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# Some properties of g-p-frames in complex Banach spaces

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**Abstract**

In this paper, we introduce the concept of dual frame of g-p-frame, and give the sufficient condition for a g-p-frame to have dual frames. Using operator theory and methods of functional analysis, we get some new properties of g-p-frame. In addition, we also characterize g-p-frame and g-q-Riesz bases by using analysis operator of g-p-Bessel sequence. 

Keywords: g-p-frame, g-q-Riesz basis, dual frames, analysis operator.
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## 1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [7] in 1952, and popularized from then on. Frames play an important role not only in pure mathematics, but also in signal processing, so it is natural to search for extensions to Banach spaces. p-frame in complex Banach spaces formally defined by Aldroubi et al. [3] in 2001. They were reintroduced and developed in 2003 by Chrisrensen and Stoeva [5]. After several years, Xiao et al. [10] generalized the g-frame and g-Riesz basis in a complex Hilbert space to a complex Banach space and obtained some basic properties. From papers [1–3, 5, 6, 8–10], it is known that some properties of g-p-frame are similar to those of p-frame, but the others are not. For example, if $\{\Lambda_j\}_{j\in J}$ is a g-p-Bessel sequence (p-Bessel sequence), then the synthesis operator $T_\Lambda$ is linear bounded, if $\{\Lambda_j\}_{j\in J}$ is a g-p-frame (p-frame), then the synthesis operator $T_\Lambda$ is surjective. As for g-q-Riesz bases, if $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz bases, then the synthesis operator $T_\Lambda$ is linear isomorphism. But q-Riesz bases do not have same property. p-frame has dual frame and every element of Banach spaces can be linear expression by p-frame and its dual, but g-p-frame does not have corresponding conclusion. So we need further study the properties of g-p-frame.

The purpose of this paper is to further develop the g-p-frames theory in complex Banach spaces. The paper is organized as follows. In Section 2, we recall the basic definitions and some important results about g-p-frame in complex Banach spaces. In Section 3, we introduce the concept of dual frames of g-p-frame, and give the sufficient condition for a g-p-frame to have dual frames. In Section 4, we give some new properties of g-p-frames in complex Banach spaces.

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## 2. Preliminaries

In this section, we introduce some definitions and important results of g-p-frame and g-q-Riesz basis we need later.

Suppose that X and Y are two complex Banach spaces and $\{Y_j\}_{j\in J}$ is a sequence of closed subspaces of Y, where J is a subset of integers $\mathbb{Z}$. Let $BL(X, Y_j)$ be the collection of all bounded linear operators from X into $Y_j$, $BL(X)$ be the collection of all bounded linear operators on X, $Y_j^*$ is the adjoint space of $Y_j$.

We introduce three sequence spaces.

(1) $l^\infty = \left\{\{a_j\}_{j\in J} : a_j \in C \text{ and } \sup_{j\in J} |a_j| < \infty\right\}$, with the norm given by

$$\|\{a_j\}_{j\in J}\|_\infty = \sup_{j\in J} |a_j|.$$

(2) $\left(\sum_{j\in J} \oplus Y_j\right)_{l^p} = \left\{\{a_j\}_{j\in J} : a_j \in Y_j \text{ and } \sum_{j\in J} \|a_j\|^p < \infty\right\}, p > 1$, corresponding norm is

$$\|\{a_j\}_{j\in J}\|_p = \left(\sum_{j\in J} \|a_j\|^p\right)^{\frac{1}{p}}.$$

(3) $\left(\sum_{j\in J} \oplus Y_j^*\right)_{l^q} = \left\{\{a_j\}_{j\in J} : a_j \in Y_j^* \text{ and } \sum_{j\in J} \|a_j\|^q < \infty\right\}, q > 1$, corresponding norm is

$$\|\{a_j\}_{j\in J}\|_q = \left(\sum_{j\in J} \|a_j\|^q\right)^{\frac{1}{q}}.$$

**Proposition 2.1** ([4]). *Let* $1 < p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}^* = \left(\sum_{j\in J} \bigoplus Y_j^*\right)_{l^q}.$$

From Proposition 2.1, we can show that the formula,

$$\langle f, g\rangle = \sum_{j\in J} \langle f_j, g_j\rangle,$$

where $f = \{f_j\}_{j\in J} \in \left(\sum_{j\in J} \oplus Y_j\right)_{l^p}$, $g = \{g_j\}_{j\in J} \in \left(\sum_{j\in J} \oplus Y_j^*\right)_{l^q}$, the inner product defines a continuous functional on $\left(\sum_{j\in J} \oplus Y_j\right)_{l^p}$ whose norm is equal to $\|g\|_q = \left(\sum_{j\in J} \|g_j\|^q\right)^{\frac{1}{q}}$.

