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On multi-valued weak quasi-contractions in b-metric spaces

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Abstract

We introduce some generalizations of the contractions for multi-valued mappings and establish some fixed point theorems for multi-valued mappings in b-metric spaces. Our results generalize and extend several known results in b-metric and metric spaces. Some examples are included which illustrate the cases when the new results can be applied while the old ones cannot.

Keywords: Fixed points, b-metric space, set-valued mapping.
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1. Introduction and preliminaries

In the papers of Bakhtin [4] and Czerwik [10, 11], the notion of b-metric space has been introduced and some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces proved. Successively, this notion has been reintroduced by Khamsi [18] and Khamsi and Hussain [19], with the name of metric-type space. Several results have appeared in metric-type spaces, we refer to [3, 9–16, 19, 23–26].

Definition 1.1. Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric with coefficient s if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space.

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Note that a metric space is included in the class of b-metric spaces. The concept of convergence in such spaces is similar to that of the standard metric spaces. The b-metric space (X, d, s) is called complete if every Cauchy sequence of elements from (X, d, s) is convergent. Some examples of b-metric spaces can be seen in [3, 10, 11].

Let (X, d, s) be a b-metric space, let us denote $CB(X)$ the collection of nonempty closed bounded subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X. For $x \in X$ and $A, B \in CL(X)$, we define

$$d(x, A) = \inf_{a\in A} d(x, a), \quad D(A, B) = \sup\{d(a, B) : a \in A\}.$$

Then the generalized Pompeiu-Hausdorff b-metric H on $CL(X)$ induced by d is defined as

$$H(A, B) = \begin{cases} \max\{D(A, B), D(B, A)\}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise} \end{cases}$$

for all $A, B \in CL(X)$. The following results are useful for some of the proofs in the paper.

Theorem 1.2 ([11]). *If (X, d, s) is a complete b-metric space with coefficient s, then $(CL(X), H)$, where H means the Pompeiu-Hausdorff b-metric induced by d, is also a complete b-metric space with coefficient s.*

Lemma 1.3 ([11]). *Let (X, d, s) be a b-metric space with coefficient s and $A, B \in CB(X)$. Then for each $a \in A$ and $\epsilon > 0$ there exists a $b \in B$ such that*

$$d(a, b) \leqslant H(A, B) + \epsilon.$$

Lemma 1.4 ([11]). *Let (X, d, s) be a b-metric space with coefficient s. For any $A, B, C \in CL(X)$ and any $x, y \in X$, we have the following:*

1. $d(x, A) \leqslant d(x, a)$ *for all* $a \in A$;
2. $d(x, B) \leqslant H(A, B)$ *for all* $x \in A$;
3. $d(x, A) \leqslant s[d(x, y) + d(y, A)]$;
4. $H(A, C) \leqslant s[H(A, B) + H(B, C)]$;
5. $d(x, A) = 0 \Leftrightarrow x \in A$.

Lemma 1.5 ([20]). *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X, d, s), having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leqslant \gamma d(x_n, x_{n-1})$$

for every $n \in \mathbb{N}$, is Cauchy.

Lemma 1.6 ([14]). *Let (X, d, s) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leqslant \lim_{n\to\infty} \inf d(x_n, y_n) \leqslant \lim_{n\to\infty} \sup d(x_n, y_n) \leqslant s^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leqslant \lim_{n\to\infty} \inf d(x_n, z) \leqslant \lim_{n\to\infty} \sup d(x_n, z) \leqslant s d(x, z).$$

Definition 1.7 ([20]). A mapping $T : X \to CB(X)$, where (X, d, s) is a b-metric space, is called closed if for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements from X and $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, and $y_n \in T(x_n)$ for every $n \in \mathbb{N}$, we have $y \in T(x)$.

Definition 1.8 ([20]). Given a b-metric space (X, d, s), the b-metric d is called $*$-continuous if for every $A \in CB(X)$, every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} d(x_n, A) = d(x, A)$.

