Quasi associated continued fractions and Hankel determinants of Dixon elliptic functions via Sumudu transform

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Abstract

In this work, Sumudu transform of Dixon elliptic functions for higher arbitrary powers $sm^N(x, \alpha); N \geqslant 1$, $sm^N(x, \alpha)cm(x, \alpha)$; $N \geqslant 0$ and $sm^N(x, \alpha)cm^2(x, \alpha); N \geqslant 0$ by considering modulus $\alpha \neq 0$ is obtained as three term recurrences and hence expanded as product of quasi associated continued fractions where the coefficients are functions of α. Secondly the coefficients of quasi associated continued fractions are used for Hankel determinants calculations by connecting the formal power series (Maclaurin series) and associated continued fractions. ⓒ2017 All rights reserved.

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1. Introduction

Jacobi elliptic functions $sn(x, k)$, $cn(x, k)$ and $dn(x, k)$, there ratios were applied with Laplace transform to calculate Hankel determinants from their continued fractions in [26] and Fourier series of those functions were used to derive the different ways of writing square and triangular numbers [26]. Continued fractions related to Hahn and orthogonal polynomials from the results of Laplace transform of Jacobi elliptic functions such as difference equations derived in [16]. Bimodular Jacobi elliptic functions treated with Laplace transform to solve as continued fractions and by using modular transformation results were traced back to Jacobi elliptic functions in [9]. Determinants of Bernoulli numbers, coefficients of Maclaurin series of Jacobi elliptic functions were calculated through continued fractions in [2] and orthogonal polynomials from Fourier series of Jacobi elliptic functions in [8].

On studying the cubic curve $x^3 + y^3 - 3\alpha xy = 1$ in [11] gave raise to elliptic functions $sm(x, \alpha)$ and $cm(x, \alpha)$ named after Dixon whose functions are doubly periodic, related properties and examples given

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in [12]. Similar cubic curve on $\alpha = 0$ is studied and led to elliptic functions in [13]. $sm(x, 0)$ and $cm(x, 0)$ are applied to conformal mapping study and addition, multiplication formulae pertaining those maps given in [1]. Dixon elliptic functions are used for representing factorial of numbers given in [3] and related to Fermat cubes, combinatorics given in [10]. Laplace transform is applied to Dixon functions and extended to various types of continued fractions in [9]. In [22] Dixon functions connecting trefoil curve and Weierstrass elliptic functions [22, 23] have been described.

Sumudu transform successor of Laplace transform in problem solving possess unique properties namely unit, scale and dimension preserving where other transforms does not satisfy, studied to solve fractional Maxwell's equations in [21]. Sumudu transform is applied for various engineering problems for their solutions in [4, 14, 15, 18] and solved with perturbation, decomposition methods in [25, 28]. New definition of Sumudu transform for trigonometric functions and infinite series expansions is derived in [5]. Symbolic C++ and Maple procedure for Sumudu transform is given in [6]. Sumudu transform of Dumont bimodular Jacobi elliptic functions from which Sumudu transform of $\tan(\alpha x)$, $\sec(\alpha x)$ and their products for arbitrary powers are derived as product of associated continued fractions and their Hankel determinants in [7]. Sumudu transform [4–7, 14, 15, 18–21, 25, 28, 29] of $f(x)$ is defined in the set

$$A = \{f(x)|\ \exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{|x|}{\tau_j}}, \text{if } x \in (-1)^j \times [0, \infty)\},$$

given by integral equation

$$S[f(x)](u) = \int_0^\infty e^{-x} f(ux) dx = 1/u \int_0^\infty e^{-x/u} f(x) dx \ ; \ u \in (-\tau_1, \tau_2). \tag{1.1}$$

In [32] Novel Integral Transform (NIT) is defined and shown as dual for Laplace, Laplace Carson and Sumudu transforms. $N_I[f(x)](\gamma) = 1/\gamma^2 S[f(x)](u)_{u=1/\gamma}$. Here γ is the NIT variable, and extended to $(-\infty, \infty)$ and showed dual to Fourier transform in [33].

In this paper Sumudu transform is applied to Dixon Elliptic Functions (DEF) namely $sm^N(x, \alpha)$, $sm^N(x, \alpha)cm(x, \alpha)$ and $sm^N(x, \alpha)cm^2(x, \alpha)$ to show their results as product of Quasi Associated Continued Fractions (QACF). Coefficients of obtained QACF are employed for Hankel determinants derivations using Heliermann correspondence connecting formal power series to associated continued fractions.

