \forall \rho, t \geqslant 0;$$

(3)

$$\zeta_2(\|u\|_{\widetilde{\Phi}}) \leqslant \int_\Omega \widetilde{\Phi}(u)dx \leqslant \zeta_3(\|u\|_{\widetilde{\Phi}}), \quad \forall u \in L^{\widetilde{\Phi}}(\Omega).$$

Lemma 2.6. *If Φ is an N-function and (2.3) holds with $l, m \in (1, N)$. Let $\zeta_4(t) = \min\{t^{l^*}, t^{m^*}\}$, $\zeta_5(t) = \max\{t^{l^*}, t^{m^*}\}$, for $t \geqslant 0$, where $l^* := \frac{lN}{N-l}$, $m^* := \frac{mN}{N-m}$. Then Φ_* satisfies*

(1)
$$l^* = \inf_{t>0} \frac{t\Phi_*'(t)}{\Phi_*(t)} \leqslant \sup_{t>0} \frac{t\Phi_*'(t)}{\Phi_*(t)} = m^*;$$

(2)
$$\zeta_4(t)\Phi_*(\rho) \leqslant \Phi_*(\rho t) \leqslant \zeta_5(t)\Phi_*(\rho), \quad \forall \rho, t \geqslant 0;$$

(3)
$$\zeta_4(\|u\|_{\Phi_*}) \leqslant \int_\Omega \Phi_*(u)dx \leqslant \zeta_5(\|u\|_{\Phi_*}), \quad \forall u \in L^{\Phi_*}(\Omega),$$

where Φ_* is the Sobolev conjugate function of Φ, which is defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds, \quad \textit{for } t \geqslant 0 \quad \textit{and} \quad \Phi_*(t) = \Phi_*(-t), \quad \textit{for } t \leqslant 0.$$

Lemma 2.7. *Under the assumptions of Lemma 2.6, the embedding from $W_0^{1,\Phi}(\Omega)$ into $L^{\Phi_*}(\Omega)$ is continuous and into $L^{\Upsilon}(\Omega)$ is compact for any N-function Υ increasing essentially more slowly than Φ_* near infinity, that is,*

$$\lim_{t\to+\infty} \frac{\Upsilon(ct)}{\Phi_*(t)} = 0$$

for any constant $c > 0$. Therefore, there exists $C_\Gamma > 0$ such that

$$\|u\|_\Gamma \leqslant C_\Gamma \|\nabla u\|_\Phi, \quad \forall u \in W_0^{1,\Phi}(\Omega). \tag{2.4}$$

Remark 2.8. By Lemma 2.3 and Lemma 2.5, assumptions (ϕ_1)-(ϕ_3) show that Φ_i $(i = 1, 2)$ and $\widetilde{\Phi}_i$ $(i = 1, 2)$ are N-functions satisfying Δ_2-condition globally. Thus $L^{\Phi_i}(\Omega) (i = 1, 2)$ and $W_0^{1,\Phi_i}(\Omega)(i = 1, 2)$ are separable and reflexive Banach spaces (see [1, 31]).

Notation. Throughout this paper, C is used to denote a positive constant which may be different in various places.

3. Existence

In this section, we present the following existence result by using mountain pass theorem.

Theorem 3.1. *Assume that (ϕ_1)-(ϕ_3), (F_0) and the following conditions hold:*

(F_1) *there exist two continuous functions ψ_i $(i = 1, 2) : [0, +\infty) \to \mathbb{R}$, which satisfy that $\Psi_i(t) := \int_0^{|t|} \psi_i(s)ds$, $t \in \mathbb{R}$ $(i = 1, 2)$ are two N-functions increasing essentially more slowly than Φ_{i*} $(i = 1, 2)$ near infinity, respectively, moreover,*

$$m_i < l_{\Psi_i} := \inf_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} \leqslant \sup_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} =: m_{\Psi_i} < +\infty, \tag{3.1}$$

such that

$$\begin{cases} |F_u(x, u, v)| \leqslant c_1(1 + \psi_1(|u|) + \widetilde{\Psi}_1^{-1}(\Psi_2(v))), \\ |F_v(x, u, v)| \leqslant c_1(1 + \widetilde{\Psi}_2^{-1}(\Psi_1(u)) + \psi_2(|v|)) \end{cases} \tag{3.2}$$

for all $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, where constant $c_1 > 0$, $\widetilde{\Psi}_i$ denote the complements of Ψ_i $(i = 1, 2)$, respectively;

(F_2)

$$\limsup_{|(u,v)|\to 0} \frac{|F(x,u,v)|}{\lambda_1\Phi_1(u)+\lambda_2\Phi_2(v)} = c_2 \ \textit{uniformly in} \quad x\in\Omega,$$

where constants $c_2 \in [0,1)$ *and* λ_i $(i=1,2)>0$ *satisfy the Poincaré inequalities in* [1] *supplied by*

$$\lambda_1\int_\Omega \Phi_1(u)dx \leqslant \int_\Omega \Phi_1(|\nabla u|)dx, \quad \forall u\in W_0^{1,\Phi_1}(\Omega)$$

and

$$\lambda_2\int_\Omega \Phi_2(v)dx \leqslant \int_\Omega \Phi_2(|\nabla v|)dx, \quad \forall v\in W_0^{1,\Phi_2}(\Omega);$$

(F_3)

$$\lim_{|(u,v)|\to+\infty} \frac{F(x,u,v)}{\Phi_1(u)+\Phi_2(v)} = +\infty \ \textit{uniformly in} \ x\in\Omega;$$

(F_4) *there exists a continuous function* $\gamma:[0,\infty)\to\mathbb{R}$ *and it satisfies that* $\Gamma(t):=\int_0^{|t|}\gamma(s)ds$, $t\in\mathbb{R}$ *is an* N*-function with*

$$1<l_\Gamma := \inf_{t>0}\frac{t\gamma(t)}{\Gamma(t)} \leqslant \sup_{t>0}\frac{t\gamma(t)}{\Gamma(t)} =: m_\Gamma < +\infty,$$

and functions $H_i(t) := |t|^{\frac{l_i l_\Gamma}{l_\Gamma - 1}}$, $t\in\mathbb{R}$ $(i=1,2)$ *increase essentially more slowly than* Φ_{i*} $(i=1,2)$ *near infinity, respectively, such that*

$$\Gamma\left(\frac{F(x,u,v)}{|u|^{l_1}+|v|^{l_2}}\right) \leqslant c_3\overline{F}(x,u,v), \quad x\in\Omega, \quad |(u,v)|\geqslant r, \tag{3.3}$$

where constants $c_3, r>0$ *and*

$$\overline{F}(x,u,v) := \frac{1}{m_1}F_u(x,u,v)u + \frac{1}{m_2}F_v(x,u,v)v - F(x,u,v), \quad \forall (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R}.$$

Then system (1.1) *possesses a nontrivial weak solution.*

Remark 3.2. Under assumptions (ϕ_1)-(ϕ_3), (F_1) and (F_4), by Lemma 2.7, the following embeddings are compact:

$$W_0^{1,\Phi_i}(\Omega)\hookrightarrow L^{\Psi_i}(\Omega), W_0^{1,\Phi_i}(\Omega)\hookrightarrow L^{l_i\widetilde{l_\Gamma}}(\Omega) \quad \text{and} \quad W_0^{1,\Phi_i}(\Omega)\hookrightarrow L^{l_i\widetilde{m_\Gamma}}(\Omega), \ \ i=1,2,$$

where $\widetilde{l_\Gamma} = \frac{l_\Gamma}{l_\Gamma - 1}$ and $\widetilde{m_\Gamma} = \frac{m_\Gamma}{m_\Gamma - 1}$.

Remark 3.3. By (2) in Lemma 2.3, assumptions (F_3) and (F_4) show

$$\lim_{|(u,v)|\to+\infty} \overline{F}(x,u,v)\to+\infty, \quad \text{uniformly in } x\in\Omega.$$

Remark 3.4. Based on the Young's inequality (2.1), $F(x,0,0)=0$ and the fact

$$F(x,u,v) = \int_0^u F_s(x,s,v)ds + \int_0^v F_t(x,0,t)dt + F(x,0,0), \quad \forall (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R},$$

equation (3.2) shows that there exists a constant $c_4>0$ such that

$$|F(x,u,v)| \leqslant c_4(|u|+|v|+\Psi_1(u)+\Psi_2(v)), \quad \forall (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R},$$

which, together with (3.1) and (2) in Lemma 2.3, shows that there exists a constant $c_5>0$ such that

$$|F(x,u,v)| \leqslant c_5(1+\Psi_1(u)+\Psi_2(v)), \quad \forall (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.4}$$

Define $W := W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ with the norm

$$\|(u,v)\| := \|u\|_{0,\Phi_1} + \|v\|_{0,\Phi_2} = \|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}.$$

Remark 2.8 shows W is a separable and reflexive Banach space. We observe that the energy functional I on W corresponding to system (1.1) is

$$I(u,v) := \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx, \quad (u,v) \in W.$$

Denote by I_i $(i = 1,2) : W \to \mathbb{R}$ the functionals

$$I_1(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx, \quad \text{and} \quad I_2(u,v) = \int_\Omega F(x,u,v)dx.$$

Then

$$I(u,v) = I_1(u,v) - I_2(u,v).$$

Under the assumptions (ϕ_1)-(ϕ_3), by similar arguments as [21], we can prove that I_1 is well-defined and of class $C^1(W,\mathbb{R})$ and

$$\langle I_1'(u,v),(\tilde{u},\tilde{v})\rangle = \int_\Omega \phi_1(|\nabla u|)\nabla u \nabla \tilde{u} dx + \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla \tilde{v} dx$$

for all $(\tilde{u},\tilde{v}) \in W$. Furthermore, under the assumption (F_1), standard arguments show that I_2 is also well-defined and of class $C^1(W,\mathbb{R})$ and

$$\langle I_2'(u,v),(\tilde{u},\tilde{v})\rangle = \int_\Omega F_u(x,u,v)\tilde{u} dx + \int_\Omega F_v(x,u,v)\tilde{v} dx$$

for all $(\tilde{u},\tilde{v}) \in W$. Therefore, I is well-defined and of class $C^1(W,\mathbb{R})$ and

$$\begin{aligned}\langle I'(u,v),(\tilde{u},\tilde{v})\rangle = &\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla \tilde{u} dx + \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla \tilde{v} dx \\ &- \int_\Omega F_u(x,u,v)\tilde{u} dx - \int_\Omega F_v(x,u,v)\tilde{v} dx\end{aligned}$$

for all $(\tilde{u},\tilde{v}) \in W$. Then, the critical points of I on W are weak solutions of system (1.1).

