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Strong convergence of some iterative algorithms for a general system of variational inequalities

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Abstract

In this paper, we introduce two iterative algorithms (one implicit algorithm and one explicit algorithm) for finding a common element of the solution set of a general system of variational inequalities for continuous monotone mappings and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. First, this system of variational inequalities is proven to be equivalent to a fixed point problem of nonexpansive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative algorithms to a common element of the solution set and the fixed point set, which is the unique solution of a certain variational inequality.

Keywords: Composite iterative algorithm, general system of variational inequatlites, continuous monotone mapping, continuous peudocontractive mapping, ρ-Lipschitzian, η-strongly monotone mapping, variational inequality, strongly positive bounded linear operator, fixed points.
2010 MSC: 47J20, 47H05, 47H09, 47H10, 49J40, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S.

A mapping $F : C \to H$ is called monotone, if

$$\langle x - y, Fx - Fy \rangle \geqslant 0, \quad \forall x,\ y \in C,$$

and F is called α-inverse-strongly monotone (see [5, 10]) if there exists a positive real number α such that

$$\langle x - y, Fx - Fy \rangle \geqslant \alpha \|Fx - Fy\|^2, \quad \forall x,\ y \in C.$$

The class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

A mapping $T : C \to H$ is said to be pseudocontractive, if

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

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and T is said to be k-strictly pseudocontractive, if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$, for all $x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass.

Let F be a nonlinear mapping of C into H. The variational inequality problem (VIP) is to find a $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C. \tag{1.1}$$

We denote the set of solutions of VIP (1.1) by $VI(C, F)$. The variational inequality problem has been extensively studied in the literature; see [3, 5, 7, 9, 10, 13, 14, 16, 18] and the references therein.

In 2008, Ceng et al. [2] considered the following general system of variational inequalities:

$$\begin{cases} \langle \lambda F_1 y^* + x^* - y^*, x - x^* \rangle \geqslant 0, & \forall x \in C, \\ \langle \nu F_2 x^* + y^* - x^*, x - y^* \rangle \geqslant 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where F_1 and F_2 are an α-inverse-strongly monotone mapping and a β-inverse-strongly monotone mapping, respectively, and $\lambda \in (0, 2\alpha)$ and $\nu \in (0, 2\beta)$ are two constants. For finding an element $Fix(S) \cap \Gamma$, where $S : C \to C$ is a nonexpansive mapping and Γ is the solution set of the problem (1.2), they introduced a relaxed extragradient method ([8]) and proved strong convergence to a common element of $Fix(S) \cap \Gamma$.

In 2016, Alofi et al. [1] also considered the problem (1.2) coupled with the fixed point problem, and introduced two composite iterative algorithms (one implicit algorithm and one explicit algorithm) based on Jung's composite iterative method [6] to find an element $Fix(T) \cap \Gamma$, where $T : C \to C$ is a k-strictly pseudocontractive mapping and Γ is the solution set of the problem (1.2), and showed strong convergence to a common element of $Fix(T) \cap \Gamma$. The following problems arise:

Question 1. Can we extend the class of inverse-strongly monotone mappings in [1, 2] to the more general class of continuous monotone mappings?

Question 2. Can we extend the class of nonexpansive mappings in [2] or the class of strictly pseudocontractive mappings in [1] to the more general class of pseudocontractive mappings?

In this paper, in order to give the affirmative answers to the above two questions, we consider a general system of variational inequalities slightly different from the problem (1.2). More precisely, we introduce the following general system of variational inequalities (GSVI) for two continuous monotone mappings F_1 and F_2 of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda F_1 x^* + x^* - y^*, x - x^* \rangle \geqslant 0, & \forall x \in C, \\ \langle \nu F_2 y^* + y^* - x^*, x - y^* \rangle \geqslant 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\lambda > 0$ and ν are two constants. The solution set of GSVI (1.3) is denoted by Ω. First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping. Second, by using Jung's composite iterative algorithms [6], we introduce a composite implicit iterative algorithm and a composite explicit iterative algorithm for finding a common element of $\Omega \cap Fix(T)$, where T is a continuous pseudocontractive mapping. Then we establish strong convergence of these two composite iterative algorithms to a common element of $\Omega \cap Fix(T)$, which is the unique solution of a certain variational inequality related to a minimization problem. As a direct consequence, we obtain strong convergence to a common element of $VI(C, F) \cap Fix(T)$, where F is a continuous monotone mapping.

2. Preliminaries and lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leqslant \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well-known that $P_C(x)$ is characterized by the property:

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geqslant 0, \quad \forall x \in H,\ y \in C. \tag{2.1}$$

In a Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x,\ y \in H. \tag{2.2}$$

We recall that:

(i) an operator A is said to be strongly positive on H, if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geqslant \overline{\gamma}\|x\|^2, \quad \forall x \in H;$$

(ii) a mapping $V : C \to H$ is said to be l-Lipschitzian, if there exists a constant $l \geqslant 0$ such that

$$\|Vx - Vy\| \leqslant l\|x - y\|, \quad \forall x,\ y \in C;$$

(iii) a mapping $G : C \to H$ is said to be ρ-strongly monotone, if there exists a constant $\rho > 0$ such that

$$\langle Gx - Gy, x - y\rangle \geqslant \rho\|x - y\|^2, \quad \forall x,\ y \in C.$$

The following lemma is an immediate consequence of an inner product.

Lemma 2.1. *In a real Hilbert space H, there holds the following inequality*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

We need the following lemmas for the proof of our main results.

Lemma 2.2 ([15])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leqslant (1 - \omega_n)s_n + \omega_n\delta_n + \nu_n, \quad \forall n \geqslant 1,$$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \omega_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1 - \omega_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} \omega_n|\delta_n| < \infty$;
(iii) $\nu_n \geqslant 0$ $(n \geqslant 1)$, $\sum_{n=1}^{\infty} \nu_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (Demiclosedness principle [4])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $(I - S)x_n \to y$, then $(I - S)x^* = y$.*

Lemma 2.4 ([11])**.** *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} > 1$. Then*

$$\langle (A - I)x - (A - I)y, x - y\rangle \geqslant (\overline{\gamma} - 1)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $A - I$ is strongly monotone with a constant $\overline{\gamma} - 1$.

Lemma 2.5 ([11]). *Assume that* A *is a strongly positive bounded linear operator on* H *with a coefficient* $\overline{\gamma} > 0$ *and* $0 < \zeta \leqslant \|A\|^{-1}$. *Then* $\|I - \zeta A\| \leqslant 1 - \zeta\overline{\gamma}$.