2. Preliminaries

Considering the cubic curve $x^3 + y^3 - 3\alpha xy = 1; (\alpha \neq -1)$ in [11] derived the two set of functions $sm(x, \alpha)$ and $cm(x, \alpha)$ which are doubly periodic also showed they are elliptic. Derivative of DEF ([11, Eqs. (1) and (3), page 171] and [9, Eqs. (1.18) and (1.19), page 9])

$$\frac{d}{dx} sm(x, \alpha) = cm^2(x, \alpha) - \alpha sm(x, \alpha) \quad \text{and} \quad \frac{d}{dx} cm(x, \alpha) = -sm^2(x, \alpha) + \alpha cm(x, \alpha), \tag{2.1}$$

with ([9, Eq. (1.21), page 10]),

$$sm(0, \alpha) = 1 \quad \text{and} \quad cm(0, \alpha) = 1, \tag{2.2}$$

and the functions satisfy the cubic curve mentioned above named Pythagorean theorem ([11, Eq. (2), page 171] and [9, Eq. (1.22), page 10])

$$sm^3(x, \alpha) + cm^3(x, \alpha) - 3\alpha sm(x, \alpha)cm(x, \alpha) = 1. \tag{2.3}$$

Continued fraction notation taken from ([17, Eq. (2.1.4 b), page 18] and ([24, Eq. (1.2.5'), page 8])

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}} \frac{a_n}{b_n} = \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3 + \cdots} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}.$$

Definition 2.1. Let $a = \{a_n\}$, $b = \{b_n\}$ and u is an indeterminate, then the continued fraction of following form is called associated continued fraction ([17, Eq. (7.2.1), page 241]), [30] and ([31, Eq. (54.1), page 208]):

$$1 + \frac{a_0 u}{1 + b_0 u +} \underset{n=1}{\overset{\infty}{\mathbf{K}}} \frac{-a_n u}{1 + b_n u}.$$

Remark 2.2. The continued fraction of the form

$$\frac{a_0 u}{c_0 + b_0 u +} \underset{n=1}{\overset{\infty}{\mathbf{K}}} \frac{-a_n u}{c_n + b_n u}$$

is called QACF. Moreover the coefficients a_0, b_0, c_0, a_n, b_n and $c_n; n \geqslant 1$ are functions of u rather constants.

Definition 2.3. Let $c = \{c_\nu\}_{\nu=1}^\infty$ be a sequence in $\mathbb{C}$. Then the following $m \times m$ matrices are defined [9, 17, 26], whose determinants are denoted by respective $H_m^{(n)}$ and χ_m:

$$H_m^{(n)} = H_m^{(n)}(c_\nu) = \det \begin{pmatrix} c_n & c_{n+1} & \cdots & c_{m+n-2} & c_{n+m-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{m+n-1} & c_{m+n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{2m+n-3} & c_{2m+n-2} \end{pmatrix},$$

$$\chi_m = \chi_m(c_\nu) = \det \begin{pmatrix} c_1 & c_2 & \cdots & c_{m-1} & c_{m+1} \\ c_2 & c_3 & \cdots & c_m & c_{m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_m & c_{m+1} & \cdots & c_{2m-2} & c_{2m} \end{pmatrix}.$$

Remark 2.4. The matrix for χ_m is obtained from the matrix for $H_{m+1}^{(1)}$ by deleting the last row and next to last column [9, 17, 26]. For $n = 1$, $H_1^{(1)} = c_1$ and $\chi_1 = c_2$. Determinants $H_m^{(n)}$ and χ_m [27] are named as persymmetric determinants (or) Turanian determinants (or) Hankel determinants.

The relation between formal power series and its associated continued fractions is given as lemma [9], ([17, Theorem 7.14, pp 244–248]), [26].