We will use the mountain pass theorem (see [30, Theorem 2.2]) to prove Theorem 3.1, and use the symmetric mountain pass theorem (see [30, Theorem 9.12]) to prove Theorem 4.1 in Section 4. By arguments in [6], it turns out that the (PS)-condition due to Palais-Smale can be replaced by $(C)_c$-condition due to Cerami in the mountain pass theorem and in the symmetric mountain pass theorem.

We recall that $I \in C^1(E,\mathbb{R})$ satisfies $(C)_c$-condition if any $(C)_c$-sequence $\{u_n\} \subset E$ has a convergent subsequence, where $(C)_c$-sequence $\{u_n\}$ means that

$$I(u_n) \to c, \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \text{ as } n \to \infty. \tag{3.5}$$

Lemma 3.5 ([30, Theorem 2.2]). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfying (PS)-condition. Suppose $I(0) = 0$ and*

(I_1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geqslant \alpha$, and*

(I_2) *there is an $e \in E \backslash B_\rho$ such that $I(e) \leqslant 0$.*

Then I possesses a critical value $c \geqslant \alpha$.

Lemma 3.6. *Suppose that (ϕ_1)-(ϕ_3), (F_1) and (F_2) hold. Then there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geqslant \alpha$.*

Proof. By (F_2), (3.4) and the fact that F is continuous, we obtain that there exist constants $\varepsilon > 0$ and $C > 0$ such that

$$|F(x,u,v)| \leqslant (1-\varepsilon)(\lambda_1\Phi_1(u)+\lambda_2\Phi_2(v)) + C(\Psi_1(u)+\Psi_2(v)), \quad \forall (x,u,v) \in \Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.6}$$

When $\|(u,v)\| \leqslant 1$, by (3.6), Poincaré inequality in (F_2), Lemma 2.4, Remark 3.2 and (2.4), we obtain

$$\begin{aligned}
I(u,v) &= \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx\\
&\geqslant \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega |F(x,u,v)|dx\\
&\geqslant \varepsilon\int_\Omega \Phi_1(|\nabla u|)dx + \varepsilon\int_\Omega \Phi_2(|\nabla v|)dx - C\int_\Omega \Psi_1(u)dx - C\int_\Omega \Psi_2(v)dx\\
&\geqslant \varepsilon\min\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\} + \varepsilon\min\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\}\\
&\quad - C\max\{\|u\|_{\Psi_1}^{l_{\Psi_1}}, \|u\|_{\Psi_1}^{m_{\Psi_1}}\} - C\max\{\|v\|_{\Psi_2}^{l_{\Psi_2}}, \|v\|_{\Psi_2}^{m_{\Psi_2}}\}\\
&\geqslant \varepsilon\min\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\} + \varepsilon\min\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\}\\
&\quad - C\max\{\|\nabla u\|_{\Phi_1}^{l_{\Psi_1}}, \|\nabla u\|_{\Phi_1}^{m_{\Psi_1}}\} - C\max\{\|\nabla v\|_{\Phi_2}^{l_{\Psi_2}}, \|\nabla v\|_{\Phi_2}^{m_{\Psi_2}}\}\\
&= \varepsilon\|\nabla u\|_{\Phi_1}^{m_1} + \varepsilon\|\nabla v\|_{\Phi_2}^{m_2} - C\|\nabla u\|_{\Phi_1}^{l_{\Psi_1}} - C\|\nabla v\|_{\Phi_2}^{l_{\Psi_2}}\\
&= \|\nabla u\|_{\Phi_1}^{m_1}(\varepsilon - C\|\nabla u\|_{\Phi_1}^{l_{\Psi_1}-m_1}) + \|\nabla v\|_{\Phi_2}^{m_2}(\varepsilon - C\|\nabla v\|_{\Phi_2}^{l_{\Psi_2}-m_2}).
\end{aligned}$$

Since $1 < m_i < l_{\Psi_i}$, we can choose positive constants ρ and α small enough such that $I(u,v) \geqslant \alpha$ for all $(u,v) \in W$ with $\|(u,v)\| = \rho$. □

Lemma 3.7. *Suppose that* (ϕ_1)-(ϕ_3) *and* (F_3) *hold. Then there is a point* $(u,v) \in W\backslash B_\rho$ *such that* $I(u,v) \leqslant 0$.

Proof. By (F_3) and the fact that F is continuous, then for any given constant $M > 0$, there exists a constant $C_M > 0$ such that

$$F(x,u,v) \geqslant M(\Phi(u)+\Phi(v)) - C_M, \quad \forall (x,u,v) \in \Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.7}$$

Now, choose $u_0 \in C_0^\infty(\Omega)\setminus\{0\}$ with $0 \leqslant u_0(x) \leqslant 1$. Then $(u_0, 0) \in W$, and by (3.7) and (2) in Lemma 2.3, when $t > 0$ we have

$$\begin{aligned}
I(tu_0,0) &= \int_\Omega \Phi_1(t|\nabla u_0|)dx - \int_\Omega F(x,tu_0,0)dx\\
&\leqslant \int_\Omega \Phi_1(t|\nabla u_0|)dx - M\int_\Omega \Phi_1(tu_0)dx + C_M|\Omega|\\
&\leqslant \Phi_1(t)\int_\Omega \max\{|\nabla u_0|^{l_1}, |\nabla u_0|^{m_1}\}dx - M\Phi_1(t)\int_\Omega \min\{|u_0|^{l_1}, |u_0|^{m_1}\}dx + C_M|\Omega|\\
&\leqslant \Phi_1(t)(\||\nabla u_0|\|_{L^{l_1}(\Omega)}^{l_1} + \||\nabla u_0|\|_{L^{m_1}(\Omega)}^{m_1} - M\|u_0\|_{m_1}^{m_1}) + C_M|\Omega|.
\end{aligned}$$

Since $M > 0$ is arbitrary and $\lim_{t\to+\infty}\Phi_1(t) = +\infty$, we can choose $M > \frac{\||\nabla u_0|\|_{L^{l_1}(\Omega)}^{l_1} + \||\nabla u_0|\|_{L^{m_1}(\Omega)}^{m_1}}{\|u_0\|_{m_1}^{m_1}}$ and large t such that $I(tu_0,0) \leqslant 0$ and $\|(tu_0,0)\| > \rho$. □

Lemma 3.8. *Suppose that* (ϕ_1)-(ϕ_3), (F_1), (F_3) *and* (F_4) *hold. Then* $(C)_c$*-sequence in* W *is bounded.*

Proof. This proof is partially motivated by [12, Lemma 4.1]. Let $\{(u_n,v_n)\}$ be a $(C)_c$-sequence of I in W. Then, for n large enough, by (3.5) and (ϕ_3), we obtain

$$c+1 \geqslant I(u_n,v_n) - \left\langle I'(u_n,v_n), \left(\frac{1}{m_1}u_n, \frac{1}{m_2}v_n\right)\right\rangle$$

$$\begin{aligned}
&= \int_\Omega \Phi_1(|\nabla u_n|)dx + \int_\Omega \Phi_2(|\nabla v_n|)dx - \int_\Omega F(x,u_n,v_n)dx \\
&\quad - \frac{1}{m_1}\int_\Omega \phi_1(|\nabla u_n|)|\nabla u_n|^2 dx - \frac{1}{m_2}\int_\Omega \phi_2(|\nabla v_n|)|\nabla v_n|^2 dx \\
&\quad + \frac{1}{m_1}\int_\Omega F_u(x,u_n,v_n)u_n dx + \frac{1}{m_2}\int_\Omega F_v(x,u_n,v_n)v_n dx \\
&= \int_\Omega \left(\Phi_1(|\nabla u_n|) - \frac{1}{m_1}\phi_1(|\nabla u_n|)|\nabla u_n|^2\right) dx + \int_\Omega \left(\Phi_2(|\nabla u_n|) - \frac{1}{m_2}\phi_2(|\nabla v_n|)|\nabla v_n|^2\right) dx \\
&\quad + \int_\Omega \left(\frac{1}{m_1}F_u(x,u_n,v_n)u_n + \frac{1}{m_2}F_v(x,u_n,v_n)v_n - F(x,u_n,v_n)\right) dx \\
&\geqslant \int_\Omega \overline{F}(x,u_n,v_n)dx.
\end{aligned} \tag{3.8}$$

To prove the boundedness of $\{(u_n,v_n)\}$, arguing by contradiction, suppose that there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $\|(u_n,v_n)\| = \|\nabla u_n\|_{\Phi_1} + \|\nabla v_n\|_{\Phi_2} \to +\infty$. Next, we discuss the problem in two cases.