2. The generalizations of Nadler contraction for multi-valued mappings

In this section, we introduce the following condition of contractions for multi-valued mappings in b-metric space (X, d). A map $T : X \to CB(X)$ is called weak quasi-contraction or (θ, k, L)-quasi-contraction if there exist constant $\theta \in (0, 1), k \in [0, 1]$ and $L \geqslant 0$ such that

$$H(Tx, Ty) \leqslant \theta M_{k,T}(x, y) + Ld(y, Tx) \tag{2.1}$$

for all $x, y \in X$, where $M_{k,T}(x, y) = \max\{d(x, y), kd(x, Tx), kd(y, Ty)\}$.

Remark 2.1. Due to the symmetry of the distance, the weak quasi-contraction condition (2.1) implicitly includes the following dual one

$$H(Tx, Ty) \leqslant \theta M_{1,T}(x, y) + Ld(x, Ty) \tag{2.2}$$

for all $x, y \in X$, obtained from (2.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y. Consequently, in the concrete applications it is necessary to check that both conditions (2.1) and (2.2) are satisfied.

Again as in [8], Aydi et al. [3, Theorem 2.2] introduced the q-set-valued quasi-contraction in the complete b-metric space. The multi-valued map $T : X \to CB(X)$ is said to be a q-multi-valued quasi-contraction if

$$H(Tx, Ty) \leqslant kM(x, y) \tag{2.3}$$

for any $x, y \in X$, where $0 \leqslant k < 1$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Recently, Miculescu and Mihail [20, Theorem 3.3] used the following version of q-set-valued quasi-contraction in the complete b-metric spaces. Let $T : X \to CB(X)$ has a property that there exist $c, d \in [0, 1]$ and $\alpha \in [0, 1)$ such that

$$H(T(x), T(y)) \leqslant \alpha N_{c,d}(x, y) \text{ for all } x, y \in X, \tag{2.4}$$

where

$$N_{c,d}(x, y) = \max\{d(x, y), cd(x, T(x)), cd(y, T(y)), \frac{d}{2}(d(x, T(y)) + d(y, T(x)))\}.$$

Remark 2.2. Notice that since

$$\frac{d(x, Ty) + d(y, Tx)}{2} \leqslant \max\{d(x, Ty), d(y, Tx)\},$$

we then have that $N_{c,d}(x, y) \leqslant M(x, y)$ for all $x, y \in X$.

The following example shows that in b-metric spaces a weak quasi-contraction may not be a q-quasi-contraction in the sense of Aydi et al. and may not be a contraction in the sense of Miculescu and Mihail.

Example 2.3. Let $X = \mathbb{R}$, $d(x, y) = (x - y)^2$ for all $x, y \in X$ and $T : X \to CB(X)$ be defined by $Tx = \{x\}$. We obtain that d is b-metric (with $s = 2$), but (X, d) is not a metric space. For $x = 0, y = 1$, and $z = 2$, we have

$$d(x, z) = 4 > 2 = d(x, y) + d(y, z).$$

Then (X, d) is a complete b-metric space. Recall that for all $x, y \in X$,

$$(x - y)^2 = H(Tx, Ty) \leqslant a \max\{d(x, y), kd(x, Tx), kd(y, Ty)\} + Ld(y, Tx) = (a + L)(x - y)^2 = (x - y)^2$$

for $a = \frac{1}{6}$, $k = 1$, and $L = \frac{5}{6}$, we have that T satisfies condition (2.1) (note that $d(y, Tx) = d(x, Ty)$). Suppose that T is a q-quasi-contraction in the sense of Aydi et al.. Thus, there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$(x - y)^2 = H(Tx, Ty) \leqslant \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} = \alpha(x - y)^2 < (x - y)^2 \text{ if } x \neq y.$$

This is a contradiction. Similarly, suppose that T is a q-quasi-contraction in the sense of Miculescu and Mihail. Thus, there exist $c, d \in [0, 1]$ and $\alpha \in [0, 1)$ such that:

$$H(T(x), T(y)) \leqslant \alpha N_{c,d}(x, y) \text{ for all } x, y \in X.$$

So,

$$\begin{aligned}(x-y)^2 = H(Tx, Ty) &\leqslant \alpha \max\{d(x, y), cd(x, Tx), cd(y, Ty), \frac{d}{2}(d(x, Ty) + d(y, Tx))\}\\ &= \alpha \max\{(x-y)^2, d(x-y)^2\} < (x-y)^2 \text{ if } x \neq y.\end{aligned}$$

This is a contradiction.