**Definition 2.2** ([10]). Let $p > 1$, $\Lambda_j \in BL(X, Y_j)$ , $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a generalized p-frame, or simply a g-p-frame, for X with respect to $\{Y_j\}_{j\in J}$. If there exist two positive constants A and B such that

$$A\|f\| \leqslant \left(\sum_{j\in J} \|\Lambda_j f\|^p\right)^{\frac{1}{p}} \leqslant B\|f\|, \quad \forall f \in X. \tag{2.1}$$

A and B are called the lower and upper frame bounds, respectively.

If the right-hand inequality of (2.1) holds, we say that $\{\Lambda_j\}_{j\in J}$ is a g-p-Bessel sequence for $X$ with respect to $\{Y_j\}_{j\in J}$ with bound $B$.

**Definition 2.3** ([10]). Let $q > 1$ , $\Lambda_j \in BL(X, Y_j), j \in J$. We call a sequence $\{\Lambda_j\}_{j\in J}$ is a generalized q-Riesz basis, or simply a g-q-Riesz basis, for $X$ with respect to $\{Y_j\}_{j\in J}$ if the sequence $\{\Lambda_j\}_{j\in J}$ satisfies the following two conditions:

(1) $\{\Lambda_j\}_{j\in J}$ is g-complete, i.e., $\{f : \Lambda_j f = 0, j \in J\} = \{0\}$.

(2) There exist two positive constants $A$ and $B$ such that for any finite subset $J_1 \subset J$ and $a_j \in Y_j^*$, $j \in J_1$ satisfying

$$A\left(\sum_{j\in J_1} \|a_j\|^q\right)^{\frac{1}{q}} \leqslant \left\|\sum_{j\in J_1} \Lambda_j^* a_j\right\| \leqslant B\left(\sum_{j\in J_1} \|a_j\|^q\right)^{\frac{1}{q}}.$$

If $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis for $X$ with respect to $\{Y_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is also a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$. But the reverse is not true.

**Definition 2.4** ([2]). Let $\{\Lambda_j\}_{j\in J}$ be a g-p-Bessel sequence for $X$ with respect to $\{Y_j\}_{j\in J}$, define the linear operators as follows:

$$U_\Lambda : X \to \left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}, \qquad U_\Lambda f = \{\Lambda_j f\}_{j\in J},$$

$$T_\Lambda : \left(\sum_{j\in J} \bigoplus Y_j^*\right)_{l^q} \to X^*, \quad T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j.$$

$U_\Lambda$ and $T_\Lambda$ are called the analysis and synthesis operators of $\{\Lambda_j\}_{j\in J}$, respectively.

If $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$ with bounds $A$ and $B$, then $A\|f\| \leqslant \|U_\Lambda f\| \leqslant B\|f\|$, i.e., $U_\Lambda$ is linear bounded operator.

**Proposition 2.5** ([10]). *If $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$, $Q$ is invertible operator in $X$. Then $\{\Lambda_j Q\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$.*

**Proposition 2.6** ([10]). *$\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis for $X$ with respect to $\{Y_j\}_{j\in J}$ with bounds $A$ and $B$ if and only if the operator $T_\Lambda$ is invertible.*

**Proposition 2.7** ([2]). *If $\{\Lambda_j\}_{j\in J}$ is a g-p-Bessel sequence for $X$ with respect to $\{Y_j\}_{j\in J}$, then*

(a) $U_\Lambda^* = T_\Lambda$;

(b) *if $\Lambda = \{\Lambda_j \in BL(X, Y_j) : j \in J\}$ is a g-p-frame for $X$ and all of $Y_j$ are reflexive, then $T_\Lambda^* = U_\Lambda$.*

**Proposition 2.8** ([2]). *If $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$, then the following are equivalent:*

(a) *$\{\Lambda_j\}_{j\in J}$ is g-q-Riesz basis for $X$ with respect to $\{Y_j\}_{j\in J}$.*

(b) *If for any $\{a_j\}_{j\in J} \in l^q(\{Y_j^*\}_{j\in J})$, $\sum_{j\in J} \Lambda_j^* a_j = 0$, then $a_j = 0$, for all $j \in J$.*

(c) *$U_\Lambda$ is surjective.*

## 3. Dual of g-p-frame

In this section, we will introduce the concept of dual frame of g-p-frame, and prove a basic fact that g-q-Riesz basis must have dual frames. In addition, we give a property of g-p-frame which have dual frame.