Case 1. Suppose that $\|\nabla u_n\|_{\Phi_1} \to +\infty$ and also $\|\nabla v_n\|_{\Phi_2} \to +\infty$. Let $\bar{u}_n = \frac{u_n}{\|\nabla u_n\|_{\Phi_1}}$ and $\bar{v}_n = \frac{v_n}{\|\nabla v_n\|_{\Phi_2}}$. Then $\{(\bar{u}_n,\bar{v}_n)\}$ is bounded in separable, reflexive Banach space W. Passing to a subsequence $\{(\bar{u}_n,\bar{v}_n)\}$, by Remark 3.2, there exists a point $(\bar{u},\bar{v}) \in W$ such that

- ⋆ $\bar{u}_n \rightharpoonup \bar{u}$ in $W_0^{1,\Phi_1}(\Omega)$, $\bar{u}_n \to \bar{u}$ in $L^{l_1\widetilde{l_\Gamma}}(\Omega)$ and in $L^{l_1\widetilde{m_\Gamma}}(\Omega)$, $\bar{u}_n(x) \to \bar{u}(x)$ a.e. in Ω;
- ⋆ $\bar{v}_n \rightharpoonup \bar{v}$ in $W_0^{1,\Phi_2}(\Omega)$, $\bar{v}_n \to \bar{v}$ in $L^{l_2\widetilde{l_\Gamma}}(\Omega)$ and in $L^{l_2\widetilde{m_\Gamma}}(\Omega)$, $\bar{v}_n(x) \to \bar{v}(x)$ a.e. in Ω.

Firstly, we assume that $[\bar{u} \neq 0] := \{x \in \Omega : \bar{u}(x) \neq 0\}$ or $[\bar{v} \neq 0] := \{x \in \Omega : \bar{v}(x) \neq 0\}$ has nonzero Lebesgue measure. It is clear that

$$|u_n| = |\bar{u}_n|\|\nabla u_n\|_{\Phi_1} \to +\infty \quad \text{in } [\bar{u} \neq 0]$$

and

$$|v_n| = |\bar{v}_n|\|\nabla v_n\|_{\Phi_2} \to +\infty \quad \text{in } [\bar{v} \neq 0].$$

Then, by (3.8), Remark 3.3 and Fatou's Lemma, we have

$$c + 1 \geqslant \int_\Omega \overline{F}(x,u_n,v_n)dx \to +\infty,$$

which is a contradiction. Next, we assume that both $[\bar{u} \neq 0]$ and $[\bar{v} \neq 0]$ have zero Lebesgue measure, that is, $\bar{u} = 0$ in $W_0^{1,\Phi_1}(\Omega)$ and $\bar{v} = 0$ in $W_0^{1,\Phi_2}(\Omega)$. By Lemma 2.4, we have

$$\begin{aligned}
\min\{\|\nabla u_n\|_{\Phi_1}^{l_1}, \|\nabla u_n\|_{\Phi_1}^{m_1}\} + \min\{\|\nabla v_n\|_{\Phi_2}^{l_2}, \|\nabla v_n\|_{\Phi_2}^{m_2}\} &\leqslant \int_\Omega \Phi_1(|\nabla u_n|)dx + \int_\Omega \Phi_2(|\nabla v_n|)dx \\
&= I(u_n,v_n) + \int_\Omega F(x,u_n,v_n)dx.
\end{aligned} \tag{3.9}$$

When n large enough, that is

$$\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2} \leqslant I(u_n,v_n) + \int_\Omega F(x,u_n,v_n)dx,$$

which is equivalent to

$$\begin{aligned}
1 &\leqslant \frac{I(u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} + \left(\int_{|(u_n,v_n)|\leqslant R} + \int_{|(u_n,v_n)|>R}\right) \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} dx \\
&= o_n(1) + \int_{|(u_n,v_n)|\leqslant R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} dx + \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} dx,
\end{aligned} \tag{3.10}$$

where R is a positive constant such that $R > r$ (see (F_4)) and

$$F(x,u,v) \geqslant 0, \quad \forall x \in \Omega, \quad |(u,v)| > R \quad (\text{by } (F_3)).$$

By the fact that F is continuous, there exists a constant $C_R > 0$ such that

$$|F(x,u,v)| < C_R, \quad \forall x \in \Omega, \quad |(u,v)| \leqslant R. \tag{3.11}$$

Then

$$\int_{|(u_n,v_n)|\leqslant R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} dx \leqslant \frac{C_R|\Omega|}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} = o_n(1). \tag{3.12}$$

Besides, it follows from Hölder's inequality that

$$\begin{aligned} &\int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2}} dx \\ &= \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{\frac{|u_n|^{l_1}}{|\bar{u}_n|^{l_1}} + \frac{|v_n|^{l_2}}{|\bar{v}_n|^{l_2}}} dx \\ &\leqslant \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{|u_n|^{l_1} + |v_n|^{l_2}} (|\bar{u}_n|^{l_1} + |\bar{v}_n|^{l_2}) dx \\ &\leqslant 2 \left\| \frac{F(x,u_n,v_n)}{|u_n|^{l_1} + |v_n|^{l_2}} \chi\{|(u_n,v_n)| > R\} \right\|_{\Gamma} \|(|\bar{u}_n|^{l_1} + |\bar{v}_n|^{l_2})\chi\{|(u_n,v_n)| > R\}\|_{\widetilde{\Gamma}}, \end{aligned} \tag{3.13}$$

where χ denotes the characteristic function which satisfies

$$\chi\{|(u_n(x),v_n(x))| > R\} = \begin{cases} 1 & \text{for } x \in \{x \in \Omega : |(u_n(x),v_n(x))| > R\}, \\ 0 & \text{for } x \in \{x \in \Omega : |(u_n(x),v_n(x))| \leqslant R\}. \end{cases}$$

For n large enough, by (3.3), (3.8) and the fact that $\overline{F}$ is continuous, we obtain

$$\int_\Omega \Gamma\left(\frac{F(x,u_n,v_n)}{|u_n|^{l_1} + |v_n|^{l_2}} \chi\{|(u_n,v_n)| > R\right) dx \leqslant c_3 \int_\Omega \overline{F}(x,u_n,v_n) dx + C \leqslant c_3(c+1) + C.$$

Then, for n large enough, by Lemma 2.4, there exists a constant $J > 0$ such that

$$\left\| \frac{F(x,u_n,v_n)}{|u_n|^{l_1} + |v_n|^{l_2}} \chi\{|(u_n,v_n)| > R\} \right\|_{\Gamma} \leqslant J. \tag{3.14}$$

Moreover, it is easy to see that

$$\|(|\bar{u}_n|^{l_1} + |\bar{v}_n|^{l_2})\chi\{|(u_n,v_n)| > R\}\|_{\widetilde{\Gamma}} \leqslant \|(|\bar{u}_n|^{l_1} + |\bar{v}_n|^{l_2})\|_{\widetilde{\Gamma}} \leqslant \||\bar{u}_n|^{l_1}\|_{\widetilde{\Gamma}} + \||\bar{v}_n|^{l_2}\|_{\widetilde{\Gamma}}.$$

By Lemma 2.3 and Lemma 2.5, (F_4) implies that N-function $\widetilde{\Gamma}$ satisfies a Δ_2-condition globally. Then, by (2.2), $\|u\|_{\widetilde{\Gamma}} \to 0$ as $\int_\Omega \widetilde{\Gamma}(|u|)dx \to 0$. It follows from Lemma 2.5 and $\star$ that

$$\begin{aligned} &\int_\Omega \widetilde{\Gamma}(|\bar{u}_n|^{l_1}) dx + \int_\Omega \widetilde{\Gamma}(|\bar{v}_n|^{l_2}) dx \\ &\qquad \leqslant \widetilde{\Gamma}(1) \int_\Omega \max\{|\bar{u}_n|^{l_1 \widetilde{l_\Gamma}}, |\bar{u}_n|^{l_1 \widetilde{m_\Gamma}}\} dx + \widetilde{\Gamma}(1) \int_\Omega \max\{|\bar{v}_n|^{l_2 \widetilde{l_\Gamma}}, |\bar{v}_n|^{l_2 \widetilde{m_\Gamma}}\} dx \\ &\qquad \leqslant \widetilde{\Gamma}(1) \left(\int_\Omega |\bar{u}_n|^{l_1 \widetilde{l_\Gamma}} dx + \int_\Omega |\bar{u}_n|^{l_1 \widetilde{m_\Gamma}} dx + \int_\Omega |\bar{v}_n|^{l_2 \widetilde{l_\Gamma}} dx + \int_\Omega |\bar{v}_n|^{l_2 \widetilde{m_\Gamma}} dx \right) = o_n(1), \end{aligned}$$

which implies

$$\|(|\bar{u}_n|^{l_1}+|\bar{v}_n|^{l_2})\chi\{|(u_n,v_n)|>R\}\|_{\widetilde{\Gamma}} \leqslant \||\bar{u}_n|^{l_1}\|_{\widetilde{\Gamma}}+\||\bar{v}_n|^{l_2}\|_{\widetilde{\Gamma}}=o_n(1). \tag{3.15}$$

By combining (3.12), (3.13), (3.14), (3.15) with (3.10), we get a contradiction.