Lemma 2.5. When an associated continued fraction converges to formal power series,

$$1 + \sum_{\nu=1}^\infty c_\nu z^\nu = 1 + \frac{a_0 z}{1 + b_0 z +} \underset{n=1}{\overset{\infty}{\mathbf{K}}} \frac{-a_n z^2}{1 + b_n z}, \quad (a_n \neq 0), \tag{2.4}$$

then,

$$H_m^{(1)}([c_\nu]) \neq 0, \quad a_m = \frac{H_m^{(1)} H_{m-2}^{(1)}}{\left(H_{m-1}^{(1)}\right)^2} \quad \text{and} \quad b_m = \frac{\chi_{m-1}}{H_{m-1}^{(1)}} - \frac{\chi_m}{H_m^{(1)}}, \quad (m \geqslant 1), \tag{2.5}$$

where $H_{-1}^{(1)} = H_0^{(1)} = 1$ and $\chi_0 = 0$. Conversely if (2.5) is true then (2.4) is true. Also,

$$H_m^{(1)}([c_\nu]) = \prod_{j=1}^m a_j^{m+1-j} = \prod_{j=0}^{m-1} a_j^{m-j}, \quad \text{and} \quad \frac{\chi_m([c_\nu])}{H_m^{(1)}([c_\nu])} = -\sum_{j=1}^m b_j, \quad (m \geqslant 1). \tag{2.6}$$

3. Main results 1: Sumudu transform of Dixon elliptic functions $(\alpha \neq 0)$

In [9] Laplace transform is applied for DEF and obtained QACF for up to third power of $sm(x, \alpha)$ and second power of $sm(x, \alpha)cm(x, \alpha)$ results given. In this paper, we generalize the previous work to

arbitrary powers and show its back traceable to results of [9]. Followed by, Hankel determinants are derived for the QACF using Lemma 2.5 by assuming $c_0 = c_n; n \geqslant 1 = 1$. Applying Sumudu transform (1.1) to higher powers of DEF, main results of this paper are given in following three theorems.

Theorem 3.1. Sumudu transform of DEF $sm^N(x, \alpha); N \geqslant 1$ is given by QACF:

(i)

$$\mathbb{S}[sm(x,\alpha)](u) = \frac{(1-5\alpha u)u}{F_1^* + 4u^3 +} \frac{-6E_4 u^6}{F_4 + G_4 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-E_{3n+1} H_{3n-2} u^6}{F_{3n+1} + G_{3n+1} u^3}. \tag{3.1}$$

(ii)

$$\mathbb{S}[sm^2(x,\alpha)](u) = \frac{2(1-7\alpha u)u^2}{F_2 + G_2 u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-E_{3n+2} H_{3n-1} u^6}{F_{3n+2} + G_{3n+2} u^3}. \tag{3.2}$$

(iii) Let $N = 3, 6, 9, 12, \cdots$

$$\mathbb{S}[sm^N(x,\alpha)](u) = \prod_{i=1}^{\frac{N}{3}} \frac{E_{3i} u^3}{F_{3i} + G_{3i} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-E_{3n+3i} H_{3n+3i-3} u^6}{F_{3n+3i} + G_{3n+3i} u^3}. \tag{3.3}$$

(iv) Let $N = 4, 7, 10, 13, \cdots$

$$\mathbb{S}[sm^N(x,\alpha)](u) = \frac{(1-5\alpha u)u}{F_1^* + 4u^3 +} \frac{-6E_4 u^6}{F_4 + G_4 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-E_{3n+1} H_{3n-2} u^6}{F_{3n+1} + G_{3n+1} u^3}$$

$$\times \prod_{i=1}^{\frac{N-1}{3}} \frac{E_{3i+1} u^3}{F_{3i+1} + G_{3i+1} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-E_{3n+3i+1} H_{3n+3i-2} u^6}{F_{3n+3i+1} + G_{3n+3i+1} u^3}. \tag{3.4}$$

(v) Let $N = 5, 8, 11, 14, \cdots$

$$\mathbb{S}[sm^N(x,\alpha)](u) = \frac{2(1-7\alpha u)}{F_2 + G_2 u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-E_{3n+2} H_{3n-1} u^6}{F_{3n+2} + G_{3n+2} u^3}$$

$$\times \prod_{i=1}^{\frac{N-2}{3}} \frac{E_{3i+2} u^3}{F_{3i+2} + G_{3i+2} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-E_{3n+3i+2} H_{3n+3i-1} u^6}{F_{3n+3i+2} + G_{3n+3i+2} u^3}, \tag{3.5}$$

where $E_{(.)}, F_{(.)}, G_{(.)}$ and $H_{(.)}$ are polynomials in u and α defined in equations (3.10), (3.11), (3.12), (3.13) respectively and $F_1^* = F_1^*(u, \alpha) = (1-5\alpha u)(1-2\alpha u)(1+\alpha u)$ [9].