**Definition 3.1.** Let $\{\Lambda_j \in BL(X, Y_j) : j \in J\}$ be a g-p-frame and $\{T_j \in BL(X^*, Y_j^*) : j \in J\}$ be a g-q-frame. If these two frames satisfy the following conditions:

$$f = \sum_{j\in J} T_j^* \Lambda_j f, \quad \forall f \in X, \tag{3.1}$$

$$f = \sum_{j\in J} \Lambda_j^* T_j f, \quad \forall f \in X^*, \tag{3.2}$$

we call $\{T_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ a pair of dual frame for X. Here, one of them is called a dual frame of other.

Let $\{\Lambda_j\}_{j\in J}$ be a g-p-Bessel sequence for X with respect to $\{Y_j\}_{j\in J}$, and $U_\Lambda$ be the analysis operator of $\{\Lambda_j\}_{j\in J}$. The adjoint operator $U_\Lambda^*$ of $U_\Lambda$ is defined by:

$$U_\Lambda^* : \left(\sum_{j\in J} \bigoplus Y_j^*\right)_{l^q} \to X^*, \quad U_\Lambda^*(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j.$$

In fact, suppose that $\{T_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ a pair of dual frame for X, $U_T$, $U_\Lambda$ are the analysis operators of $\{T_j\}_{j\in J}$, $\{\Lambda_j\}_{j\in J}$, respectively. Then (3.1), (3.2) imply that

$$I_X = U_T^* U_\Lambda, \quad I_{X^*} = U_\Lambda^* U_T.$$

**Lemma 3.2.** *Suppose $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$, $\{T_j\}_{j\in J}$ is a g-q-frame for $X^*$ with respect to $\{Y_j^*\}_{j\in J}$. If $f = \sum_{j\in J} \Lambda_j^* T_j f$, for all $f \in X^*$, then*

$$f = \sum_{j\in J} T_j^* \Lambda_j f, \quad \forall f \in X.$$

*Proof.* Let us define operator,

$$T : X \to X, \quad Tf = \sum_{j\in J} T_j^* \Lambda_j f, \quad \forall f \in X.$$

Assume the upper bounds of $\{\Lambda_j\}_{j\in J}, \{T_j\}_{j\in J}$ are $B_1, B_2$, respectively, then

$$\begin{aligned}
\|Tf\| &= \sup_{g\in X^*, \|g\|=1} |\langle Tf, g\rangle| \\
&= \sup_{g\in X^*, \|g\|=1} \left|\left\langle \sum_{j\in J} T_j^* \Lambda_j f, g\right\rangle\right| \\
&= \sup_{g\in X^*, \|g\|=1} \left|\sum_{j\in J} \langle \Lambda_j f, T_j g\rangle\right| \\
&\leqslant \sup_{g\in X^*, \|g\|=1} \left(\sum_{j\in J} \|\Lambda_j f\|^p\right)^{\frac{1}{p}} \left(\sum_{j\in J} \|T_j g\|^q\right)^{\frac{1}{q}} \\
&\leqslant B_1 B_2 \|f\|.
\end{aligned}$$

Hence $\|T\| \leqslant B_1 B_2$, $T \in BL(X)$. For any $f \in X, g \in X^*$, we have

$$\langle Tf, g\rangle = \left\langle \sum_{j\in J} T_j^* \Lambda_j f, g \right\rangle = \sum_{j\in J} \langle \Lambda_j f, T_j g\rangle,$$

$$\langle f, g\rangle = \left\langle f, \sum_{j\in J} \Lambda_j^* T_j g \right\rangle = \sum_{j\in J} \langle \Lambda_j f, T_j g\rangle.$$

So $\langle Tf, g\rangle = \langle f, g\rangle$, for all $f \in X, g \in X^*$, which implies that $T = I_X$. □