Case 2. Suppose that $\|\nabla u_n\|_{\Phi_1} \leqslant C$ or $\|\nabla v_n\|_{\Phi_2} \leqslant C$ for some $C>0$ and all $n \in \mathbb{N}$. Without loss of generality, we assume that $\|\nabla u_n\|_{\Phi_1} \to +\infty$ and $\|\nabla v_n\|_{\Phi_2} \leqslant C$, for some $C>0$ and all $n \in \mathbb{N}$. Let $\bar{u}_n = \frac{u_n}{\|\nabla u_n\|_{\Phi_1}}$ and $\bar{v}_n = \frac{v_n}{\|\nabla u_n\|_{\Phi_1}}$. Then $\|\bar{u}_n\|_{0,\Phi_1}=1$ and $\|\bar{v}_n\|_{0,\Phi_2} \to 0$. Passing to a subsequences $\{(\bar{u}_n,\bar{v}_n)\}$, by Remark 3.2, there exist $\bar{u} \in W_0^{1,\Phi_1}(\Omega)$ and $v \in W_0^{1,\Phi_2}(\Omega)$ such that

- ⋆ $\bar{u}_n \rightharpoonup \bar{u}$ in $W_0^{1,\Phi_1}(\Omega)$, $\bar{u}_n \to \bar{u}$ in $L^{l_1\widetilde{l_\Gamma}}(\Omega)$ and in $L^{l_1\widetilde{m_\Gamma}}(\Omega)$, $\bar{u}_n(x) \to \bar{u}(x)$ a.e. in Ω;
- ⋆ $\bar{v}_n \to 0$ in $W_0^{1,\Phi_2}(\Omega)$, $\bar{v}_n \to 0$ in $L^{l_2\widetilde{l_\Gamma}}(\Omega)$ and in $L^{l_2\widetilde{m_\Gamma}}(\Omega)$, $\bar{v}_n(x) \to 0$ a.e. in Ω;
- ⋆ $v_n \rightharpoonup v$ in $W_0^{1,\Phi_2}(\Omega)$, $v_n \to v$ in $L^{l_2\widetilde{l_\Gamma}}(\Omega)$ and in $L^{l_2\widetilde{m_\Gamma}}(\Omega)$, $v_n(x) \to v(x)$ a.e. in Ω.

Similarly, we firstly assume that $[\bar{u} \neq 0]$ has nonzero Lebesgue measure. We can see that

$$|u_n| = |\bar{u}_n|\|\nabla u_n\|_{\Phi_1} \to +\infty \quad \text{in } [\bar{u} \neq 0].$$

Then, by (3.8), Remark 3.3 and Fatou's Lemma, we get a contradiction by

$$c+1 \geqslant \int_\Omega \overline{F}(x,u_n,v_n)dx \to +\infty.$$

Next, we suppose that $[\bar{u} \neq 0]$ has zero Lebesgue measure, that is, $\bar{u}=0$ in $W_0^{1,\Phi_1}(\Omega)$. By Lemma 2.5 and ⋆, we have

$$\begin{aligned}
\min\left\{\||v_n|^{l_2}\|_{\widetilde{\Gamma}}^{\widetilde{l_\Gamma}}, \||v_n|^{l_2}\|_{\widetilde{\Gamma}}^{\widetilde{m_\Gamma}}\right\} &\leqslant \int_\Omega \widetilde{\Gamma}(|v_n|^{l_2})dx \\
&\leqslant \widetilde{\Gamma}(1)\int_\Omega \max\{|v_n|^{l_2\widetilde{l_\Gamma}}, |v_n|^{l_2\widetilde{m_\Gamma}}\}dx \\
&\leqslant \widetilde{\Gamma}(1)\left(\int_\Omega |v_n|^{l_2\widetilde{l_\Gamma}}dx + \int_\Omega |v_n|^{l_2\widetilde{m_\Gamma}}dx\right) \to C,
\end{aligned}$$

which shows that there exists a constant $L>0$ such that

$$\||v_n|^{l_2}\|_{\widetilde{\Gamma}} \leqslant L, \quad \forall n \in \mathbb{N}. \tag{3.16}$$

When n large enough, (3.9) changed into

$$\|\nabla u_n\|_{\Phi_1}^{l_1} + M \leqslant I(u_n,v_n) + \int_\Omega F(x,u_n,v_n)dx + M,$$

where M is a positive constant with $M > 4JL$ (see (3.14) and (3.16)). Then, by (3.11), (3.14), (3.15), (3.16) and Hölder's inequality, above estimate means

$$\begin{aligned}
1 &\leqslant \frac{I(u_n,v_n)+M}{\|\nabla u_n\|_{\Phi_1}^{l_1}+M} + \int_\Omega \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1}+M}dx \\
&= o_n(1) + \int_{|(u_n,v_n)|\leqslant R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1}+M}dx + \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1}+M}dx \\
&= o_n(1) + \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{\|\nabla u_n\|_{\Phi_1}^{l_1}+M}dx \\
&\leqslant o_n(1) + \int_{|(u_n,v_n)|>R} \frac{F(x,u_n,v_n)}{|u_n|^{l_1}+|v_n|^{l_2}}\left(|\bar{u}_n|^{l_1}+\frac{1}{M}|v_n|^{l_2}\right)dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant o_n(1) + 2\left\| \frac{F(x,u_n,v_n)}{|u_n|^{l_1}+|v_n|^{l_2}}\chi\{|(u_n,v_n)|>R\}\right\|_{\Gamma} \left\| \left(|\bar{u}_n|^{l_1} + \frac{1}{M}|v_n|^{l_2}\right)\chi\{|(u_n,v_n)|>R\}\right\|_{\tilde{\Gamma}} \\
&\leqslant o_n(1) + 2J\left(\||\bar{u}_n|^{l_1}\|_{\tilde{\Gamma}} + \frac{1}{M}\||v_n|^{l_2}\|_{\tilde{\Gamma}}\right) \\
&\leqslant o_n(1) + 2J\left(o_n(1) + \frac{L}{M}\right) = o_n(1) + \frac{2JL}{M} < o_n(1) + \frac{1}{2},
\end{aligned}
$$

which is a contradiction. □

Lemma 3.9. *Suppose that* (ϕ_1)-(ϕ_3), (F_1), (F_3) *and* (F_4) *hold. Then I satisfies* $(C)_c$*-condition.*

Proof. Let $\{(u_n,v_n)\}$ be any $(C)_c$-sequence of I in W. Lemma 3.8 shows $\{(u_n,v_n)\}$ is bounded. Passing to a subsequence $\{(u_n,v_n)\}$, by Remark 3.2, there exists a point $(u,v)\in W$ such that

⋆ $u_n \rightharpoonup u$ in $W_0^{1,\Phi_1}(\Omega)$, $u_n \to u$ in $L^{\Psi_1}(\Omega)$, $u_n(x)\to u(x)$ a.e. in Ω;

⋆ $v_n \rightharpoonup v$ in $W_0^{1,\Phi_2}(\Omega)$, $v_n \to v$ in $L^{\Psi_2}(\Omega)$, $v_n(x)\to v(x)$ a.e. in Ω.

Now, we define operators

$\mathcal{F}: W_0^{1,\Phi_1}(\Omega)\to (W_0^{1,\Phi_1}(\Omega))^*$ by $\langle\mathcal{F}(u),\tilde{u}\rangle := \int_\Omega \phi_1(|\nabla u|)\nabla u\nabla\tilde{u}dx,\ u,\tilde{u}\in W_0^{1,\Phi_1}(\Omega)$,

and

$\mathcal{G}: W_0^{1,\Phi_2}(\Omega)\to (W_0^{1,\Phi_2}(\Omega))^*$ by $\langle\mathcal{G}(v),\tilde{v}\rangle := \int_\Omega \phi_2(|\nabla v|)\nabla v\nabla\tilde{v}dx,\ v,\tilde{v}\in W_0^{1,\Phi_2}(\Omega)$.