The aim of this paper is to obtain sufficient conditions for the existence of fixed point for the multi-valued mappings which satisfy condition (2.1) in b-metric spaces.

3. Main results

The following theorem is our main result, which can be regarded as an extension of Nadler's fixed point theorem [21] in b-metric space.

Theorem 3.1. *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *weak quasi-contraction for which there exist* $\theta \in (0, 1), k \in [0, 1]$ *and* $L \geqslant 0$ *such that*

$$H(Tx, Ty) \leqslant \theta \max\{d(x, y), kd(x, Tx), kd(y, Ty)\} + Ld(y, Tx) \tag{3.1}$$

for all $x, y \in X$. *Then there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *which converges to some point* $x^* \in X$ *such that* $x_{n+1} \in T(x_n)$ *for every* $n \in \mathbb{N}$. *Also,* x^* *is a fixed point of* T *if any of the following conditions are satisfied:*

(i) T *is closed;*

(ii) d *is* $*$*-continuous;*

(iii) $s\theta k < 1$.

Proof. Let $x_0 \in X$. Choose $x_1 \in Tx_0$. Let

$$\epsilon = \frac{1-\theta}{1+\theta} H(Tx_0, Tx_1).$$

If $H(Tx_0, Tx_1) = 0$, we obtain $Tx_0 = Tx_1$ and $x_1 \in Tx_1$. In this case the proof is completed. So, we may assume $\epsilon > 0$. From Lemma 1.3 we obtain that there is a point $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leqslant H(Tx_0, Tx_1) + \epsilon = \frac{2}{1+\theta} H(Tx_0, Tx_1).$$

Similarly, there is a point $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leqslant H(Tx_1, Tx_2) + \epsilon,$$

where

$$\epsilon = \frac{1-\theta}{1+\theta} H(Tx_1, Tx_2).$$

If $H(Tx_1, Tx_2) = 0$, we obtain $Tx_2 = Tx_1$ and $x_2 \in Tx_2$. In this case, the proof is completed. So, we may assume $\epsilon > 0$. Hence,

$$d(x_2, x_3) \leqslant \frac{2}{1+\theta} H(Tx_1, Tx_2).$$

Continuing this process we produce a sequence (x_n) of points of X such that

$$x_{n+1} \in Tx_n \text{ for every } n \in \mathbb{N}, \tag{3.2}$$

and

$$d(x_n, x_{n+1}) \leqslant \frac{2}{1+\theta} H(Tx_{n-1}, Tx_n) \text{ for every } n \in \mathbb{N}. \tag{3.3}$$

From condition (3.1), we obtain

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leqslant \theta \max\{d(x_{n-1}, x_n), kd(x_{n-1}, Tx_{n-1}), kd(x_n, Tx_n)\} + Ld(x_n, Tx_{n-1}) \\ &\leqslant \theta \max\{d(x_{n-1}, x_n), kd(x_{n-1}, x_n), kd(x_n, x_{n+1})\} + Ld(x_n, x_n) \\ &= \theta \max\{d(x_{n-1}, x_n), kd(x_n, x_{n+1})\}. \end{aligned}$$

If $\max\{d(x_{n-1}, x_n), kd(x_n, x_{n+1})\} = kd(x_n, x_{n+1})$, from (3.3) we obtain a contradiction $1 < \frac{2\theta k}{1+\theta}$. So,

$$d(x_n, x_{n+1}) \leqslant \frac{2\theta}{1+\theta} d(x_{n-1}, x_n).$$

Now, since $\frac{2\theta}{1+\theta} < 1$, from Lemma 1.5 we obtain that the sequence (x_n) is a Cauchy sequence. Since (X, d, s) is complete, the sequence (x_n) converges to some point $x^* \in X$.

(i) Suppose that T is closed. From Definition 1.7 and (3.2) we obtain $x^* \in Tx^*$.