**Lemma 3.3.** *Suppose $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$, $\{T_j\}_{j\in J}$ is a g-q-frame for $X^*$ with respect to $\{Y_j^*\}_{j\in J}$. If $f=\sum\limits_{j\in J} T_j^*\Lambda_j f$, for all $f\in X$, then*

$$f=\sum_{j\in J}\Lambda_j^* T_j f,\quad \forall f\in X^*.$$

*Proof.* The proving process is similar to Lemma 3.2, so we omit the proof. □

**Theorem 3.4.** *Suppose $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis for $X$ with respect to $\{Y_j\}_{j\in J}$ with bounds $A, B$. Then $\{\Lambda_j\}_{j\in J}$ must have dual frame, and the dual frame is g-p-Riesz basis for $X^*$ with respect to $\{Y_j^*\}_{j\in J}$ with bounds $\frac{1}{B},\frac{1}{A}$.*

*Proof.* Since $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis for X with respect to $\{Y_j\}_{j\in J}$, by Proposition 2.6, synthesis operator $T_\Lambda$ is invertible. Then for every $f\in X^*$, there exists $\{g_j\}_{j\in J}\in\left(\sum\limits_{j\in J}\bigoplus Y_j^*\right)_{l^q}$ such that $f=\sum\limits_{j\in J}\Lambda_j^* g_j$. Let us define the operator,

$$K_j: X^*\to Y_j^*,\quad K_j(f)=g_j.$$

Therefore

$$f=\sum_{j\in J}\Lambda_j^* K_j f,\quad \forall f\in X^*. \tag{3.3}$$

Since $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis, for any $K_j f\in Y_j^*$, we have

$$A\Big(\sum_{j\in J}|K_j f|^q\Big)^{\frac{1}{q}}\leqslant\Big\|\sum_{j\in J}\Lambda_j^*(K_j f)\Big\|\leqslant B\Big(\sum_{j\in J}|K_j f|^q\Big)^{\frac{1}{q}}.$$

Hence

$$\frac{1}{B}\|f\|\leqslant\Big(\sum_{j\in J}|K_j f|^q\Big)^{\frac{1}{q}}\leqslant\frac{1}{A}\|f\|,$$

i.e., $\{K_j\}_{j\in j}$ is a g-q-frame for $X^*$ with respect to $\{Y_j^*\}_{j\in J}$. By Lemma 3.2, we can get

$$f=\sum_{j\in J}K_j^*\Lambda_j f,\quad \forall f\in X. \tag{3.4}$$

By Proposition 2.6, $T_\Lambda$ is invertible and $U_K=T_\Lambda^{-1}$, therefore $U_K$ is invertible. Furthermore, by Proposition 2.7, $U_K^*=T_K$ is invertible, therefore $\{K_j\}_{j\in J}$ is a g-q-Riesz basis.

Combining (3.3) and (3.4), $\{K_j\}_{j\in J}$ is the dual of $\{\Lambda_j\}_{j\in J}$ with bounds $\frac{1}{B},\frac{1}{A}$. □

**Theorem 3.5.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-p-frame with bounds $A$ and $B$ and have dual frame $\{T_j\}_{j\in J}$, assume $J_1\subset J$ and $I_X-\sum\limits_{j\in J_1}T_j^*\Lambda_j$ is invertible. Then $\{\Lambda_j\}_{j\in J\setminus J_1}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J\setminus J_1}$ with bounds $A\Big\|(I_X-\sum\limits_{j\in J_1}T_j^*\Lambda_j)^{-1}\Big\|_X^{-1}$ and $B$.*

*Proof.* Assume $\{T_j\}_{j\in J}$ is dual frame of $\{\Lambda_j\}_{j\in J}$, for any $f\in X$ we have

$$f=\sum_{j\in J}T_j^*\Lambda_j f=\sum_{j\in J\setminus J_1}T_j^*\Lambda_j f+\sum_{j\in J_1}T_j^*\Lambda_j f,$$

thus

$$I_X f-\sum_{j\in J_1}T_j^*\Lambda_j f=\sum_{j\in J\setminus J_1}T_j^*\Lambda_j f.$$