The following lemma can be easily proven, and therefore, we omit the proof. (see [16]).

Lemma 2.6. *Let* H *be a real Hilbert space. Let* $G : H \to H$ *be a* ρ*-Lipschitzian and* η*-strongly monotone mapping with constants* ρ, $\eta > 0$. *Let* $0 < \mu < \frac{2\eta}{\rho^2}$ *and* $0 < t < \sigma \leqslant 1$. *Then* $S := \sigma I - t\mu G : H \to H$ *is a contractive mapping with constant* $\sigma - t\tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [17], respectively.

Lemma 2.7 ([17]). *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $F : C \to H$ *be a continuous monotone mapping. Then, for* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$\langle y - z, Fz\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C.$$

For $r > 0$ *and* $x \in H$, *define* $F_r : H \to C$ *by*

$$F_r x = \left\{ z \in C : \langle y - z, Fz\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) F_r *is single-valued;*
(ii) F_r *is firmly nonexpansive, that is,*

$$\|F_r x - F_r y\|^2 \leqslant \langle x - y, F_r x - F_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) $Fix(F_r) = VI(C, F)$;
(iv) $VI(C, F)$ *is a closed convex subset of* C.

Lemma 2.8 ([17]). *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : C \to H$ *be a continuous pseudocontractive mapping. Then, for* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leqslant 0, \quad \forall y \in C.$$

For $r > 0$ *and* $x \in H$, *define* $T_r : H \to C$ *by*

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leqslant 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) T_r *is single-valued;*
(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leqslant \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) $Fix(T_r) = Fix(T)$;
(iv) $Fix(T)$ *is a closed convex subset of* C.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace of H;
- $A : C \to C$ is a strongly positive linear bounded self-adjoint operator with a constant $\overline{\gamma} \in (1, 2)$;
- $V : C \to C$ is l-Lipschitzian with constant $l \in [0, \infty)$;

- $G : C \to C$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leqslant \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- F_1 and $F_2 : C \to H$ are continuous monotone mappings;
- Ω is the solution set of GSVI (1.3) for F_1 and F_2;
- $F_{1\lambda} : H \to C$ is a mapping defined by

$$F_{1\lambda}x = \left\{ z \in C : \langle y - z, F_1 z\rangle + \frac{1}{\lambda}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C \right\}$$

  for $\lambda > 0$;
- $F_{2\nu} : H \to C$ is a mapping defined by

$$F_{2\nu}x = \left\{ z \in C : \langle y - z, F_2 z\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C \right\}$$

  for $\nu > 0$;
- $R : H \to C$ is a mapping defined by $Rx = F_{1\lambda}F_{2\nu}x$ for each $x \in H$;
- $T : C \to C$ is a continuous pseudocontractive mapping such that $\mathrm{Fix}(T) \neq \emptyset$;
- $T_{r_t} : H \to C$ is a mapping defined by

$$T_{r_t}x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \leqslant 0, \quad \forall y \in C \right\}$$

  for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leqslant 0, \quad \forall y \in C \right\}$$

  for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $\Omega \cap \mathrm{Fix}(T) \neq \emptyset$.

By Lemma 2.7 and Lemma 2.8, we note that $F_{1\lambda}$, $F_{2\nu}$, T_{r_t}, and T_{r_n} are nonexpansive and

$$\mathrm{Fix}(T_{r_n}) = \mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t}).$$

First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping.

Proposition 3.1. *Let C be a closed convex subset of a real Hilbert space H. For given x^*, $y^* \in C$, (x^*, y^*) is a solution of GSVI (1.3) for continuous monotone mappings F_1 and F_2 if and only if x^* is a fixed point of the mapping $R : H \to C$ defined by*

$$Rx = F_{1\lambda}F_{2\nu}x, \quad \forall x \in H,$$

where $y^* = F_{2\nu}x^*$.

Proof.

$$\begin{cases} \langle \lambda F_1 x^* + x^* - y^*, x - x^*\rangle \geqslant 0, & \forall x \in C, \\ \langle \nu F_2 y^* + y^* - x^*, x - y^*\rangle \geqslant 0, & \forall x \in C, \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} & \langle \lambda F_1 x^* + x^* - y^*, x - x^*\rangle \geqslant 0, \quad \forall x \in C, \\ \Longleftrightarrow & \langle x - x^*, \lambda F_1 x^*\rangle + \langle x - x^*, x^* - y^*\rangle \geqslant 0, \quad \forall x \in C, \\ \Longleftrightarrow & \langle x - x^*, F_1 x^*\rangle + \frac{1}{\lambda}\langle x - x^*, x^* - y^*\rangle \geqslant 0, \quad x \in C, \\ \Longleftrightarrow & x^* = F_{1\lambda}y^* \end{cases}$$

and

$$\begin{cases} & \langle \nu F_2 y^* + y^* - x^*, x - y^* \rangle \geqslant 0, \quad \forall x \in C, \\ \Longleftrightarrow & \langle x - y^*, \nu F_2 y^* \rangle + \langle x - y^*, y^* - x^* \rangle \geqslant 0, \quad \forall x \in C, \\ \Longleftrightarrow & \langle x - y^*, F_2 y^* \rangle + \frac{1}{\nu} \langle x - y^*, y^* - x^* \rangle \geqslant 0, \quad x \in C, \\ \Longleftrightarrow & y^* = F_{2\nu} x^* \end{cases}$$

$\Longleftrightarrow$

$$x^* = F_{1\lambda} y^* = F_{1\lambda} F_{2\nu} x^* = Rx^*.$$

□

Remark 3.2. We note that since the mappings $F_{1\lambda}$ and $F_{2\nu}$ are firmly nonexpansive by Lemma 2.7, the mapping $R : H \to C$ in Proposition 3.1 is nonexpansive.

Now, we introduce the following composite algorithm that generates a net $\{x_t\}$ in an implicit way:

$$x_t = (I - \theta_t A) T_{r_t} R x_t + \theta_t [t\gamma V x_t + (I - t\mu G) T_{r_t} R x_t], \tag{3.1}$$

where $t \in (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\})$ and $\theta_t \in (0, \|A\|^{-1}]$.

For $t \in (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\})$ and $\theta_t \in (0, \|A\|^{-1}]$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = (I - \theta_t A) T_{r_t} R x + \theta_t [t\gamma V x + (I - t\mu G) T_{r_t} R x], \quad \forall x \in C.$$

By the same argument as in [6] along with Lemma 2.5 and Lemma 2.6, it is easy to see that Q_t is a contractive mapping with constant $1 - \theta_t(\overline{\gamma} - 1 + t(\tau - \gamma l))$. By the Banach Contraction Principle, Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$, which can be proved by the same method as in [6]. We include only the proof of (iv).