(ii) Suppose that d is $*$-continuous. Then, we have

$$\lim_{n\to\infty} d(x_n, Tx^*) = d(x^*, Tx^*). \tag{3.4}$$

From Lemma 1.4 and (3.1) we have

$$\begin{aligned} d(x_{n+1}, Tx^*) \leqslant H(Tx_n, Tx^*) &\leqslant \theta \max\{d(x_n, x^*), kd(x_n, Tx_n), kd(x^*, Tx^*)\} + Ld(x^*, Tx_n) \\ &\leqslant \theta \max\{d(x_n, x^*), kd(x_n, x_{n+1}), kd(x^*, Tx^*)\} + Ld(x^*, x_{n+1}). \end{aligned}$$

Hence, using (3.4) we obtain

$$d(x^*, Tx^*) \leqslant \theta k d(x^*, Tx^*).$$

Since $\theta k < 1$, we conclude that $d(x^*, Tx^*) = 0$ and from Lemma 1.4 we obtain $x^* \in Tx^*$.

(iii) We follow some ideas from [20]. Let

$$d(x^*, T(x^*)) \leqslant \overline{\lim_{n\to\infty}} d(x_n, T(x^*)).$$

Then there exists a subsequence (x_{n_k}) of (x_n) such that for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $d(u, T(u)) - \epsilon \leqslant d(x_{n_{k+1}}, T(u))$ for every $k \geqslant k_0$. Since

$$d(x_{n+1}, Tx^*) \leqslant \theta \max\{d(x_n, x^*), kd(x_n, x_{n+1}), kd(x^*, Tx^*)\} + Ld(x^*, x_{n+1}),$$

using Lemma 1.6, we have

$$\frac{1}{s} d(x^*, Tx^*) \leqslant \theta k d(x^*, Tx^*).$$

Since $s\theta k < 1$, from the above inequality, we conclude that $d(x^*, T(x^*)) = 0$, i.e. $x^* \in T(x^*)$, so T has a fixed point. Now, let

$$d(x^*, Tx^*) > \overline{\lim_{n\to\infty}} d(x_n, T(x^*)).$$

Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have

$$d(x_n, Tx^*) \leqslant d(x^*, Tx^*).$$

So,

$$\begin{aligned} d(x^*, Tx^*) &\leqslant s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &\leqslant s[d(x^*, x_{n+1}) + \theta \max\{d(x^*, x_n), kd(x_n, x_{n+1}), kd(x^*, Tx^*)\} + Ld(x^*, x_{n+1})]. \end{aligned}$$

From the above inequality, when $n \to \infty$, we obtain

$$d(x^*, Tx^*) \leqslant s\theta k d(x^*, Tx^*).$$

Since $s\theta k < 1$, we obtain $x^* \in Tx^*$. □

4. Some applications

We shall present some applications of Theorem 3.1 in b-metric spaces.

Corollary 4.1 (Version of Nadler's fixed point theorem in b-metric spaces, [21]). *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *a mapping satisfying*

$$H(Tx, Ty) \leqslant \theta d(x, y) \tag{4.1}$$

for all $x, y \in X$, *where* $\theta \in (0, 1)$. *Then* T *has a fixed point.*

Proof. Put $k = L = 0$ in Theorem 3.1. □

Corollary 4.1 improves the next result by Czerwik [11].

Corollary 4.2. *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *a mapping satisfying*

$$H(Tx, Ty) \leqslant \lambda d(x, y)$$

for all $x, y \in X$, *where* $\lambda \in (0, \frac{1}{s})$. *Then* T *has a fixed point.*

Example 4.3. Let $X = [1, +\infty)$ be equipped with the complete b-metric d such that $d(x, y) = (x - y)^2$ for all $x, y \in X$ (with coefficient $s = 2$). Define $T : X \to CB(X)$ by $Tx = [1, 1 + \frac{4x}{5}]$ for all $x \in X$. Also, take $\theta = \frac{9}{16}$. We have

$$H(Tx, Ty) \leqslant \theta d(x, y)$$

for all $x, y \in X$, that is (4.1) holds. All hypotheses of Corollary 4.1 are satisfied and $x = 1$ is a fixed point of T.

On the other hand, Corollary 4.2 is not applicable. For $x = 2$ and $y = 1$, we have $H(Tx, Ty) = \frac{16}{25}$, $d(x, y) = 1$, so

$$H(Tx, Ty) > \lambda d(x, y) \text{ for all } \lambda \in [0, \frac{1}{2}).$$

Also, we may not apply the main result of Aydi et al. [3, Theorem 2.2]. Again, for $x = 2$ and $y = 1$, we have

$$d(x, Tx) = 0, d(y, Ty) = 0, d(x, Ty) = 0, d(y, Tx) = 0,$$

so

$$H(Tx, Ty) > \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $\lambda \in [0, \frac{1}{s^2+s})$.