Moreover, we have

$$\begin{aligned}\Big\|I_X f-\sum_{j\in J_1} T_j^*\Lambda_j f\Big\|_X &= \Big\|\sum_{j\in J\setminus J_1} T_j^*\Lambda_j f\Big\|_X\\ &=\sup_{g\in X^*,\|g\|=1}\Big|\Big\langle \sum_{j\in J\setminus J_1} T_j^*\Lambda_j f, g\Big\rangle\Big|\\ &=\sup_{g\in X^*,\|g\|=1}\Big|\sum_{j\in J\setminus J_1}\langle \Lambda_j f, T_j g\rangle\Big|\\ &\leqslant \sup_{g\in X^*,\|g\|=1}\Big(\sum_{j\in J\setminus J_1}\|\Lambda_j f\|^p\Big)^{\frac{1}{p}}\Big(\sum_{j\in J\setminus J_1}\|T_j g\|^q\Big)^{\frac{1}{q}}\\ &\leqslant \frac{1}{A}\Big(\sum_{j\in J\setminus J_1}\|\Lambda_j f\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

If $I_X-\sum_{j\in J_1} T_j^*\Lambda_j$ is invertible on $X$, then

$$A\|(I_X-\sum_{j\in J_1} T_j^*\Lambda_j)^{-1}\|_X^{-1}\|f\|\leqslant A\|(I_X-\sum_{j\in J_1} T_j^*\Lambda_j)f\|_X\leqslant (\sum_{j\in J\setminus J_1}\|\Lambda_j f\|^p)^{\frac{1}{p}},$$

and

$$\Big(\sum_{j\in J\setminus J_1}\|\Lambda_j f\|^p\Big)^{\frac{1}{p}}\leqslant\Big(\sum_{j\in J}\|\Lambda_j f\|^p\Big)^{\frac{1}{p}}\leqslant B\|f\|.$$

Hence $\{\Lambda_j\}_{j\in J\setminus J_1}$ is a g-p-frame with bounds $A\Big\|(I_X-\sum_{j\in J_1} T_j^*\Lambda_j)^{-1}\Big\|_X^{-1}$ and $B$. □

## 4. Some properties of g-p-frame in complex Banach spaces

In this section, we will introduce some new properties of g-p-frame and g-q-Riesz basis.

**Theorem 4.1.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-p-Bessel sequence with bound $B$ and $m=\{m_j\}_{j\in J}\in l^\infty$, then $\{m_j\Lambda_j\}_{j\in J}$ is a g-p-Bessel sequence with bound $\|m\|_\infty B$.*

*Proof.* It is easy to show that, for all $f\in X$

$$\Big(\sum_{j\in J}\|m_j\Lambda_j f\|^p\Big)^{\frac{1}{p}}\leqslant\|m\|_\infty\Big(\sum_{j\in J}\|\Lambda_j f\|^p\Big)^{\frac{1}{p}}\leqslant\|m\|_\infty B\|f\|.$$

□

**Theorem 4.2.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-p-frame with bounds $A, B$, and $T\in BL(X), T_j\in BL(Y_j)$ be bound invertible operators. Suppose that*

$$0<m=\inf_{j\in J}\frac{1}{\|T_j^{-1}\|_{Y_j}}\leqslant\sup_{j\in J}\|T_j\|_{Y_j}=M<\infty,$$

*and $\Gamma_j=T_j\Lambda_j T\in BL(X,Y_j)$ for each $j\in J$. Then $\{\Gamma_j\in BL(X,Y_j): j\in J\}$ is a g-p-frame with bounds $mA\|T^{-1}\|_X^{-1}, MB\|T\|_X$.*

*Proof.* Assume that $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ with bounds A and B. Then for any $f \in X$, we have

$$\begin{aligned}\left(\sum_{j\in J}\left\|\Gamma_j f\right\|_{Y_j}^p\right)^{\frac{1}{p}} &= \left(\sum_{j\in J}\left\|T_j(\Lambda_j Tf)\right\|_{Y_j}^p\right)^{\frac{1}{p}}\\ &\geqslant \left(\sum_{j\in J}\frac{1}{\left\|T_j^{-1}\right\|_{Y_j}^p}\left\|\Lambda_j(Tf)\right\|_{Y_j}^p\right)^{\frac{1}{p}}\\ &\geqslant mA\|Tf\|_X\\ &\geqslant mA\|T^{-1}\|_X^{-1}\|f\|.\end{aligned}$$

Similarly we have

$$(\sum_{j\in J}\left\|\Gamma_j f\right\|_{Y_j}^p)^{\frac{1}{p}} \leqslant MB\|T\|_X\|f\|.$$

It follows that $\{\Gamma_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ with bounds $mA\|T^{-1}\|_X^{-1}, MB\|T\|_X$. □

*Remark* 4.3. If $T_j = E_j$ for any $j \in J$, where $E_j$ is an identity operator on $Y_j$, from Theorem 4.2, we obtain [10, theorem 2.4]; if X and $\{Y_j\}_{j\in J}$ are Hilbert spaces and $p = 2$, we obtain [8, lemma 2.13].