Proposition 3.3. *Let $\{x_t\}$ be defined via (3.1). Then*

(i) *$\{x_t\}$ is bounded for $t \in (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\})$;*
(ii) *$\lim_{t\to 0} \|x_t - T_{r_t} R x_t\| = 0$ provided $\lim_{t\to 0} \theta_t = 0$;*
(iii) *$\lim_{t\to 0} \|x_t - y_t\| = 0$, where $y_t = t\gamma V x_t + (I - t\mu G) T_{r_t} R x_t$;*
(iv) *$\lim_{t\to 0} \|x_t - R x_t\| = 0$;*
(v) *$x_t : (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\}) \to H$ is locally Lipschitzian provided $\theta_t : (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\}) \to (0, \|A\|^{-1}]$ is locally Lipschitzian, and $r_t : (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\}) \to (0, \infty)$ is locally Lipschitzian;*
(vi) *x_t defines a continuous path from $(0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\})$ into H provided $\theta_t : (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\}) \to (0, \|A\|^{-1}]$ is continuous, and $r_t : (0, \min\{1, \frac{2-\overline{\gamma}}{\tau - \gamma l}\}) \to (0, \infty)$ is continuous.*

Proof. (iv) Let $z_t = R x_t$, let $y_t = t\gamma V x_t + (I - t\mu G) T_{r_t} R x_t$ and let $p \in \Omega \cap \mathrm{Fix}(T)$. Since $z_t = R x_t$, $p = Rp$ and $T_{r_t} p = p$, from Lemma 2.1 we have

$$\begin{aligned} \|y_t - p\|^2 &= \|t(\gamma V x_t - \mu G p) + (I - t\mu G) T_{r_t} z_t - (I - t\mu G) p\|^2 \\ &\leqslant (t\|\gamma V x_t - \mu G p\| + \|(I - t\mu G) T_{r_t} z_t - (I - t\mu G) T_{r_t} p\|)^2 \\ &\leqslant t\|\gamma V x_t - \mu G p\|^2 + (1 - t\tau)\|z_t - p\|^2 + 2t(1 - t\tau)\|\gamma V x_t - \mu G p\|\|z_t - p\|. \end{aligned} \tag{3.2}$$

Moreover, from (2.2), we deduce

$$\begin{aligned} \|z_t - p\|^2 =& \|R x_t - p\|^2 \\ &\leqslant \|x_t - p\|^2 \\ &= \langle x_t - p, x_t - p \rangle \\ &= \langle z_t - p, x_t - p \rangle + \langle x_t - z_t, x_t - p \rangle \\ &\leqslant \frac{1}{2}[\|x_t - p\|^2 + \|z_t - p\|^2 - \|x_t - z_t\|^2] + \|x_t - p\|\|x_t - z_t\|, \end{aligned}$$

and hence

$$\|z_t - p\|^2 \leqslant \|x_t - p\|^2 - \|x_t - z_t\|^2 + 2\|x_t - p\|\|x_t - z_t\|. \tag{3.3}$$

Thus, from (3.2) and (3.3), we derive

$$\begin{aligned}\|y_t - p\|^2 &\leqslant t\|\gamma V x_t - \mu G p\|^2 + (1 - t\tau)\|z_t - p\|^2 + 2t(1 - t\tau)\|\gamma V x_t - \mu G p\|\|z_t - p\| \\ &\leqslant t\|\gamma V x_t - \mu G p\|^2 + \|x_t - p\|^2 - (1 - t\tau)(\|x_t - z_t\|^2 - 2\|x_t - p\|\|x_t - z_t\|) \\ &\quad + 2t(1 - t\tau)\|\gamma V x_t - \mu G p\|\|z_t - p\|.\end{aligned}$$

This implies that

$$\begin{aligned}&(1 - t\tau)\|x_t - z_t\|(\|x_t - z_t\| - 2\|x_t - p\|) \\ &\quad\leqslant t\|\gamma V x_t - \mu G p\|^2 + (\|x_t - p\| + \|y_t - p\|)(\|x_t - p\| - \|y_t - p\|) + 2t\|\gamma V x_t - \mu G p\|\|z_t - p\| \\ &\quad\leqslant t\|\gamma V x_t - \mu G p\|^2 + (\|x_t - p\| + \|y_t - p\|)\|x_t - y_t\| + 2t\|\gamma V x_t - \mu G p\|\|z_t - p\|.\end{aligned}$$

Since $t \to 0$ and $\|x_t - y_t\| \to 0$ by (iii), we get

$$\lim_{t\to 0} \|x_t - z_t\|(\|x_t - z_t\| - 2\|x_t - p\|) = 0.$$

In general, $\lim_{t\to 0}(\|x_t - z_t\| - 2\|x_t - p\|) \neq 0$. So, we conclude

$$\lim_{t\to 0} \|x_t - Rx_t\| = \lim_{t\to 0} \|x_t - z_t\| = 0.$$

□

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.4) below.

Theorem 3.4. *Let the net $\{x_t\}$ be defined via (3.1). If $\lim_{t\to 0} \theta_t = 0$, then x_t converges strongly to $\widetilde{x}$ in $\Omega \cap \mathrm{Fix}(T)$ as $t \to 0$, which solves the variational inequality*

$$\langle (A - I)\widetilde{x}, \widetilde{x} - p\rangle \leqslant 0, \quad \forall p \in \Omega \cap \mathrm{Fix}(T). \tag{3.4}$$

Equivalently, we have

$$P_{\Omega\cap \mathrm{Fix}(T)}(2I - A)\widetilde{x} = \widetilde{x}.$$

Proof. We first note that the uniqueness of a solution of the variational inequality (3.4) is a consequence of the strong monotonicity of $A - I$ (by Lemma 2.4). See [1, 6] for this fact.