Corollary 4.4 (Version of fixed point theorem by Berinde [5] and Abbas et al. [1] in b-metric spaces). *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *a weak contraction, i.e., there exist* $\theta \in (0, 1)$ *and* $L \geqslant 0$ *such that*

$$H(Tx, Ty) \leqslant \theta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X. \tag{4.2}$$

Then T *has a fixed point.*

Proof. Put $k = 0$ in Theorem 3.1. □

Example 4.5. Let $T : [0, 1] \to [0, 1]$ defined by $Tx = \{x\}$ for all $x \in [0, 1]$. Then

(i) T does not satisfy the contractive condition (2.3) of Aydi et al. [3];

(ii) T does not satisfy the contractive condition (2.4) of Miculescu and Mihail [20];

(iii) T satisfies condition (4.2) with $\theta \in (0, 1)$ arbitrary and $L \geqslant 1 - \theta$.

A weak contraction has always at least one fixed point and there exist weak contractions that have infinitely many fixed points.

Corollary 4.6 (Version of fixed point theorem by Kannan [17] in b-metric spaces)**.** *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *a mapping for which there exists* $\lambda \in (0, \frac{1}{2s})$ *such that*

$$H(Tx, Ty) \leqslant \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{4.3}$$

Then T *has a unique fixed point.*

Proof. Put in Theorem 3.1 $k = 1$. Let $x, y \in X$ be arbitrary taken. We have to discuss three possible cases, but due to the symmetry of $M_{1,T}(x, y)$, it suffices to consider only two of them.

1. $M_{1,T}(x, y) = d(x, y)$. Then $d(x, Tx) \leqslant d(x, y)$ and $d(y, Ty) \leqslant d(x, y)$, hence from condition (4.3) we obtain

$$\lambda[d(x, Tx) + d(y, Ty)] \leqslant 2\lambda M_{1,T}(x, y).$$

Then conditions (2.1) and (2.2) are obviously satisfied (with $\theta = 2\lambda$ and $L = 0$).

2. $M_{1,T}(x, y) = d(x, Tx)$. Then $d(y, Ty) \leqslant d(x, Tx)$, so from condition (4.3) we have

$$\lambda[d(x, Tx) + d(y, Ty)] \leqslant 2\lambda M_{1,T}(x, y).$$

Then condition (2.1) holds for $\theta = 2\lambda$ and $L = 0$. So (2.2) also holds.

Therefore both (2.1) and (2.2) hold with $\theta = 2\lambda$ and $L = 0$. Now, from Theorem 3.1 we conclude that the mapping T has a fixed point x^* if $2\lambda s < 1$ (condition (iii) in Theorem 3.1). Let y^* be a fixed point of the mapping T. Then, from condition (4.3) we obtain

$$d(x^*, y^*) \leqslant H(Tx^*, y^*) \leqslant \lambda[d(x^*, Tx^*) + d(y^*, Ty^*)] = 0,$$

so $x^* = y^*$. □

Corollary 4.7 (Version of fixed point theorem by Chatterjea [6] in b-metric spaces)**.** *Let* (X, d, s) *be a complete b-metric space and* $T : X \to CB(X)$ *a mapping for which there exists* $\lambda \in (0, \frac{1}{s+s^2})$ *such that*

$$H(Tx, Ty) \leqslant \lambda[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{4.4}$$

Then T *has a unique fixed point.*

Proof. By (4.4) and triangle rule we have

$$\begin{aligned} H(Tx, Ty) &\leqslant \lambda s[d(x, y) + s(d(y, Tx) + H(Tx, Ty))] + \lambda d(y, Tx) \\ &\leqslant \lambda s d(x, y) + \lambda(s^2 + 1)d(y, Tx) + \lambda s^2 H(Tx, Ty). \end{aligned}$$