**Theorem 4.4.** *Let $\{T_j\}_{j\in J}$ be a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ with bounds A, B, and $\{\Lambda_j\}_{j\in J}$ be a g-p-Bessel sequence for X with respect to $\{Y_j\}_{j\in J}$ with bound M $(M < A)$. Then $\{T_j \pm \Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ with bounds $(A - M), (B + M)$.*

*Proof.* Assume that $U_\Lambda, U_T$ are analysis operators of $\{\Lambda_j\}_{j\in J}$, $\{T_j\}_{j\in J}$, respectively, then for any $f \in X$ , we have

$$\begin{aligned}\left(\sum_{j\in J}\|(T_j \pm \Lambda_j)f\|^p\right)^{\frac{1}{p}} &= \|\{(T_j \pm \Lambda_j)f\}_{j\in J}\|\\ &= \|\{T_j f\}_{j\in J} \pm \{\Lambda_j f\}_{j\in J}\|\\ &= \|U_T f \pm U_\Lambda f\|\\ &\leqslant \|U_T f\| + \|U_\Lambda f\|\\ &\leqslant (B + M)\|f\|.\end{aligned}$$

Similarly we have

$$\begin{aligned}\left(\sum_{j\in J}\|(T_j \pm \Lambda_j)f\|^p\right)^{\frac{1}{p}} &= \|\{(T_j \pm \Lambda_j)f\}_{j\in J}\|\\ &= \|\{T_j f\}_{j\in J} \pm \{\Lambda_j f\}_{j\in J}\|\\ &= \|U_T f \pm U_\Lambda f\|\\ &\geqslant \|U_T f\| - \|U_\Lambda f\|\\ &\geqslant (A - M)\|f\|.\end{aligned}$$

Hence, $\{T_j \pm \Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ with bounds $(A - M), (B + M)$. □

**Theorem 4.5.** *Let $p > 1$, $\Lambda_j \in BL(X, Y_j) : j \in J$, then $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ if and only if $\{\Lambda_j\}_{j\in J}$ is g-complete and $R(U_\Lambda)$ is closed subspace of $\left(\sum_{j\in J}\bigoplus Y_j\right)_{l^p}$.*

*Proof.* First we prove the necessity. Since $\{\Lambda_j\}_{j\in J}$ is a g-p-frame, $\Lambda_j f = 0$, for all $j \in J$, then we have $f = 0$. Hence $\{\Lambda_j\}_{j\in J}$ is g-complete.

Next we prove that $R(U_\Lambda)$ is closed. Let $\{g_n : n \geqslant 1\}$ be a Cauchy sequence in $R(U_\Lambda)$. There exist some $f_n \in X$, $n \geqslant 1$ such that $g_n = U_\Lambda f_n$. Then for any $\varepsilon > 0$, there exists $K > 0$ such that, for any $p, m > K$, $\|U_\Lambda f_p - U_\Lambda f_m\| < \varepsilon$ , since $\|U_\Lambda f\| \geqslant A\|f\|$. We have

$$A\|f_p - f_m\| \leqslant \|U_\Lambda f_p - U_\Lambda f_m\| < \varepsilon.$$

Hence $\{f_n : n \geqslant 1\}$ is a Cauchy sequence in X. Since X is Banach space, there exists $f_0 \in X$ such that $\|f_n - f_0\| \to 0$ as $n \to \infty$. It follows that

$$\|U_\Lambda f_n - U_\Lambda f_0\| \leqslant B\|f_n - f_0\| \to 0, \quad n \to \infty.$$

Hence $g_n = U_\Lambda f_n \to U_\Lambda f_0 \in R(U_\Lambda)$ as $n \to \infty$. Therefore, $R(U_\Lambda)$ is closed.