Proof. For $N \geqslant 0$:

$$\mathbb{S}[sm^N(x,\alpha)](u) = A_N = \int_0^\infty e^{-x} sm^N(xu, \alpha) dx, \tag{3.6}$$

$$\mathbb{S}[sm^N(x,\alpha)cm(x,\alpha)](u) = B_N = \int_0^\infty e^{-x} sm^N(xu, \alpha) cm(xu, \alpha) dx, \tag{3.7}$$

$$\mathbb{S}[sm^N(x,\alpha)cm^2(x,\alpha)](u) = C_N = \int_0^\infty e^{-x} sm^N(xu, \alpha) cm^2(xu, \alpha) dx. \tag{3.8}$$

Integrating by parts, (3.6) reduces to the following difference equations using (2.1), (2.2), (2.3) in A_N and C_{N-1} with $A_0 = 1$:

$$A_1 = uC_0 - \alpha u A_1,$$
$$A_2 = 2uC_1 - 2\alpha u A_2,$$
$$A_3 = 3uC_2 - 3\alpha u A_3,$$
$$\vdots$$
$$A_N = NuC_{N-1} - N\alpha u A_N.$$

Solving for A_N using recurrences of (3.7) and (3.8) leads to QACF:

$$\frac{A_N}{A_{N-3}} = \frac{E_N u^3}{F_N + G_N u^3 - H_N u^3 \frac{A_{N+3}}{A_N}}, \quad (N \geqslant 3), \tag{3.9}$$

where,

$$E_N = E_N(u, \alpha) = (N-2)(N-1)N(1 - (2N+3)\alpha u), \quad (N \geqslant 3), \tag{3.10}$$

$$F_N = F_N(u, \alpha) = (1 - (2N-3)\alpha u)(1 - (2N+3)\alpha u)(1 + N\alpha u)(1 - 2N\alpha u), \quad (N \geqslant 2), \tag{3.11}$$

$$G_N = G_N(u, \alpha) = (1 - (2N+3)\alpha u)(N-1)^2 N + (1 - (2N-3)\alpha u)N(N+1)^2, \quad (N \geqslant 2), \tag{3.12}$$

$$H_N = H_N(u, \alpha) = N(N+1)(N+2)(1 - (2N-3)\alpha u), \quad (N \geqslant 3). \tag{3.13}$$

For $N = 1, 2$ and 3

$$A_1 = \frac{(1 - 5\alpha u)u}{(1 - 5\alpha u)(1 - 2\alpha u)(1 + \alpha u) + 4u^3 - 6u^3 \frac{A_4}{A_1}}, \tag{3.14}$$

$$A_2 = \frac{2(1 - 7\alpha u)u^2}{F_2 + G_2 u^3 - H_2 u^3 \frac{A_5}{A_2}}, \tag{3.15}$$

$$A_3 = \frac{E_3 u^3}{F_3 + G_3 u^3 - H_3 u^3 \frac{A_6}{A_3}}. \tag{3.16}$$

Now (3.1) and (3.2) are immediate consequences from respective (3.14) and (3.15) iterating with (3.9). Equations (3.3), (3.4), (3.5) are obtained by continuous iteration of (3.9) starting with (3.16) and after mathematical calculations and simplification completes the proof. □

Remark 3.2. Multiplying u to (3.1) and (3.2) gives the result in [9] ([9, Theorems 22 and 23, page 65]). Multiplying (3.3) with u and for $N = 3$ gives the result in [9] ([9, Theorem 24, page 65]). Results derived in equations (3.3), (3.4), (3.5) are new with the existing literature.

Theorem 3.3. Sumudu transform of DEF $sm^N(x, \alpha)cm(x, \alpha); N \geqslant 0$ is given by QACF:

(i)

$$\mathbb{S}[cm(x, \alpha)] = \frac{P_0^*}{Q_0^* + 2u^3 +} \frac{-6P_3 u^6}{Q_3 + R_3 u^3 +} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{-S_{3n-3}P_{3n}u^6}{Q_{3n} + R_{3n}u^3}. \tag{3.17}$$

(ii)

$$\mathbb{S}[sm(x, \alpha)cm(x, \alpha)] = \frac{P_1^* u}{Q_1^* + 12u^3 +} \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{-S_{3n-2}P_{3n+1}u^6}{Q_{3n+1} + R_{3n+1}u^3}. \tag{3.18}$$