Then, we have

$$
\begin{aligned}
\langle\mathcal{F}(u_n),u_n-u\rangle &= \int_\Omega \phi_1(|\nabla u_n|)\nabla u_n\nabla(u_n-u)dx \\
&= \langle I'(u_n,v_n),(u_n-u,0)\rangle + \int_\Omega F_u(x,u_n,v_n)(u_n-u)dx.
\end{aligned} \tag{3.17}
$$

Equation (3.5) shows that

$$
|\langle I'(u_n,v_n),(u_n-u,0)\rangle| \leqslant \|I'(u_n,v_n)\|\|u_n-u\|_{0,\Phi_1}\to 0. \tag{3.18}
$$

By (F_1) and Hölder's inequality, we get

$$
\begin{aligned}
\left|\int_\Omega F_u(x,u_n,v_n)(u_n-u)dx\right| &\leqslant c_1\int_\Omega (1+\psi_1(|u_n|)+\widetilde{\Psi}_1^{-1}(\Psi_2(v_n)))|u_n-u|dx \\
&\leqslant 2c_1\|1+\psi_1(|u_n|)+\widetilde{\Psi}_1^{-1}(\Psi_2(v_n))\|_{\widetilde{\Psi}_1}\|u_n-u\|_{\Psi_1}.
\end{aligned} \tag{3.19}
$$

Condition (F_1) shows that functions Ψ_1 and $\widetilde{\Psi}_1$ are N-functions satisfying Δ_2-condition globally, which together with the convexity of N-function, Lemma 2.4, Remark 3.2 and the boundedness of $\{(u_n,v_n)\}$, implies that

$$
\int_\Omega \widetilde{\Psi}_1(1+\psi_1(|u_n|)+\widetilde{\Psi}_1^{-1}(\Psi_2(v_n)))dx \leqslant C\int_\Omega (1+\Psi_1(u_n)+\Psi_2(v_n))dx\leqslant C,
$$

which, together with Lemma 2.4 again, shows that

$$
\|1+\psi_1(|u_n|)+\widetilde{\Psi}_1^{-1}(\Psi_2(v_n))\|_{\widetilde{\Psi}_1}\leqslant C \tag{3.20}
$$

for some $C>0$. Moreover, ⋆ shows that

$$
\|u_n-u\|_{\Psi_1}\to 0. \tag{3.21}
$$

Then, combining (3.18), (3.19), (3.20), (3.21) with (3.17), we obtain

$$
\langle\mathcal{F}(u_n),u_n-u\rangle \to 0,\quad \text{as } n\to\infty.
$$

By [12, Proposition A.3], $\mathcal{F}$ is of the class (S_+), that is, if a sequence $\{u_n\} \subset W_0^{1,\Phi_1}(\Omega)$ satisfying

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty} \langle \mathcal{F}(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W_0^{1,\Phi_1}(\Omega)$. Thus $u_n \to u$ in $W_0^{1,\Phi_1}(\Omega)$. Similarly, we can obtain that $v_n \to v$ in $W_0^{1,\Phi_2}(\Omega)$. Therefore, $\{(u_n, v_n)\} \to (u, v)$ in W. □

Proof of Theorem 3.1. By Lemmas 3.6, 3.7, 3.9 and the obvious fact $I(0) = 0$, all conditions of Lemma 3.5 hold. Then system (1.1) possesses a nontrivial weak solution which is a critical point of I. □

4. Multiplicity

In this section, we present the following multiplicity result by using symmetric mountain pass theorem.

Theorem 4.1. *Assume that* (ϕ_1)-(ϕ_3), (F_0), (F_1),

(F_4) *and the following conditions hold:*

(F_5)

$$\lim_{|(u,v)|\to+\infty} \frac{F(x,u,v)}{|u|^{m_1} + |v|^{m_2}} = +\infty \ \textit{uniformly in} \ x \in \Omega;$$

(F_6) $F(x, -u, -v) = F(x, u, v)$, *for all* $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

Then system (1.1) *possesses infinitely many weak solutions* $\{(u_k, v_k)\}$ *such that*

$$I(u_k, v_k) := \int_\Omega \Phi_1(|\nabla u_k|)dx + \int_\Omega \Phi_2(|\nabla v_k|)dx - \int_\Omega F(x, u_k, v_k)dx \to +\infty, \textit{as } k \to \infty.$$

Now, we display the symmetric mountain pass theorem as follows.

Lemma 4.2 ([30, Theorem 9.12])**.** *Let* E *be an infinite-dimensional Banach space and let* $I \in C^1(E, \mathbb{R})$ *be even, satisfy* (PS)*-condition, and* $I(0) = 0$. *If* $E = V \oplus X$, *where* V *is finite dimensional, and* I *satisfies*

(I_1) *there are constants* ρ, $\alpha > 0$ *such that* $I|_{\partial B_\rho \cap X} \geqslant \alpha$, *and*

(I_2) *for each finite dimensional subspace* $\widetilde{E} \subset E$, *there is an* $R = R(\widetilde{E})$ *such that* $I \leqslant 0$ *on* $\widetilde{E}\backslash B_{R(\widetilde{E})}$, *where* $B_r = \{u \in E : \|u\| < r\}$,

then I *possesses an unbounded sequence of critical values.*

Since $W_0^{1,\Phi_i}(\Omega)$ $(i = 1, 2)$ are reflexive and separable Banach spaces, then there exist sequences $\{e_{ij} : j \in \mathbb{N}\} \subset W_0^{1,\Phi_i}(\Omega)$ $(i = 1, 2)$ and $\{e^*_{ij} : j \in \mathbb{N}\} \subset W_0^{1,\Phi_i}(\Omega)^*$ $(i = 1, 2)$ such that

$$W_0^{1,\Phi_i}(\Omega) = \overline{\operatorname{span}\{e_{ij} : j = 1, 2, \cdots\}}, \quad W_0^{1,\Phi_i}(\Omega)^* = \overline{\operatorname{span}\{e^*_{ij} : j = 1, 2, \cdots\}}, \quad i = 1, 2, \tag{4.1}$$

and

$$e^*_{in}(e_{im}) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases} \quad i = 1, 2, \tag{4.2}$$

(see [37, Section 17]). Define

$$Y_{i(k)} := \operatorname{span}\{e_{ij} : j = 1, \cdots, k\}, \quad Z_{i(k)} := \overline{\operatorname{span}\{e_{ij} : j = k + 1, \cdots\}}, \quad i = 1, 2. \tag{4.3}$$

Since the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^{\Psi_i}(\Omega)$ $(i = 1, 2)$ are compact, then, with a similar discussion as [13, Lemma 2.10], we can get

$$\alpha_{i(k)} := \sup\left\{\|z\|_{\Psi_i} : \|z\|_{0,\Phi_i} = 1, z \in Z_{i(k)}\right\} \to 0, \quad i = 1, 2, \text{ as } k \to \infty. \tag{4.4}$$

Lemma 4.3. *Let* $Y_{i(k)}$ *and* $Z_{i(k)}$ *be the subsets of* $W_0^{1,\Phi_i}(\Omega)$ *defined by* (4.3). *Then*

$$W_0^{1,\Phi_i}(\Omega) = Y_{i(k)} \oplus Z_{i(k)}, \quad i = 1, 2, \quad k \in \mathbb{N}.$$

Proof. Let us prove that $W_0^{1,\Phi_1}(\Omega) = Y_{1(k)} \oplus Z_{1(k)}$, $k \in \mathbb{N}$. Then, with the same arguments, we can prove that $W_0^{1,\Phi_2}(\Omega) = Y_{2(k)} \oplus Z_{2(k)}$, $k \in \mathbb{N}$. It is clear that both $Y_{1(k)}$ and $Z_{1(k)}$ are closed subspaces of $W_0^{1,\Phi_1}(\Omega)$ for $k \in \mathbb{N}$. For any $x \in W_0^{1,\Phi_1}(\Omega)$, by (4.1), there exists a sequence $\{x_n\} \subset \operatorname{span}\{e_{1j} : j = 1, 2, \cdots\}$ which converges to x. Let

$$x_n = \sum_{l=1}^{N(n)} a_{l,n} e_{1l}, \quad \text{where } a_{l,n} \in \mathbb{R}, N(n) \in \mathbb{N} \text{ and } N(n) \geqslant k.$$

Since $\{x_n\}$ is a Cauchy sequence, for any given $\delta > 0$ there exists an N such that

$$\|x_n - x_m - 0\|_{0,\Phi_1} = \|x_n - x_m\|_{0,\Phi_1} = \|\sum_{l=1}^{N(n)} a_{l,n} e_{1l} - \sum_{l=1}^{N(m)} a_{l,m} e_{1l}\|_{0,\Phi_1} < \delta, \quad (m, n > N). \tag{4.5}$$

According to the continuity of $e_{1j}^* \in W_0^{1,\Phi_1}(\Omega)^*$ $(j = 1, \cdots, k)$ and (4.2), for every $\varepsilon > 0$, by (4.5), we can choose $\delta > 0$ small enough such that

$$|e_{1j}^*(x_n - x_m) - e_{1j}^*(0)| = |e_{1j}^*(\sum_{l=1}^{N(n)} a_{l,n} e_{1l} - \sum_{l=1}^{N(m)} a_{l,m} e_{1l})| = |a_{j,n} - a_{j,m}| < \varepsilon,$$

which means that sequences $\{a_{j,n} : n = 1, 2, \cdots\}$ $(j = 1, \cdots, k)$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, then there exist $a_j \in \mathbb{R}(j = 1, \cdots, k)$ such that sequences $\{a_{j,n}\}$ converge to a_j, $j = 1, \cdots, k$, as $n \to \infty$. Now we can choose a sequence $\{\tilde{x}_n\} \subset \operatorname{span}\{e_{1j} : j = 1, 2, \cdots\}$ which satisfies

$$\tilde{x}_n = \sum_{j=1}^{k} a_j e_{1j} + \sum_{l=k+1}^{N(n)} a_{l,n} e_{1l}.$$

We conclude that the sequence $\{\tilde{x}_n\}$ converges to x, because the sequence $\{x_n\}$ converges to x and

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \tilde{x}_n\|_{0,\Phi_1} &= \lim_{n\to\infty} \|\sum_{l=1}^{k} a_{l,n} e_{1l} - \sum_{j=1}^{k} a_j e_{1j}\|_{0,\Phi_1} \\
&= \lim_{n\to\infty} \|\sum_{j=1}^{k} (a_{j,n} - a_j) e_{1j}\|_{0,\Phi_1} \\
&\leqslant \lim_{n\to\infty} \sum_{j=1}^{k} |(a_{j,n} - a_j)| \|e_{1j}\|_{0,\Phi_1} = 0.
\end{aligned}$$

Let

$$y = \sum_{j=1}^{k} a_j e_{1j} \quad \text{and} \quad z_n = \sum_{l=k+1}^{N(n)} a_{l,n} e_{1l}.$$

Then the sequence $\{z_n\} \subset \operatorname{span}\{e_{1j} : j = k+1, \cdots\}$ converges to $x - y$, which implies $x - y \in Z_{1(k)}$. Note that $y \in Y_{1(k)}$ and $x = y + (x - y)$. We get $W_0^{1,\Phi_1}(\Omega) = Y_{1(k)} + Z_{1(k)}$. Now, we prove $Y_{1(k)} \cap Z_{1(k)} = \{0\}$.