Next we prove the sufficiency. Assume that $U_\Lambda$ is analysis operator of $\{\Lambda_j\}_{j\in J}$. Since $\{\Lambda_j\}_{j\in J}$ is g-complete, $U_\Lambda$ is injective and consequently $U_\Lambda$ is bijective from X to $R(U_\Lambda)$. Obviously, $R(U_\Lambda)$ is linear closed subspace of $\left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}$, so $R(U_\Lambda)$ is a Banach space. Let $\overline{U_\Lambda} : X \to R(U_\Lambda)$ and $U_\Lambda(f) = \overline{U_\Lambda}(f)$, for all $f \in X$. By the bounded inverse theorem, there are two positive constants $\|U_\Lambda^{-1}\|^{-1}$ and B such that

$$\|U_\Lambda^{-1}\|^{-1}\|f\| \leqslant \|U_\Lambda f\| \leqslant B\|f\|, \quad \forall f \in X.$$

That is,

$$\|U_\Lambda^{-1}\|^{-1}\|f\| \leqslant \left(\sum_{j\in J} \|\Lambda_j f\|^p\right)^{\frac{1}{p}} \leqslant B\|f\|, \quad \forall f \in X.$$

Hence $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}$. □

**Theorem 4.6.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-p-frame. Then $\{\Lambda_j\}_{j\in J}$ is a g-q-Riesz basis if and only if $U_\Lambda$ is invertible.*

*Proof.* First we prove the necessity. Assume that $U_\Lambda$ is analysis operator of $\{\Lambda_j\}_{j\in J}$. Since $\{\Lambda_j\}_{j\in J}$ is g-q-Riesz basis for X with respect to $\{Y_j\}_{j\in J}$, from Proposition 2.8, $U_\Lambda$ is surjective. $\{\Lambda_j\}_{j\in J}$ is g-p-frame, $\Lambda_j f = 0$, for all $j \in J$, we have $f = 0$. Hence $U_\Lambda$ is injective. Therefore, $U_\Lambda$ is invertible.

Next we prove the sufficiency. Suppose that $U_\Lambda$ is invertible. Obviously it is injective and consequently $\{\Lambda_j\}_{j\in J}$ is g-complete. Since $U_\Lambda$ is invertible, by [6, theorem 4.12], $U_\Lambda^*$ is invertible, and

$$U_\Lambda^*(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j, \quad \forall \{g_j\}_{j\in J} \in \left(\sum_{j\in J} \bigoplus Y_j^*\right)_{l^q}.$$

There exist two positive constant A and B such that

$$A\left(\sum_{j\in J_1} \|g_j\|_{Y_j^*}^q\right)^{\frac{1}{q}} \leqslant \left\|\sum_{j\in J_1} \Lambda_j^* g_j\right\| \leqslant B\left(\sum_{j\in J_1} \|g_j\|_{Y_j^*}^q\right)^{\frac{1}{q}}.$$

□

**Theorem 4.7.** *Let $p > 1$, $\{\Lambda_j \in L(X, Y_j) : j \in J\}$, then $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for X with respect to $\{Y_j\}_{j\in J}$ if and only if X is isomorphic to a subspace M of $\left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}$.*

*Proof.* Define the analysis operator of $\{\Lambda_j\}_{j\in J}$ as

$$U_\Lambda : X \to \left(\sum_{j\in J} \bigoplus Y_j\right)_{l^p}, \quad U_\Lambda f = \{\Lambda_j f\}_{j\in J}.$$

By definition of g-p-frame, $U_\Lambda$ is injective and consequently is bijective from $X$ to $R(U_\Lambda)$. So $X$ is isomorphic to $R(U_\Lambda)$, which is a subspace of $\left(\sum_{j\in J}\bigoplus Y_j\right)_{l^p}$.

Conversely, assume that $M$ is a subspace of $\left(\sum_{j\in J}\bigoplus Y_j\right)_{l^p}$ and $T$ is an isomorphism from $X$ onto $M$. We can define an operator by:

$$P_j : M \to Y_j, \quad P_j(\{y_i\}_{i\in J}) = y_j, \quad \forall j \in J.$$

Put $\Lambda_j = P_j T$ , then $\Lambda_j : X \to Y_j$, for all $j \in J$ is bounded linear operator such that

$$\{\Lambda_j f\}_{j\in J} = \{P_j T f\}_{j\in J} \in \left(\sum_{j\in J}\bigoplus Y_j\right)_{l^p}, \quad \forall j \in J.$$

Also, for any $f \in X$, we have

$$\frac{\|f\|}{\|T^{-1}\|} \leqslant \|\{\Lambda_j f\}_{j\in J}\| = \|\{P_j T f\}_{j\in J}\| = \|Tf\| \leqslant \|T\|\|f\|.$$

This shows that $\{\Lambda_j\}_{j\in J}$ is a g-p-frame for $X$ with respect to $\{Y_j\}_{j\in J}$. □

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