(iii)

$$\mathbb{S}[sm^2(x, \alpha)cm(x, \alpha)] = \frac{2P_2^* u^2}{Q_2 + R_2 u^3 +} \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{-S_{3n-1}P_{3n+2}u^6}{Q_{3n+2} + R_{3n+2}u^3}. \tag{3.19}$$

(iv) Let $N = 3, 6, 9, 12, \cdots$

$$\mathbb{S}[sm^N(x, \alpha)cm(x, \alpha)] = \frac{P_0^*}{Q_0^* + 2u^3 +} \frac{-6P_3 u^6}{Q_3 + R_3 u^3 +} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{-S_{3n-3}P_{3n}u^6}{Q_{3n} + R_{3n}u^3}$$

$$\times \prod_{i=1}^{\frac{N}{3}} \frac{P_{3i}u^3}{Q_{3i} + R_{3i}u^3 +} \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{-S_{3n+3i-3}P_{3n+3i}u^6}{Q_{3n+3i} + R_{3n+3i}u^3}. \tag{3.20}$$

(v) Let $N = 4, 7, 10, 13, \cdots$

$$\mathbb{S}[sm^N(x, \alpha)cm(x, \alpha)] = \frac{P_1^* u}{Q_1^* + 12u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-S_{3n-2}P_{3n+1}u^6}{Q_{3n+1} + R_{3n+1}u^3}$$

$$\times \prod_{i=1}^{\frac{N-1}{3}} \frac{P_{3i+1}u^3}{Q_{3i+1} + R_{3i+1}u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-S_{3n+3i-2}P_{3n+3i+1}u^6}{Q_{3n+3i+1} + R_{3n+3i+1}u^3}. \tag{3.21}$$

(vi) Let $N = 5, 8, 11, 14, \cdots$

$$\mathbb{S}[sm^N(x, \alpha)cm(x, \alpha)] = \frac{2P_2^* u^2}{Q_2 + R_2 u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-S_{3n-1}P_{3n+2}u^6}{Q_{3n+2} + R_{3n+2}u^3}$$

$$\times \prod_{i=1}^{\frac{N-2}{3}} \frac{P_{3i+2}u^3}{Q_{3i+2} + R_{3i+2}u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-S_{3n+3i-1}P_{3n+3i+2}u^6}{Q_{3n+3i+2} + R_{3n+3i+2}u^3}, \tag{3.22}$$

where $P_{(.)}$, $Q_{(.)}$, $R_{(.)}$ and $S_{(.)}$ are polynomials in u and α given by equations (3.24), (3.25), (3.26), (3.27), respectively [9] and P_0^*, Q_0^*, P_1^*, Q_1^* and P_2^* are polynomials in u and α given by equations (3.31), (3.32), (3.33), (3.34), (3.35), respectively.

Proof. Using equations (2.1), (2.2), (2.3), evaluating by parts, (3.7) reduces to the following difference equations in A_{N-1} and B_N:

$$B_0 = 1 - uA_2 + \alpha uB_0,$$
$$B_1 = u - 2uA_3 + 3\alpha uB_1,$$
$$B_2 = 2uA_1 - 3uA_4 + 5\alpha uB_2,$$
$$B_3 = 3uA_2 - 4uA_5 + 7\alpha uB_3,$$
$$\vdots$$
$$B_N = NuA_{N-1} - (N+1)uA_{N+2} + (2N+1)\alpha uB_N.$$

Solving for B_N using recurrences of (3.6) and (3.8) leads to QACF:

$$\frac{B_N}{B_{N-3}} = \frac{P_N u^3}{Q_N + R_N u^3 - S_N u^3 \frac{B_{N+3}}{B_N}}, \quad (N \geqslant 3), \tag{3.23}$$

where,

$$P_N = P_N(u, \alpha) = (N-2)(N-1)N(1 - (2N+4)\alpha u)(1 + (N+2)\alpha u), \quad (N \geqslant 3), \tag{3.24}$$

$$Q_N = Q_N(u, \alpha) = (1 - (2N-2)\alpha u)(1 + (N-1)\alpha u)(1 - (2N+4)\alpha u)$$
$$\times (1 + (N+2)\alpha u)(1 - (2N+1)\alpha u), \quad (N \geqslant 2), \tag{3.25}$$

$$R_N = R_N(u, \alpha) = (N-1)N^2(1 - (2N+4)\alpha u)(1 + (N+2)\alpha u)$$
$$+ (N+1)^2(N+2)(1 - (2N-2)\alpha u)(1 + (N-1)\alpha u), \quad (N \geqslant 2), \tag{3.26}$$

$$S_N = S_N(u, \alpha) = (N+1)(N+2)(N+3)(1 - (2N-2)\alpha u)(1 + (N-1)\alpha u), \quad (N \geqslant 1). \tag{3.27}$$