Let $x \in Y_{1(k)} \cap Z_{1(k)}$. Then there exists a sequence $\{z_n\} \subset \operatorname{span}\{e_{1j} : j = k+1, \cdots\}$ which converges to $x = \sum_{l=1}^{k} a_l e_{1l}$. Let

$$z_n = \sum_{l=k+1}^{N(n)} a_{l,n} e_{1l}.$$

By the continuity of $e_{1j}^* \in W_0^{1,\Phi_1}(\Omega)^*$ $(j = 1, \cdots, k)$ and (4.2), we have

$$\lim_{n\to\infty} e_{1j}^*(z_n) = \lim_{n\to\infty} e_{1j}^* \left(\sum_{l=k+1}^{N(n)} a_{l,n} e_{1l} \right) = 0 = e_{1j}^*(\sum_{l=1}^{k} a_l e_{1l}) = a_j, \quad \text{for } j = 1, \cdots, k,$$

which implies $x = 0$. Therefore, $W_0^{1,\Phi_1}(\Omega) = Y_{1(k)} \oplus Z_{1(k)}$, $k \in \mathbb{N}$. □

Lemma 4.4. *For Banach space $W = W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$, there exits a sequence $\{\eta_{(j)}\} \subset W$ defined by*

$$\eta_{(j)} = \begin{cases} (e_{1n}, 0) & \text{if } j = 2n-1, \\ (0, e_{2n}) & \text{if } j = 2n, \quad \text{for } n \in \mathbb{N}, \end{cases} \tag{4.6}$$

such that

(1)
$$W = \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}},$$

(2)
$$W = Y_k \oplus Z_k,$$

where
$$Y_k := \operatorname{span}\{\eta_{(j)} : j = 1, \cdots, k\} \quad \text{and} \quad Z_k := \overline{\operatorname{span}\{\eta_{(j)} : j = k+1, \cdots\}}.$$

Proof.
(1) Since W is complete, then it is obvious that $\overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}} \subseteq W$. Now, we prove that $W \subseteq \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}}$. For every $(u, v) \in W$, by (4.1), there exist sequences $\{u_n\} \subset \operatorname{span}\{e_{1j} : j = 1, 2, \cdots\}$ and $\{v_n\} \subset \operatorname{span}\{e_{2j} : j = 1, 2, \cdots\}$ which converge to u in $W_0^{1,\Phi_1}(\Omega)$ and v in $W_0^{1,\Phi_2}(\Omega)$, respectively. Let

$$u_n = \sum_{j=1}^{N_1(n)} a_{j,n} e_{1j} \quad \text{and } v_n = \sum_{j=1}^{N_2(n)} b_{j,n} e_{2j}, \quad \text{where} \quad a_{j,n}, b_{j,n} \in \mathbb{R} \quad \text{and} \quad N_1(n), N_2(n) \in \mathbb{N}.$$

Then
$$(u_n, v_n) = \left(\sum_{j=1}^{N_1(n)} a_{j,n} e_{1j}, \sum_{j=1}^{N_2(n)} b_{j,n} e_{2j} \right) = \sum_{j=1}^{N_1(n)} a_{j,n} (e_{1j}, 0) + \sum_{j=1}^{N_2(n)} b_{j,n} (0, e_{2j}).$$

By (4.6) and last equality, we get $\{(u_n, v_n)\} \subset \operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}$ and

$$\|(u_n, v_n) - (u, v)\| = \|(u_n - u, v_n - v)\| = \|u_n - u\|_{0,\Phi_1} + \|v_n - v\|_{0,\Phi_2} \to 0, \text{ as } n \to \infty,$$

which implies that $(u, v) \in \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}}$. So, $W \subseteq \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}}$. Therefore, $W = \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots\}}$.
(2) Combining Lemma 4.3 with (4.6), we can see that

$$\eta_{(n)} \notin \overline{\operatorname{span}\{\eta_{(j)} : j = 1, 2, \cdots \text{ and } j \neq n\}}, \quad \forall n \in \mathbb{N}.$$

Then there exists (see [37, Section 17]) a sequence $\{\eta_{(j)}^*\} \subset W^*$ such that

$$\eta^*_{(n)}(\eta_{(m)}) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

With the same discussion as Lemma 4.3, we can obtain that $W = Y_k \oplus Z_k$. □

Lemma 4.5. *Suppose that* (ϕ_1)-(ϕ_3) *and* (F_1) *hold. Then there are constants* $\rho, \alpha > 0$ *and* $k \in \mathbb{N}$ *such that* $I\mid_{\partial B_\rho \cap Z_{2k}} \geqslant \alpha$.

Proof. For $(u, v) \in Z_{2k}$, by (3.4), (4.4) and Lemma 2.4, we have

$$\begin{aligned}
I(u,v) &= \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx \\
&\geqslant \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - c_5\int_\Omega \Psi_1(u)dx - c_5\int_\Omega \Psi_2(v)dx - c_5|\Omega| \\
&\geqslant \min\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\} + \min\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\} \\
&\quad - c_5\max\{\|u\|_{\Psi_1}^{l_{\Psi_1}}, \|u\|_{\Psi_1}^{m_{\Psi_1}}\} - c_5\max\{\|v\|_{\Psi_2}^{l_{\Psi_2}}, \|v\|_{\Psi_2}^{m_{\Psi_2}}\} - c_5|\Omega| \\
&\geqslant \min\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\} + \min\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\} \\
&\quad - c_5\max\{\alpha_{1(k)}^{l_{\Psi_1}}\|\nabla u\|_{\Phi_1}^{l_{\Psi_1}}, \alpha_{1(k)}^{m_{\Psi_1}}\|\nabla u\|_{\Phi_1}^{m_{\Psi_1}}\} - c_5\max\{\alpha_{2(k)}^{l_{\Psi_2}}\|\nabla v\|_{\Phi_2}^{l_{\Psi_2}}, \alpha_{2(k)}^{m_{\Psi_2}}\|\nabla v\|_{\Phi_2}^{m_{\Psi_2}}\} - c_5|\Omega|.
\end{aligned}$$

Since $\alpha_{i(k)} \to 0$, $i = 1, 2$, as $k \to \infty$, then above inequality implies that there exist constants $\rho > 0$, lager $k \in \mathbb{N}$ and $\alpha > 0$ such that $I\mid_{\partial B_\rho \cap Z_{2k}} \geqslant \alpha$. □

Lemma 4.6. *Suppose that* (ϕ_1)-(ϕ_3) *and* (F_5) *hold. Then for each finite dimensional subspace* $\widetilde{W} \subset W$*, there exists a positive constant* $R = R(\widetilde{W})$ *such that* $I \leqslant 0$ *on* $\widetilde{W} \backslash B_{R(\widetilde{W})}$.

Proof. For each finite dimensional subspace $\widetilde{W} \subset W$, one has $\widetilde{W} \subseteq W_1 \times W_2$, where W_1 and W_2 are finite dimensional subspaces of $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively. Since any two norms in finite dimensional space is equivalent, then there exist positive constants $d_1, d_2.d_3, d_4$ such that

$$\begin{aligned}
d_1\|\nabla u\|_{\Phi_1} \leqslant \|u\|_{L^{m_1}(\Omega)} \leqslant d_2\|\nabla u\|_{\Phi_1}, \quad \forall u \in W_1, \\
d_3\|\nabla v\|_{\Phi_2} \leqslant \|v\|_{L^{m_2}(\Omega)} \leqslant d_4\|\nabla v\|_{\Phi_2}, \quad \forall v \in W_2.
\end{aligned} \tag{4.7}$$

Moreover, (F_5) and the continuity of function F imply that for any given constant $M > \max\left\{\frac{2}{d_1^{m_1}}, \frac{2}{d_3^{m_2}}\right\}$, there exists a constant $C_M > 0$ such that

$$F(x,u,v) \geqslant M(|u|^{m_1} + |v|^{m_2} - C_M, \quad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{4.8}$$