Next, we prove that $x_t \to \widetilde{x}$ as $t \to 0$. Let $z_t = Rx_t$. Observing $\mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t})$ (by Lemma 2.8 (iii)) and $\mathrm{Fix}(R) = \Omega$ (by Proposition 3.1), from (3.1), we write for given $p \in \Omega \cap \mathrm{Fix}(T)$,

$$\begin{aligned}x_t - p &= (I - \theta_t A)T_{r_t} z_t - (I - \theta_t A)T_{r_t} p + \theta_t[t\gamma V x_t + (I - t\mu G)T_{r_t} z_t - p] + \theta_t(I - A)p \\ &= (I - \theta_t A)(T_{r_t} z_t - T_{r_t} p) + \theta_t[t(\gamma V x_t - \mu G p) + (I - t\mu G)T_{r_t} z_t - (I - t\mu G)p] + \theta_t(I - A)p,\end{aligned}$$

to derive that

$$\begin{aligned}\|x_t - p\|^2 &= \langle (I - \theta_t A)(T_{r_t} z_t - T_{r_t} p), x_t - p\rangle + \theta_t[t\langle \gamma V x_t - \mu G p, x_t - p\rangle \\ &\quad + \langle (I - t\mu G)T_{r_t} z_t - (I - t\mu G)p, x_t - p\rangle] + \theta_t\langle (I - A)p, x_t - p\rangle \\ &\leqslant (1 - \theta_t\overline{\gamma})\|z_t - p\|\|x_t - p\| \\ &\quad + \theta_t[(1 - t\tau)\|z_t - p\|\|x_t - p\| + t\gamma l\|x_t - p\|^2 + t\langle \gamma V p - \mu G p, x_t - p\rangle] \\ &\quad + \theta_t\langle (I - A)p, x_t - p\rangle \\ &\leqslant (1 - \theta_t\overline{\gamma})\|x_t - p\|^2 + \theta_t[(1 - t\tau)\|x_t - p\|^2 + t\gamma l\|x_t - p\|^2 + t\langle \gamma V p - \mu G p, x_t - p\rangle] \\ &\quad + \theta_t\langle (I - A)p, x_t - p\rangle \\ &= [1 - \theta_t(\overline{\gamma} - 1 + t(\tau - \gamma l))]\|x_t - p\|^2 + \theta_t(t\langle \gamma V p - \mu G p, x_t - p\rangle + \langle (I - A)p, x_t - p\rangle).\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \leqslant \frac{1}{\overline{\gamma} - 1 + t(\tau - \gamma l)}(t\langle \gamma Vp - \mu Gp, x_t - p\rangle + \langle (I-A)p, x_t - p\rangle). \tag{3.5}$$

Since $\{x_t\}$ is bounded as $t \to 0$ (by Proposition 3.3 (i)), there exists a subsequence $\{t_n\}$ in $(0, \min\{1, \frac{2-\overline{\gamma}}{\tau-\gamma l}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$. First of all, we prove that $x^* \in \Omega \cap \mathrm{Fix}(T)$. To this end, we divide its proof into four steps.

Step 1. From Proposition 3.3 (iv), we know that $\lim_{n\to\infty} \|x_{t_n} - Rx_{t_n}\| = \lim_{n\to\infty} \|x_{t_n} - z_{t_n}\| = 0$.

Step 2. We show that $\lim_{n\to\infty} \|u_{t_n} - z_{t_n}\| = 0$, where $u_{t_n} = T_{r_{t_n}} z_{t_n}$. Indeed, from Proposition 3.3 (ii) and Step 1, it follows that

$$\|u_{t_n} - z_{t_n}\| \leqslant \|u_{t_n} - x_{t_n}\| + \|x_{t_n} - z_{t_n}\| \to 0 \quad (\text{as } n \to \infty).$$

Step 3. We show that $x^* \in \Omega$. In fact, since $x_n \rightharpoonup x^*$ and $x_n - Rx_n \to 0$ by Step 1, from Lemma 2.3 (Demiclosedness principle), we get $x^* = Rx^*$, that is, $x^* \in \mathrm{Fix}(R)$. Thus, by Proposition 3.1, we have $x^* \in \Omega$.

Step 4. We have $x^* \in \mathrm{Fix}(T)$ by the same argument as in the proof of [18, Theorem 3.1]. We include its proof for the sake of completeness. In fact, from the definition of $u_{t_n} = T_{r_{t_n}} z_{t_n}$, we have

$$\langle y - u_{t_n}, Tu_{t_n}\rangle - \frac{1}{r_{t_n}}\langle y - u_{t_n}, (1 + r_{t_n})u_{t_n} - z_{t_n}\rangle \leqslant 0, \quad \forall y \in C. \tag{3.6}$$

Put $w_t = tv + (1-t)x^*$ for all $t \in (0, 1]$ and $v \in C$. Then, $w_t \in C$ and from (3.6) and pseudocontractivity of T, it follows that

$$\begin{aligned}
\langle u_{t_n} - w_t, Tw_t\rangle &\geqslant \langle u_{t_n} - w_t, Tw_t\rangle + \langle w_t - u_{t_n}, Tu_{t_n}\rangle - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, (1 + r_{t_n})u_{t_n} - z_{t_n}\rangle \\
&= -\langle w_t - u_{t_n}, Tw_t - Tu_{t_n}\rangle - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, u_{t_n} - z_{t_n}\rangle - \langle w_t - u_{t_n}, u_{t_n}\rangle \\
&\geqslant -\|w_t - u_{t_n}\|^2 - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, u_{t_n} - z_{t_n}\rangle - \langle w_t - u_{t_n}, u_{t_n}\rangle \\
&= -\langle w_t - u_{t_n}, w_t\rangle - \langle w_t - u_{t_n}, \frac{u_{t_n} - z_{t_n}}{r_{t_n}}\rangle.
\end{aligned} \tag{3.7}$$

By Step 2, we get $\frac{u_{t_n} - z_{t_n}}{r_{t_n}} \to 0$ as $n \to \infty$. Moreover, since $x_{t_n} \rightharpoonup x^*$, by Step 1 and Step 2, we have $u_{t_n} \rightharpoonup x^*$ as $n \to \infty$. Therefore, from (3.7), as $n \to \infty$, it follows that

$$\langle x^* - w_t, Tw_t\rangle \geqslant \langle x^* - w_t, w_t\rangle,$$

and hence

$$-\langle v - x^*, Tw_t\rangle \geqslant -\langle v - x^*, w_t\rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is continuous, we get

$$-\langle v - x^*, Tx^*\rangle \geqslant -\langle v - x^*, x^*\rangle, \quad \forall v \in C.$$

Now, let $v = Tx^*$. Then we obtain $x^* = Tx^*$ and hence $x^* \in \mathrm{Fix}(T)$. Therefore, $x^* \in \Omega \cap \mathrm{Fix}(T)$.