After simple computations, we get

$$H(Tx, T, y) \leqslant \frac{\lambda s}{1 - \lambda s^2} d(x, y) + \frac{\lambda(s^2 + 1)}{1 - \lambda s^2} d(y, Tx),$$

which is (4.2), with $\theta = \frac{\lambda s}{1-\lambda s^2}$ (since $\lambda < \frac{1}{s+s^2}$) and $L = \frac{\lambda(s^2+1)}{1-\lambda s^2} \geqslant 0$. So, from Corollary 4.4 we obtain that T has a fixed point x^*. If y^* is also a fixed point of T, from condition (4.4) we obtain

$$d(x^*,y^*) \leqslant H(Tx^*,y^*) \leqslant \lambda[d(x^*,Ty^*)+d(y^*,Tx^*)] \leqslant \lambda[d(x^*,y^*)+d(y^*,x^*)] < 2\lambda d(x^*,y^*) < d(x^*,y^*).$$

It is a contradiction if $x^* \neq y^*$. □

Corollary 4.8 (Generalizations of fixed point theorem by Reich in b-metric spaces, [22]). *Let (X,d,s) be a complete b-metric space and $T : X \to CB(X)$ a mapping satisfying*

$$H(Tx,Ty) \leqslant \theta \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

for all $x,y \in X$, where $\theta \in (0,\frac{1}{s})$. Then T has a fixed point.

Proof. Put $k = 1, L = 0$ in Theorem 3.1. □

Corollary 4.9 (Version of fixed point theorem by Ćirić [7, 8]). *Let (X,d,s) be a complete b-metric space and $T : X \to CB(X)$ a mapping satisfying*

$$H(Tx,Ty) \leqslant \alpha M(x,y), \tag{4.5}$$

where $\alpha \in (0,\frac{1}{2s})$. Then T has a fixed point.

Proof. Let $x,y \in X$ be arbitrary taken. We will use Theorem 3.1. We have to discuss five possible cases, but due to the symmetry of $M(x,y)$, it suffices to consider only three of them.

1. $M(x,y) = d(x,y)$. Then $M(x,y) = M_{1,T}(x,y)$, so condition (2.1) and its dual condition (2.2) are obviously satisfied with $\theta = \alpha$ and $L = 0$.

2. $M(x,y) = d(x,Tx)$. And in this case we have $M(x,y) = M_{1,T}(x,y)$, so $\theta = \alpha$ and $L = 0$.

3. $M(x,y) = d(y,Tx)$. From (4.5) we get

$$H(Tx,Ty) \leqslant \alpha d(y,Tx) \leqslant \theta M_{1,T}(x,y) + \alpha d(y,Tx),$$

so (2.1) holds with $\theta \in (0,1)$ and $L = \alpha$. Since,

$$H(Tx,Ty) \leqslant \alpha M(x,y) = \alpha d(y,Tx) \leqslant \alpha s[d(y,Ty)+H(Ty,Tx)] \leqslant \alpha s[M_{1,T}(x,y)+H(Tx,Ty)],$$

we get

$$H(Tx,Ty) \leqslant \frac{\alpha s}{1-\alpha s} M_{1,T}(x,y).$$

So, dual (2.2) also holds for all $\theta = \frac{\alpha s}{1-\alpha s}$ and $L = 0$. Therefore both (2.1) and its dual (2.2) hold with

$$\theta = \max\{\alpha, 0, \frac{\alpha s}{1-\alpha s}\} = \frac{\alpha s}{1-\alpha s}, \quad L = \max\{0,\alpha\} = \alpha.$$

Since $\alpha \in (0,\frac{1}{2s})$, we obtain that $\theta s < 1$ and $L > 0$. Therefore, from Theorem 3.1, it follows that T has a fixed point. □

Remark 4.10.

(i) Note that $(0,\frac{1}{s+s^2}) \subseteq (0,\frac{1}{2s})$ implies that Corollary 4.9 implies the main result in [3, Theorem 2.2].

(ii) In [2, Theorem 2.2] Amini-Harandi proved the following result in metric spaces.

Theorem 4.11. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a mapping satisfying*

$$H(Tx,Ty) \leqslant \alpha M(x,y),$$

where $\alpha \in (0,\frac{1}{2})$. Then T has a fixed point.

Note that from Corollary 4.9 we obtain Theorem 4.11.

Problem 4.12. Does the conclusion of Corollary 4.9 remain true for any $\alpha \in [\frac{1}{2s},1)$?

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