For $N = 0, 1$ and 2:

$$B_0 = \frac{P_0^*}{Q_0^* + 2u^3 - 6u^3 \frac{B_3}{B_0}}, \tag{3.28}$$

$$B_1 = \frac{P_1^* u}{Q_1^* + 12u^3 - S_1 u^3 \frac{B_4}{B_1}}, \tag{3.29}$$

$$B_2 = \frac{2P_2^* u^2}{Q_2 + R_2 u^3 - S_2 u^3 \frac{B_5}{B_2}},\tag{3.30}$$

where,

$$P_0^* = P_0^*(u, \alpha) = (1 - 4\alpha u)(1 + 2\alpha u),\tag{3.31}$$

$$Q_0^* = Q_0^*(u, \alpha) = (1 - 4\alpha u)(1 + 2\alpha u)(1 - \alpha u),\tag{3.32}$$

$$P_1^* = P_1^*(u, \alpha) = (1 - 6\alpha u)(1 + 3\alpha u),\tag{3.33}$$

$$Q_1^* = Q_1^*(u, \alpha) = (1 - 6\alpha u)(1 + 3\alpha u)(1 - 3\alpha u),\tag{3.34}$$

$$P_2^* = P_2^*(u, \alpha) = (1 - 8\alpha u)(1 + 4\alpha u).\tag{3.35}$$

Hence equations (3.17), (3.18), (3.19) follow from (3.28), (3.29), (3.30) respectively iterating with (3.23). And equations (3.20), (3.21), (3.22) are obtained from continuous iteration with (3.23) and simplification which completes the proof. □

Remark 3.4. Multiplying u to equations (3.17), (3.18), (3.19) gives results in [9] ([9, Theorems 25–27, pp 66–67]). Equations (3.20), (3.21), (3.22) are new with existing literature.

Theorem 3.5. Sumudu transform of DEF $sm^N(x, \alpha)cm^2(x, \alpha); N \geqslant 0$ is given by QACF:

(i)

$$\mathbb{S}[cm^2(x, \alpha)] = \frac{T_0^*}{V_0^* + W_0^* u^3 +} \frac{-X_0^* T_3 u^6}{V_3 + W_3 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-X_{3n-3} T_{3n} u^6}{V_{3n} + W_{3n} u^3}.\tag{3.36}$$

(ii)

$$\mathbb{S}[sm(x, \alpha)cm^2(x, \alpha)] = \frac{T_1^*}{V_1^* + W_1^* u^3 +} \frac{-X_1^* T_4 u^6}{V_4 + W_4 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-X_{3n-2} T_{3n+1} u^6}{V_{3n+1} + W_{3n+1} u^3}.\tag{3.37}$$

(iii)

$$\mathbb{S}[sm^2(x, \alpha)cm^2(x, \alpha)] = \frac{T_2^* u^2}{V_2 + W_2 u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-X_{3n-1} T_{3n+2} u^6}{V_{3n+2} + W_{3n+2} u^3}.\tag{3.38}$$

(iv) Let $N = 3, 6, 9, 12, \cdots$

$$\mathbb{S}[sm^N(x, \alpha)cm^2(x, \alpha)] = \frac{T_0^*}{V_0^* + W_0^* u^3 +} \frac{-X_0^* T_3 u^6}{V_3 + W_3 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-X_{3n-3} T_{3n} u^6}{V_{3n} + W_{3n} u^3}$$
$$\times \prod_{i=1}^{\frac{N}{3}} \frac{T_{3i} u^3}{V_{3i} + W_{3i} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-X_{3n+3i-3} T_{3n+3i} u^6}{V_{3n+3i} + W_{3n+3i} u^3}.\tag{3.39}$$