Now, we substitute x^* for p in (3.5) to obtain

$$\|x_{t_n} - x^*\|^2 \leqslant \frac{1}{\overline{\gamma} - 1 + t_n(\tau - \gamma l)}(t_n\langle \gamma Vx^* - \mu Gx^*, x_{t_n} - x^*\rangle + \langle (I-A)x^*, x_{t_n} - x^*\rangle). \tag{3.8}$$

Note that $x_{t_n} \rightharpoonup x^*$ and $\lim_{n\to\infty} t_n = 0$. This fact and the inequality (3.8) imply that $x_{t_n} \to x^*$ strongly.

Finally, we prove that x^* is a solution of the variational inequality (3.4). In fact, putting x_{t_n} in place of x_t in (3.5) and taking the limit as $t_n \to 0$, we obtain

$$\|x^* - p\|^2 \leqslant \frac{1}{\overline{\gamma} - 1}\langle (I - A)p, x^* - p\rangle, \quad \forall p \in \Omega \cap \mathrm{Fix}(T).$$

In particular, x^* solves the following variational inequality

$$x^* \in \Omega \cap \mathrm{Fix}(T), \quad \langle (A - I)p, x^* - p\rangle \leqslant 0, \quad \forall p \in \Omega \cap \mathrm{Fix}(T),$$

or the equivalent dual variational inequality (see [12])

$$x^* \in \Omega \cap \mathrm{Fix}(T), \quad \langle (A - I)x^*, x^* - p\rangle \leqslant 0, \quad \forall p \in \Omega \cap \mathrm{Fix}(T).$$

That is, $x^* \in \Omega \cap \mathrm{Fix}(T)$ is a solution of the variational inequality (3.4). Hence $x^* = \widetilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$. Therefore $x_t \to \widetilde{x}$ as $t \to 0$. The variational inequality (3.4) can be written as

$$\langle (2I - A)\widetilde{x} - \widetilde{x}, \widetilde{x} - p\rangle \geqslant 0, \quad \forall p \in \Omega \cap \mathrm{Fix}(T).$$

So, by (2.1), this is equivalent to the fixed point equation

$$P_{\Omega \cap \mathrm{Fix}(T)}(2I - A)\widetilde{x} = \widetilde{x}.$$

This completes the proof. □

Taking $G \equiv I$, the identity mapping, $\mu = 1$ and $\gamma = 1$ in Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let* $\{x_t\}$ *be defined by*

$$x_t = (I - \theta_t A)T_{r_t} R x_t + \theta_t[tVx_t + (1 - t)T_{r_t} R x_t].$$

If $\lim_{t\to 0} \theta_t = 0$, *then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to* $\widetilde{x}$ *in* $\Omega \cap \mathrm{Fix}(T)$, *which is the unique solution of the variational inequality* (3.4).

Taking $T \equiv I$, $G \equiv I$, $\mu = 1$ and $\gamma = 1$ in Theorem 3.4, we have the following corollary.

Corollary 3.6. *Let* $\{x_t\}$ *be defined by*

$$x_t = (I - \theta_t A)Rx_t + \theta_t[tVx_t + (1 - t)Rx_t].$$

If $\lim_{t\to 0} \theta_t = 0$, *then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to* $\widetilde{x} \in \Omega$, *which is the unique solution of the variational inequality*

$$\langle (A - I)\widetilde{x}, \widetilde{x} - p\rangle \leqslant 0, \quad \forall p \in \Omega. \tag{3.9}$$

Proof. If $T \equiv I$, then T_r in Lemma 2.8 is the identity mapping. Thus the result follows from Theorem 3.4. □

Now, we propose the following composite algorithm which generates a sequence in an explicit way:

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G)T_{r_n} R x_n, \\ x_{n+1} = (I - \beta_n A)T_{r_n} R x_n + \beta_n y_n, \quad \forall n \geqslant 0, \end{cases} \tag{3.10}$$

where $\{\alpha_n\} \in [0, 1]$; $\{\beta_n\} \subset (0, 1]$; $\{r_n\} \subset (0, \infty)$; and $x_0 \in C$ is an arbitrary initial guess, and establish strong convergence of this sequence to $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$, which is the unique solution of the variational inequality (3.4).

Theorem 3.7. *Let $\{x_n\}$ be the sequence generated by the explicit algorithm (3.10). Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [0, 1]$ *and* $\{\beta_n\} \subset (0, 1]$, $\alpha_n \to 0$ *and* $\beta_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *and* $|\beta_{n+1} - \beta_n| \leqslant o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition);*

(C4) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$, which is the unique solution of the variational inequality (3.4).

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n\tau < 1$, $\beta_n\overline{\gamma} < 1$ and $\frac{2\beta_n(\overline{\gamma}-1)}{1-\beta_n} < 1$ for all $n \geqslant 0$. Let $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$ be the unique solution of the variational inequality (3.4). (The existence of $\widetilde{x}$ follows from Theorem 3.4).

From now, we put $z_n = Rx_n$ and $y_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu G)T_{r_n}Rx_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu G)T_n z_n$. Let $p \in \Omega \cap \mathrm{Fix}(T)$. Then $p = T_{r_n}p$ by Lemma 2.8 (iii) and $p = Rp$ by Proposition 3.1. Moreover, from nonexpansivity of F, it follows that

$$\|z_n - p\| = \|Rx_n - Rp\| \leqslant \|x_n - p\|.$$

We divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. First of all, by (3.10), we deduce

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n\gamma Vx_n + (I - \alpha_n\mu G)T_{r_n}z_n - p\| \\
&= \|\alpha_n(\gamma Vx_n - \mu Gp) + (I - \alpha_n\mu G)T_{r_n}z_n - (I - \alpha_n\mu G)T_{r_n}p\| \\
&\leqslant (1 - \alpha_n(\tau - \gamma l))\|z_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\| \\
&\leqslant (1 - \alpha_n(\tau - \gamma l))\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(I - \beta_n A)T_{r_n}z_n + \beta_n y_n - p\| \\
&= \|(I - \beta_n A)T_{r_n}z_n - (I - \beta_n A)T_{r_n}p + \beta_n(y_n - p) + \beta_n(I - A)p\| \\
&\leqslant \|(I - \beta_n A)T_{r_n}z_n - (I - \beta_n A)T_{r_n}p\| + \beta_n\|y_n - p\| + \beta_n\|I - A\|\|p\| \\
&\leqslant (1 - \beta_n\overline{\gamma})\|z_n - p\| + \beta_n[(1 - \alpha_n(\tau - \gamma l))\|z_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\|] + \beta_n\|I - A\|\|p\| \\
&\leqslant (1 - \beta_n\overline{\gamma})\|x_n - p\| + \beta_n[(1 - \alpha_n(\tau - \gamma l))\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\|] + \beta_n\|I - A\|\|p\| \\
&\leqslant (1 - \beta_n(\overline{\gamma} - 1))\|x_n - p\| + \beta_n(\|\gamma Vp - \mu Gp\| + \|I - A\|\|p\|) \\
&= (1 - \beta_n(\overline{\gamma} - 1))\|x_n - p\| + \beta_n(\overline{\gamma} - 1)\frac{\|\gamma Vp - \mu Gp\| + \|I - A\|\|p\|}{\overline{\gamma} - 1} \\
&\leqslant \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Gp\| + \|I - A\|\|p\|}{\overline{\gamma} - 1}\right\}.
\end{aligned}$$