Then, by (4.7), (4.8) and Lemma 2.4, when $(u, v) \in \widetilde{W}$ we have

$$\begin{aligned}
I(u,v) &= \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx \\
&\leqslant \max\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\} + \max\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\} - M\int_\Omega (|u|^{m_1} + |v|^{m_2})dx + C_M|\Omega| \\
&\leqslant \|\nabla u\|_{\Phi_1}^{l_1} + \|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{l_2} + \|\nabla v\|_{\Phi_2}^{m_2} - M\|u\|_{L^{m_1}(\Omega)}^{m_1} - M\|v\|_{L^{m_2}(\Omega)}^{m_2} + C_M|\Omega| \\
&\leqslant \|\nabla u\|_{\Phi_1}^{l_1} + \|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{l_2} + \|\nabla v\|_{\Phi_2}^{m_2} - Md_1^{m_1}\|\nabla u\|_{\Phi_1}^{m_1} - Md_3^{m_2}\|\nabla v\|_{\Phi_2}^{m_2} + C_M|\Omega| \\
&= \|\nabla u\|_{\Phi_1}^{l_1} + \|\nabla v\|_{\Phi_2}^{l_2} - \|\nabla u\|_{\Phi_1}^{m_1}(Md_1^{m_1} - 1) - \|\nabla v\|_{\Phi_2}^{m_2}(Md_3^{m_2} - 1) + C_M|\Omega|.
\end{aligned}$$

Note that $l_i \leqslant m_i$ $(i = 1, 2)$. Then the above inequality implies that

$$\lim_{r\to\infty} \sup_{(u,v)\in\partial B_r \cap \widetilde{W}} I(u,v) = -\infty.$$

Thus, there exists an $R = R(\widetilde{W})$ such that $I \leqslant 0$ on $\widetilde{W} \backslash B_{R(\widetilde{W})}$. □

Proof of Theorem 4.1. Let $E = W$, $V = Y_{2k}$, and $X = Z_{2k}$. Obviously, $I(0) = 0$ and (F_6) implies I is even. By Lemmas 3.9, 4.5 and 4.6, all conditions of Lemma 4.2 hold. Then system (1.1) possesses infinitely many weak solutions $\{(u_k, v_k)\}$ which are critical points of I such that $I(u_k, v_k) \to +\infty$, as $k \to \infty$. □

5. Comparing with Theorem 1.1

In order to compare our results with Theorem 1.1, in this section, we present the results for equation (1.2), which correspond to Theorem 3.1 and Theorem 4.1.

Theorem 5.1. *Assume that conditions $(\phi_1)'$-$(\phi_3)'$, (f_0) in Theorem 1.1 and the following conditions hold:*

(f_1) *there exist a continuous function $\psi : [0, +\infty) \to \mathbb{R}$, which satisfies that $\Psi := \int_0^{|t|} \psi(s)ds,\ t \in \mathbb{R}$ is an N-function increasing essentially more slowly than Φ_* near infinity, moreover,*

$$m < l_\Psi := \inf_{t>0} \frac{t\psi(t)}{\Psi(t)} \leqslant \sup_{t>0} \frac{t\psi(t)}{\Psi(t)} =: m_\Psi < +\infty,$$

such that

$$|f(x,t)| \leqslant C(1 + \psi(|t|))$$

for all $(x, t) \in \Omega \times \mathbb{R}$, where constant $C > 0$;

(f_2)

$$\limsup_{t\to 0} \frac{|F(x,t)|}{\Phi(t)} = \lambda < \lambda_1, \quad \textit{uniformly in } x \in \Omega,$$

where and in the sequel $F(x,t) = \int_0^t f(x,s)ds, t \in \mathbb{R}$ and $\lambda_1 > 0$ satisfies the Poincaré inequality given by

$$\lambda_1 \int_\Omega \Phi(u)dx \leqslant \int_\Omega \Phi(|\nabla(u)|)dx, \quad \forall u \in W_0^{1,\Phi}(\Omega);$$

(f_3)

$$\lim_{t\to\infty} \frac{F(x,t)}{\Phi(t)} = +\infty \quad \textit{uniformly in } x \in \Omega;$$

(f_4) *there exists a continuous function $\gamma : [0, \infty) \to \mathbb{R}$ and it satisfies that $\Gamma(t) := \int_0^{|t|} \gamma(s)ds,\ t \in \mathbb{R}$ is an N-function with*

$$1 < l_\Gamma := \inf_{t>0} \frac{t\gamma(t)}{\Gamma(t)} \leqslant \sup_{t>0} \frac{t\gamma(t)}{\Gamma(t)} =: m_\Gamma < +\infty,$$

and function $H(t) := |t|^{\frac{ll_\Gamma}{l_\Gamma - 1}},\ t \in \mathbb{R}$ increases essentially more slowly than Φ_* near infinity such that

$$\Gamma\left(\frac{F(x,t)}{|t|^l}\right) \leqslant C\overline{F}(x,t), \quad x \in \Omega, \quad |t| \geqslant R,$$

where constants $C, R > 0$ and

$$\overline{F}(x,t) := tf(x,t) - mF(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Then (1.2) possesses a nontrivial weak solution.

Theorem 5.2. *Assume that $(\phi_1)'$-$(\phi_3)'$, (f_0), (f_1), (f_4) and the following conditions hold:*

(f_5)

$$\lim_{t\to\infty} \frac{F(x,t)}{|t|^m} = +\infty \quad \textit{uniformly in } x \in \Omega;$$

(f_6) $F(x,-t) = F(x,t)$, *for all* $(x,t) \in \Omega \times \mathbb{R}$.

Then (1.2) *possesses infinitely many weak solutions* $\{u_k\}$ *such that*

$$J(u_k) := \int_\Omega \Phi(|\nabla u_k|)dx - \int_\Omega F(x,u_k)dx \to +\infty, \quad \text{as } k \to \infty.$$

Remark 5.3. Theorem 5.1 improves the result (i) of Theorem 1.1 in Section 1. In fact, by (2) in Lemma 2.3 and (2) in Lemma 2.6, $(f_1)'$ shows that for any given constant $c > 0$, it holds that

$$\lim_{t\to\infty} \frac{\Psi(ct)}{\Phi_*(t)} \leqslant \frac{\Psi(c)\max\{t^{l_\Psi}, t^{m_\Psi}\}}{\Phi_*(1)\min\{t^{l^*}, t^{m^*}\}} = 0,$$

which implies Ψ increases essentially more slowly than Φ_*. Moreover, comparing $(f_1)'$ with (f_1), we can see that condition $m < l^*$ is not necessary in (f_1), see example below. It is obvious that $(f_2)'$ is equivalent to (f_2). Since $(f_3)'$ is equivalent to (f_5), which, together with (2) in Lemma 2.3, implies (f_3). In $(f_4)'$, condition $\frac{N}{l} < l_r$ shows $\frac{ll_r}{l_r - 1} < l^*$, which implies that function $H(t) := |t|^{\frac{ll_r}{l_r-1}}, t \in \mathbb{R}$ increases essentially more slowly than Φ_*.

Next, in order to offer an example that satisfies our conditions but do not satisfy the conditions in Theorem 1.1, we firstly need the following lemma.

Lemma 5.4. *Under the assumptions* $(\phi_1)'$-$(\phi_3)'$, *let* $\underline{m} := \liminf_{t\to+\infty} \frac{t\phi(t)}{\Phi(t)}$. *Then, function* $\Upsilon(t) := |t|^p, t \in \mathbb{R}$ *increases essentially more slowly than* Φ_* *near infinity, where* $1 < p < \underline{m}^* := \frac{\underline{m}N}{N-\underline{m}}$.

Proof. Choose $a > 0$ such that $a^* := \frac{aN}{N-a} = p$. It follows from the fact $p < \underline{m}^*$ that $a < \underline{m}$ and $p < (\frac{a+\underline{m}}{2})^* := \frac{\frac{a+\underline{m}}{2}N}{N - \frac{a+\underline{m}}{2}}$. Then, there exists a constant $K > 0$ such that

$$\frac{t\phi(t)}{\Phi(t)} \geqslant \frac{1}{2}(a + \underline{m}), \quad \forall t \geqslant K,$$

which implies that

$$\Phi(t) \geqslant C_1|t|^{\frac{1}{2}(a+\underline{m})}, \quad \forall t \geqslant K$$

for some $C_1 > 0$. Then, by the definition of Φ_*, when $t \geqslant \Phi(K)$ we have

$$\begin{aligned}
\Phi_*^{-1}(t) &= \Phi_*^{-1}(\Phi(K)) + \int_{\Phi(K)}^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds \\
&\leqslant \Phi_*^{-1}(\Phi(K)) + \left(\frac{1}{C_1}\right)^{\frac{2}{a+\underline{m}}} \int_{\Phi(K)}^t s^{(\frac{2}{a+\underline{m}} - \frac{N+1}{N})} ds \\
&= \Phi_*^{-1}(\Phi(K)) + \left(\frac{1}{C_1}\right)^{\frac{2}{a+\underline{m}}} \frac{N(a+\underline{m})}{2N-(a+\underline{m})} \left(t^{\frac{2N-(a+\underline{m})}{N(a+\underline{m})}} - \Phi(K)^{\frac{2N-(a+\underline{m})}{N(a+\underline{m})}}\right) \\
&\leqslant C_2 t^{\frac{2N-(a+\underline{m})}{N(a+\underline{m})}}
\end{aligned}$$

for some $C_2 > 0$, which implies that

$$\Phi_*(t) \geqslant \left(\frac{1}{C_2}\right)^{\frac{N(a+\underline{m})}{2N-(a+\underline{m})}} t^{\frac{N(a+\underline{m})}{2N-(a+\underline{m})}} = \left(\frac{1}{C_2}\right)^{(\frac{a+\underline{m}}{2})^*} t^{(\frac{a+\underline{m}}{2})^*}, \quad \forall t \geqslant \Phi_*^{-1}(\Phi(K)).$$