By induction, we derive

$$\|x_n - p\| \leqslant \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Gp\| + \|I - A\|\|p\|}{\overline{\gamma} - 1}\right\}, \quad \forall n \geqslant 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Gx_n\}$, $\{z_n\}$, $\{T_{r_n}z_n\}$, $\{GT_{r_n}z_n\}$, $\{Vx_n\}$, $\{AT_{r_n}z_n\}$ and $\{y_n\}$. As a consequence with the control condition (C1), we get

$$\|x_{n+1} - T_{r_n} z_n\| = \beta_n \|y_n - AT_{r_n} z_n\| \to 0 \quad (n \to \infty). \tag{3.11}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To this end, let $z_n = Rx_n$, $z_{n-1} = Rx_{n-1}$, $u_n = T_{r_n} z_n$ and $u_{n-1} = T_{r_{n-1}} z_{n-1}$. Then we derive

$$\langle y - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leqslant 0, \quad \forall y \in C, \tag{3.12}$$

and

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 + r_n)u_n - z_n\rangle \leqslant 0, \quad \forall y \in C. \tag{3.13}$$

Putting $y = u_n$ in (3.12) and $y = u_{n-1}$ in (3.13), we obtain

$$\langle u_n - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leqslant 0, \tag{3.14}$$

and

$$\langle u_{n-1} - u_n, Tu_n\rangle - \frac{1}{r_n}\langle u_{n-1} - u_n, (1 + r_n)u_n - z_n\rangle \leqslant 0. \tag{3.15}$$

Adding up (3.14) and (3.15), we have

$$\langle u_n - u_{n-1}, Tu_{n-1} - Tu_n\rangle - \langle u_n - u_{n-1}, \frac{(1 + r_{n-1})u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{(1 + r_n)u_n - z_n}{r_n}\rangle \leqslant 0,$$

which implies that

$$\langle u_n - u_{n-1}, (u_n - Tu_n) - (u_{n-1} - Tu_{n-1})\rangle - \langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n}\rangle \leqslant 0.$$

Now, using the fact that T is pseudocontractive, we get

$$\langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n}\rangle \geqslant 0,$$

and hence

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - z_{n-1} - \frac{r_{n-1}}{r_n}(u_n - z_n)\rangle \geqslant 0. \tag{3.16}$$

Without loss of generality, let us assume that there exists a real number $r_n > b > 0$, for all $n \geqslant 0$. Then, by (3.16), we have

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &\leqslant \langle u_n - u_{n-1}, z_n - z_{n-1} + (1 - \frac{r_{n-1}}{r_n})(u_n - z_n)\rangle \\ &\leqslant \|u_n - u_{n-1}\| \left\{ \|z_n - z_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - z_n\| \right\},\end{aligned}$$

and hence

$$\|T_{r_n} z_n - T_{r_{n-1}} z_{n-1}\| \leqslant \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_1, \tag{3.17}$$

where $M_1 = \sup\{\|u_n - z_n\| : n \geqslant 0\}$.

Now, simple calculations yield that

$$\begin{aligned}y_n - y_{n-1} &= \alpha_n \gamma V x_n + (I - \alpha_n \mu G)T_{r_n} z_n - \alpha_{n-1}\gamma V x_{n-1} - (I - \alpha_{n-1}\mu G)T_{r_{n-1}} z_{n-1} \\ &= (\alpha_n - \alpha_{n-1})(\gamma V x_{n-1} - \mu G T_{r_{n-1}} z_{n-1}) + \alpha_n \gamma (V x_n - V x_{n-1}) \\ &\quad + (I - \alpha_n \mu G)T_{r_n} z_n - (I - \alpha_n \mu G)T_{r_{n-1}} z_{n-1}.\end{aligned}$$

By (3.17) and Lemma 2.6, we obtain

$$\begin{aligned}\|y_n - y_{n-1}\| \leqslant & |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|GT_{r_{n-1}}z_{n-1}\|) \\ & + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\tau\alpha_n)\|T_{r_n}z_n - T_{r_{n-1}}z_{n-1}\| \\ \leqslant & |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|GT_{r_{n-1}}z_{n-1}\|) \\ & + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\tau\alpha_n)\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_1 \\ = & |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|GT_{n-1}z_{n-1}\|) \\ & + (1-\alpha_n(\tau - \gamma l)\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2) \\ \leqslant & \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \frac{1}{b}|r_n - r_{n-1}|M_1,\end{aligned} \tag{3.18}$$

where $M_2 = \sup\{\gamma\|Vx_n\| + \mu\|GT_{r_n}z_n\| : n \geqslant 0\}$. By (3.18) and Lemma 2.5, we derive

$$\begin{aligned}\|x_{n+1} - x_n\| = & \|(I - \beta_n A)T_{r_n}z_n + \beta_n y_n - (I - \beta_{n-1}A)T_{r_{n-1}}z_{n-1} - \beta_{n-1}y_{n-1}\| \\ \leqslant & \|(I - \beta_n A)(T_{r_n}z_n - T_{r_{n-1}}z_{n-1})\| \\ & + |\beta_n - \beta_{n-1}|\|A\|\|T_{r_{n-1}}z_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1}\| \\ \leqslant & (1-\beta_n\overline{\gamma})\|T_{r_n}z_n - T_{r_{n-1}}z_{n-1}\| \\ & + \beta_n(\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \frac{1}{b}|r_n - r_{n-1}|M_1) + |\beta_n - \beta_{n-1}|M_3 \\ \leqslant & (1-\beta_n\overline{\gamma})(\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|M_1) \\ & + \beta_n(\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \frac{1}{b}|r_n - r_{n-1}|M_1) + |\beta_n - \beta_{n-1}|M_3 \\ \leqslant & (1-\beta_n(\overline{\gamma} - 1))\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_3 \\ & + |\alpha_n - \alpha_{n-1}|M_3 + \frac{2}{b}|r_n - r_{n-1}|M_1 \\ \leqslant & (1-\beta_n(\overline{\gamma} - 1))\|x_n - x_{n-1}\| + (o(\beta_n) + \sigma_{n-1})M_3 \\ & + |\alpha_n - \alpha_{n-1}|M_2 + \frac{2}{b}|r_n - r_{n-1}|M_1,\end{aligned} \tag{3.19}$$