Thus, for any constant $c > 0$, we have

$$\lim_{t\to+\infty} \frac{\Upsilon(ct)}{\Phi_*(t)} \leqslant \lim_{t\to+\infty} c^p C_2^{(\frac{a+\underline{m}}{2})^*} t^{[p-(\frac{a+\underline{m}}{2})^*]} = 0,$$

which implies that $\Upsilon(t) = |t|^p, t \in \mathbb{R}$ increases essentially more slowly than Φ_* near infinity. $\square$

Example 5.5. In (1.2), let $N = 6$, $\phi(|t|)t = 2t + 4t^3, t > 0$ and $f(x,t) = 5|t|^3 t, t \in \mathbb{R}$. Then some simple computations show that

$$\Phi(t) = t^2 + t^4 \text{ and } F(x,t) = \overline{F}(x,t) = |t|^5, \quad t \in \mathbb{R},$$

and

$$l = 2, \quad m = \underline{m} = 4, \quad l^* = 3 \quad \text{and} \quad m^* = \underline{m}^* = 12.$$

Clearly, assumptions $(\phi_1)'$-$(\phi_3)'$, (f_0), (f_2) and (f_3) hold. In (f_1), choose function $\psi(t) = 10t^9$, $t \geqslant 0$ satisfying $\Psi(t) = |t|^{10}, t \in \mathbb{R}$ and $l_\Psi = m_\Psi = 10$. Then, (f_1) holds because Lemma 5.4 shows Ψ increases essentially more slowly than Φ_* near infinity. However, the fact $l^* < m$ implies that $(f_1)'$ does not hold. In (f_4), choose function $\gamma(t) = \frac{4}{3}t^{\frac{1}{3}}$, $t \geqslant 0$ satisfying $\Gamma(t) = |t|^{\frac{4}{3}}$, $t \in \mathbb{R}$ and $l_\Psi = m_\Psi = \frac{4}{3}$. Then,

$$\Gamma\left(\frac{F(x,t)}{|t|^l}\right) = |t|^4 \leqslant |t|^5, \forall |t| \geqslant 1,$$

and Lemma 5.4 shows $H(t) = |t|^{\frac{l l_\Gamma}{l_\Gamma - 1}} = t^8, t \in \mathbb{R}$ increases essentially more slowly than Φ_* near infinity.

Remark 5.6. In [12], based on Lieberman's interior regularity and boundary regularity results (see [25, 26]), maximum principle (see [29]) and other tools, Carvalho et al. investigated some important properties of solutions like (ii) in Theorem 1.1 under a stronger condition $(\phi_4)'$. However, in the system case, those methods for the scalar case may not be useful any more. To extend the result (ii) in Theorem 1.1 to system (1.1), new methods maybe need to be established. We will try to do it in our future work.

6. Examples

In this section, we present some examples to illustrate our main results. For system (1.1), ϕ_i $(i = 1,2)$ can be chosen from the following cases which satisfy all (ϕ_1)-(ϕ_3) type conditions:

1. Let $\phi(t) = t^{p-1}$, $t > 0$, $1 < p+1 < N$. In this case, simple computations show that $l = m = p+1$.
2. Let $\phi(t) = t^{p-1} + t^{q-1}$, $t > 0$, $1 < p+1 < q+1 < N < \frac{(p+1)(q+1)}{q-p}$. In this case, simple computations show that $l = p+1$, $m = q+1$.
3. Let $\phi(t) = 2p(1+t^2)^{p-1}$, $t > 0$, $1 \leqslant p < \frac{N}{2}$. In this case, simple computations show that $l = 2$, $m = 2p$.
4. Let $\phi(t) = \frac{t^{q-1}}{\log(1+t^p)}$, $t > 0$, $1 < p+1 < q+1 < N < \frac{(q-p+1)(q+1)}{p}$. In this case, simple computations show that $l = q-p+1$, $m = q+1$.
5. Let $\phi(t) = t^{q-1}\log(1+t^p)$, $t > 0$, $p, q > 0$ and $p+q+1 < N < \frac{(q+1)(p+q+1)}{p}$. In this case, simple computations show that $l = q+1$, $m = p+q+1$.

Based on this fact, it is easy to choose ϕ_i $(i = 1,2)$ and N such that (ϕ_1)-(ϕ_3) hold with $l_i^* > m_i \geqslant 4$ $(i = 1,2)$, and $\max\{\frac{N}{l_1}, \frac{N}{l_2}\} < \min\{\frac{m_1}{m_1 - l_1}, \frac{m_2}{m_2 - l_2}\}$. Then,

$$F(x,u,v) = |u|^{m_1}\log(1+|u|) + |v|^{m_2}\log(1+|v|) + |u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}},$$

satisfies (F_0)-(F_6), where constant $\epsilon > 0$ satisfying $\epsilon < \frac{2l_1^* l_2^* - m_1 l_2^* - m_2 l_1^*}{l_1^* + l_2^*}$ and $\max\{\frac{N}{l_1}, \frac{N}{l_2}\} < \min\{\frac{m_1}{m_1 - l_1 + \epsilon}, \frac{m_2}{m_2 - l_2 + \epsilon}\}$. In fact,

$$F_u(x,u,v) = m_1|u|^{m_1-2}u\log(1+|u|) + \frac{|u|^{m_1-1}u}{1+|u|} + \frac{m_1+\epsilon}{2}|u|^{\frac{m_1+\epsilon-4}{2}}|v|^{\frac{m_2+\epsilon}{2}}u,$$

$$F_v(x,u,v) = m_2|v|^{m_2-2}v\log(1+|v|) + \frac{|v|^{m_2-1}v}{1+|v|} + \frac{m_2+\epsilon}{2}|u|^{\frac{m_2+\epsilon}{2}}|v|^{\frac{m_2+\epsilon-4}{2}}v,$$

then

$$\overline{F}(x,u,v) = \frac{|u|^{m_1+1}}{m_1(1+|u|)} + \frac{|v|^{m_2+1}}{m_2(1+|v|)} + \frac{(m_1+m_2)\epsilon}{2m_1m_2}|u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}} \geqslant \frac{|u|^{m_1+1}}{m_1(1+|u|)} + \frac{|v|^{m_2+1}}{m_2(1+|v|)}.$$

It is obvious that F satisfies (F_0) and (F_6). Since

$$\lim_{|(u,v)|\to 0} \frac{F(x,u,v)}{|u|^{m_1}+|v|^{m_2}} = 0 \quad \text{and} \quad \lim_{|(u,v)|\to+\infty} \frac{F(x,u,v)}{|u|^{m_1}+|v|^{m_2}} = +\infty,$$

which, together with (2) in Lemma 2.3, shows that (F_2), (F_3) and (F_5) hold. Since $0 < \epsilon < \frac{2l_1^*l_2^*-m_1l_2^*-m_2l_1^*}{l_1^*+l_2^*}$, by the Young's inequality, there exist $\psi_i(t) = t^{a_i-1}$, $t > 0$, $m_i < a_i < l_i^*$, $(i = 1,2)$ such that (F_1) holds. Next, we check (F_4). Choose $\gamma(t) = kt^{k-1}$, $t > 0$, where $\max\{\frac{N}{l_1}, \frac{N}{l_2}\} < k \leqslant \min\{\frac{m_1}{m_1-l_1+\epsilon}, \frac{m_2}{m_2-l_2+\epsilon}\}$. Then $\Gamma(t) = |t|^k$, $t \in \mathbb{R}$ and $l_\Gamma = m_\Gamma = k$. Since $\max\{\frac{N}{l_1}, \frac{N}{l_2}\} < k$, similar arguments as Remark 5.3 show that $H_i(t) := |t|^{\frac{l_i l_\Gamma}{l_\Gamma - 1}}, t \in \mathbb{R}$ $(i = 1,2)$ increase essentially more slowly than Φ_{i*} $(i = 1,2)$ near infinity, respectively. Moreover,

$$\begin{aligned}
&\limsup_{|(u,v)|\to\infty} \left(\frac{|F(x,u,v)|}{|u|^{l_1}+|v|^{l_2}}\right)^k \frac{1}{\overline{F}(x,u,v)} \\
&\leqslant \limsup_{|(u,v)|\to\infty} \frac{\left(|u|^{m_1}\log(1+|u|) + |v|^{m_2}\log(1+|v|) + |u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}}\right)^k}{\left(|u|^{l_1}+|v|^{l_2}\right)^k \left(\frac{|u|^{m_1+1}}{m_1(1+|u|)} + \frac{|v|^{m_2+1}}{m_2(1+|v|)}\right)} \\
&\leqslant C_k \limsup_{|(u,v)|\to\infty} \frac{|u|^{km_1}(\log(1+|u|))^k + |v|^{km_2}(\log(1+|v|))^k + |u|^{k(m_1+\epsilon)} + |v|^{k(m_2+\epsilon)}}{\frac{|u|^{kl_1+m_1+1}}{m_1(1+|u|)} + \frac{|v|^{kl_2+m_2+1}}{m_2(1+|v|)}} \\
&< +\infty,
\end{aligned}$$

which shows that there exist constants $c_3, r > 0$ such that (3.3) holds.

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