where $M_3 = \sup\{\|A\|\|T_{r_n}z_n\| + \|y_n\| : n \geqslant 0\}$. By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\omega_n = \beta_n(\overline{\gamma} - 1)$, $\omega_n\delta_n = M_4 o(\beta_n)$ and $r_n = (\sigma_{n-1}M_3 + |\alpha_n - \alpha_{n-1}|M_2 + \frac{2}{b}|r_n - r_{n-1}|M_1)$, from (3.19) we deduce

$$s_{n+1} \leqslant (1-\omega_n)s_n + \omega_n\delta_n + r_n.$$

Hence, by the conditions (C2), (C3), (C4), and Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. Indeed, from (3.11) and condition (C1), we derive

$$\begin{aligned}\|x_{n+1} - y_n\| \leqslant & \|x_{n+1} - T_{r_n}z_n\| + \|T_{r_n}z_n - y_n\| \\ = & \beta_n\|y_n - AT_{r_n}z_n\| + \alpha_n\|\gamma Vx_n - \mu GT_{r_n}z_n\| \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. In fact, by Step 2 and Step 3, we get

$$\|x_n - y_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad (\text{as } n \to \infty).$$

Step 5. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$. By taking x_n and z_n instead of x_t and z_t in the proof of Proposition 3.3 (iv), respectively, the result follows from the proof of Proposition 3.3 (iv) together with Step 4.

Step 6. We show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$, where $u_n = T_{r_n} z_n$. In fact, from (3.11) and Step 2, we have

$$\|x_n - u_n\| = \|x_n - T_{r_n} z_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n} z_n\| \to 0 \quad (\text{as } n \to \infty).$$

Step 7. We show that $\lim_{n\to\infty}\|u_n - z_n\| = 0$, where $u_n = T_{r_n} z_n$. In fact, from Step 5 and Step 6, we have

$$\|u_n - z_n\| \leqslant \|u_n - x_n\| + \|x_n - z_n\| \to 0 \quad (\text{as } n \to \infty).$$

Step 8. We show that $\limsup_{n\to\infty}\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\rangle \leqslant 0$. To this end, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\rangle = \lim_{k\to\infty}\langle (I-A)\widetilde{x}, x_{n_k} - \widetilde{x}\rangle.$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup p$. Take x_{n_k} and z_{n_k} in place of x_{t_n} and z_{t_n} in Step 3 and Step 4 of proof of Theorem 3.4. Then, from Step 3 and Step 4 in proof of Theorem 3.4 along with Step 5 and Step 7, we derive $p \in \Omega \cap \mathrm{Fix}(T)$. Hence, from (3.4), we conclude

$$\limsup_{n\to\infty}\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\rangle = \lim_{k\to\infty}\langle (I-A)\widetilde{x}, x_{n_k} - \widetilde{x}\rangle = \langle (I-A)\widetilde{x}, p - \widetilde{x}\rangle \leqslant 0.$$

Step 9. We show that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. Note that $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$. Let $z_n = Rx_n$. By (3.10), $\widetilde{x} = R\widetilde{x}$, and $\widetilde{x} = T_{r_n}\widetilde{x}$, we deduce

$$\begin{aligned} y_n - \widetilde{x} &= (I - \alpha_n \mu G)T_{r_n} R x_n - (I - \alpha_n \mu G)T_{r_n} R\widetilde{x} + \alpha_n(\gamma V x_n - \mu G\widetilde{x}) \\ &= (I - \alpha_n \mu G)T_{r_n} z_n - (I - \alpha_n \mu G)T_{r_n}\widetilde{x} + \alpha_n(\gamma V x_n - \mu G\widetilde{x}), \end{aligned}$$

and

$$x_{n+1} - \widetilde{x} = (I - \beta_n A)(T_{r_n} z_n - T_{r_n}\widetilde{x}) + \beta_n(y_n - \widetilde{x}) + \beta_n(I - A)\widetilde{x}.$$

Applying Lemma 2.1, Lemma 2.5 and Lemma 2.6, we obtain

$$\begin{aligned} \|y_n - \widetilde{x}\|^2 &= \|(I - \mu\alpha_n G)T_{r_n} z_n - (I - \mu\alpha_n G)T_{r_n}\widetilde{x} + \alpha_n(\gamma V x_n - \mu G\widetilde{x})\|^2 \\ &\leqslant \|(I - \mu\alpha_n G)T_{r_n} z_n - (I - \mu\alpha_n G)T_{r_n}\widetilde{x}\|^2 + 2\alpha_n\langle \gamma V x_n - \mu G\widetilde{x}, y_n - \widetilde{x}\rangle \\ &\leqslant (1 - \alpha_n\tau)^2\|x_n - \widetilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\| \\ &\leqslant \|x_n - \widetilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\|^2 &= \|(I - \beta_n A)(T_{r_n} z_n - T_{r_n}\widetilde{x}) + \beta_n(y_n - \widetilde{x}) + \beta_n(I - A)\widetilde{x}\|^2 \\ &\leqslant \|(I - \beta_n A)(T_{r_n} z_n - T_{r_n}\widetilde{x})\|^2 + 2\beta_n\langle y_n - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + 2\beta_n\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\leqslant (1 - \beta_n\overline{\gamma})^2\|z_n - \widetilde{x}\|^2 + 2\beta_n\|y_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| + 2\beta_n\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\leqslant (1 - \beta_n\overline{\gamma})^2\|x_n - \widetilde{x}\|^2 + \beta_n(\|y_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2) + 2\beta_n\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\leqslant (1 - \beta_n\overline{\gamma})^2\|x_n - \widetilde{x}\|^2 + \beta_n[\|x_n - \widetilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\|] \\ &\quad + \beta_n\|x_{n+1} - \widetilde{x}\|^2 + 2\beta_n\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &= [(1 - \beta_n\overline{\gamma})^2 + \beta_n]\|x_n - \widetilde{x}\|^2 + 2\alpha_n\beta_n\|\gamma V x_n - \mu F\widetilde{x}\|\|y_n - \widetilde{x}\| \\ &\quad + \beta_n\|x_{n+1} - \widetilde{x}\|^2 + 2\beta_n\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle. \end{aligned} \tag{3.20}$$

It then follows from (3.20) that

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\|^2 &\leqslant \frac{(1 - \beta_n\overline{\gamma})^2 + \beta_n}{1 - \beta_n}\|x_n - \widetilde{x}\|^2 + \frac{\beta_n}{1 - \beta_n}[2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\| + 2\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\rangle] \\ &= \left(1 - \frac{2\beta_n(\overline{\gamma} - 1)}{1 - \beta_n}\right)\|x_n - \widetilde{x}\|^2 \\ &\quad + \frac{2\beta_n(\overline{\gamma} - 1)}{1 - \beta_n}\cdot\frac{1}{2(\overline{\gamma} - 1)}[2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\| + \beta_n\overline{\gamma}^2\|x_n - \widetilde{x}\|^2 \\ &\quad + 2\langle (A - I)\widetilde{x}, \widetilde{x} - x_{n+1}\rangle] \\ &\leqslant (1 - \omega_n)\|x_n - \widetilde{x}\|^2 + \omega_n\delta_n, \end{aligned}$$

where

$$\omega_n = \frac{2\beta_n(\overline{\gamma}-1)}{1-\beta_n} \quad \text{and} \quad \delta_n = \frac{1}{2(\overline{\gamma}-1)}[2\alpha_n M_4 + +\beta_n\overline{\gamma}^2 M_5 + 2\langle (A-I)\widetilde{x}, \widetilde{x}-x_{n+1}\rangle],$$

where $M_4 = \sup\{\|\gamma V x_n - \mu G\widetilde{x}\|\|y_n - \widetilde{x}\| : n \geqslant 0\}$ and $M_5 = \sup\{\|x_n - \widetilde{x}\|^2 :\geqslant 0\}$. It can be easily seen from conditions (C1) and (C2), and Step 8 that $\omega_n \to 0$, $\sum_{n=0}^{\infty}\omega_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leqslant 0$. From Lemma 2.2 with $\nu_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. This completes the proof. □

Taking $G \equiv I$, $\mu = 1$, and $\gamma = 1$ in Theorem 3.7, we obtain the following corollary.

Corollary 3.8. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} y_n = \alpha_n V x_n + (1-\alpha_n)T_{r_n}Rx_n, \\ x_{n+1} = (I-\beta_n A)T_{r_n}Rx_n + \beta_n y_n, \quad \forall n \geqslant 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the conditions (C1)-(C4) *in Theorem 3.7. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$, which is the unique solution of the variational inequality (3.4).*

Taking $T \equiv I$, $G \equiv I$, $\mu = 1$ and $\gamma = 1$ in Theorem 3.7, we have the following corollary.

Corollary 3.9. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} y_n = \alpha_n V x_n + (1-\alpha_n)Rx_n, \\ x_{n+1} = (I-\beta_n A)Rx_n + \beta_n y_n, \quad \forall n \geqslant 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1)-(C3) *in Theorem 3.7. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega$, which is the unique solution of the variational inequality (3.9).*

Taking $F_1 = F_2 = F$, $\lambda = \nu$ and $x^* = y^*$ in GSVI (1.3), we have the following result.

Corollary 3.10. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I-\alpha_n\mu G)T_{r_n}F_\lambda x_n, \\ x_{n+1} = (I-\beta_n A)T_{r_n}F_\lambda x_n + \beta_n y_n, \quad \forall n \geqslant 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the conditions (C1)-(C4) *in Theorem 3.7. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, which is the unique solution of the variational inequality*

$$\langle (A-I)\widetilde{x}, \widetilde{x}-p\rangle \leqslant 0, \quad \forall p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T).$$

Proof. If $F_1 = F_2 = F$, $\lambda = \nu$ and $x^* = y^*$ in GSVI (1.3), then GSVI (1.3) reduces to the classical variational inequality problem VIP (1.1) for a continuous monotone mapping F and $Rx = F_\lambda x$ in Proposition 3.1. Thus the result follows from Theorem 3.7. □

Remark 3.11.

1) The $\widetilde{x} \in \Omega \cap \mathrm{Fix}(T)$ in our results is the unique solution of minimization problem

$$\min_{x\in D} \frac{1}{2}\langle (A-I)x, x\rangle, \tag{3.21}$$

where the constraint set D is $\Omega \cap \mathrm{Fix}(T)$. In fact, the variational inequality (3.4) is the optimality condition for the minimization problem (3.21). Thus, for finding an element of $\Omega \cap \mathrm{Fix}(T)$, where T is a continuous pseudocontractive mapping, and F_1 and F_2 are continuous monotone mappings, Theorem 3.4, Corollary 3.5, Theorem 3.7 and Corollary 3.8 are new ones different from previous those introduced by some authors (for example, see [1, 2]).

2) Corollary 3.6 and Corollary 3.9 are also new results for finding an element of Ω, where F_1 and F_2 are continuous monotone mappings.

3) Using the same method as in [18], we can replace F_λ by F_{r_n} in Corollary 3.10 along with the condition (C4) on $\{r_n\}$. In this case, Corollary 3.10 is a new one, which improves, supplements and develops [14, Theorem 3.1] and Theorem 3.1 of Zegeye and Shahzad [18] in the following aspects:

   (a) The ρ-Lipschitzian and η-strongly monotone mapping G with constants ρ, $\eta > 0$ is used to develop our iterative method by virtue of Yamada's hybrid steepest-descent method [16].
   (b) The contractive mapping f with constant $\xi \in (0, 1)$ in [14, 18] is extended to the case of a Lipschitzian mapping V with constant $l \geqslant 0$.
   (c) The strongly positive linear bounded self-adjoint operator A is used to consider the minimization problem (3.21) whose the constraint set D is $VI(C, F) \cap Fix(T)$.

4) For finding an element of $VI(C, F) \cap Fix(T)$, Corollary 3.10 also improves, supplements and develops the corresponding results of [3, 5, 7, 13] in the following aspects together with (a), (b) and (c) in 3):

   (1) The inverse-strongly monotone mapping F in [3, 5, 7, 13] is extended to the case of the continuous monotone mapping F.
   (2) The nonexpansive mapping S in [3, 5, 13] or the strictly pseudocontractive mapping T in [7] is extended to the case of a continuous pseudocontractive mapping T.

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