(v) Let $N = 4, 7, 10, 13, \cdots$

$$\mathbb{S}[sm^N(x, \alpha)cm^2(x, \alpha)] = \frac{T_1^*}{V_1^* + W_1^* u^3 +} \frac{-X_1^* T_4 u^6}{V_4 + W_4 u^3 +} \overset{\infty}{\underset{n=2}{\mathbf{K}}} \frac{-X_{3n-2} T_{3n+1} u^6}{V_{3n+1} + W_{3n+1} u^3}$$
$$\times \prod_{i=1}^{\frac{N-1}{3}} \frac{T_{3i+1} u^3}{V_{3i+1} + W_{3i+1} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-X_{3n+3i-2} T_{3n+3i+1} u^6}{V_{3n+3i+1} + W_{3n+3i+1} u^3}.\tag{3.40}$$

(vi) Let $N = 5, 8, 11, 14, \cdots$

$$\mathbb{S}[sm^N(x,\alpha)cm^2(x,\alpha)] = \frac{T_2^* u^2}{V_2 + W_2 u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-X_{3n-1} T_{3n+2} u^6}{V_{3n+2} + W_{3n+2} u^3}$$

$$\times \prod_{i=1}^{\frac{N-2}{3}} \frac{T_{3i+2} u^3}{V_{3i+2} + W_{3i+2} u^3 +} \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{-X_{3n+3i-1} T_{3n+3i+2} u^6}{V_{3n+3i+2} + W_{3n+3i+2} u^3}, \tag{3.41}$$

where $T_{(.)}$, $V_{(.)}$, $W_{(.)}$ and $X_{(.)}$ are polynomials in u and α given by equations (3.43), (3.44), (3.45), (3.46), respectively and T_0^*, V_0^*, W_0^*, X_0^*, T_1^*, V_1^*, W_1^*, W_1^* and T_2^* are polynomials in u and α given by equations (3.50), (3.51), (3.52), (3.53), (3.54), (3.55), (3.56), (3.57), (3.58), respectively.

Proof. Integrating by parts and using (2.1), (2.2), (2.3), equation (3.8) reduces to the following difference equations in B_{N-1} and C_N:

$$C_0 = 1 - 2uB_2 + 2\alpha u C_0,$$
$$C_1 = uB_0 - 3uB_3 + 4\alpha u C_1,$$
$$C_2 = 2uB_1 - 4uB_4 + 6\alpha u C_2,$$
$$C_3 = 3uB_2 - 5uB_5 + 8\alpha u C_3,$$
$$\vdots$$
$$C_N = NuB_{N-1} - (N+2)uB_{N+2} + (2N+2)\alpha u C_N.$$

Solving for C_N using recurrences of (3.6) and (3.7) yields QACF only in C_N:

$$\frac{C_N}{C_{N-3}} = \frac{T_N u^3}{V_N + W_N u^3 - X_N u^3 \frac{C_{N+3}}{C_N}}, \quad (N \geqslant 3), \tag{3.42}$$

where,

$$T_N = T_N(u,\alpha) = (N-2)(N-1)N(1+(N+1)\alpha u)(1+(N+4)\alpha u)$$
$$\times (1-(2N+5)\alpha u), \quad (N \geqslant 3), \tag{3.43}$$
$$V_N = V_N(u,\alpha) = (1+(N+1)\alpha u)(1+(N-2)\alpha u)(1-(2N-1)\alpha u)$$
$$\times (1-(2N+2)\alpha u)(1+(N+4)\alpha u)(1-(2N+5)\alpha u), \quad (N \geqslant 1), \tag{3.44}$$
$$W_N = W_N(u,\alpha) = (1+(N-2)\alpha u)(1-(2N+5)\alpha u)(1+(N+4)\alpha u)N^2(N+1)$$
$$+ (1+(N-2)\alpha u)(1-(2N-1)\alpha u)$$
$$\times (1+(N+4)\alpha u)(N+1)(N+2)^2, \quad (N \geqslant 1), \tag{3.45}$$
$$X_N = X_N(u,\alpha) = (N+2)(N+3)(N+4)(1+(N+1)\alpha u)$$
$$\times (1+(N-2)\alpha u)(1-(2N-1)\alpha u), \quad (N \geqslant 1). \tag{3.46}$$

For $N = 0, 1$ and 2:

$$C_0 = \frac{T_0^*}{V_0^* + W_0^* u^3 - X_0^* u^3 \frac{C_3}{C_0}}, \tag{3.47}$$

$$C_1 = \frac{T_1^* u}{V_1^* + W_1^* u^3 - X_1^* u^3 \frac{C_4}{C_1}}, \tag{3.48}$$

$$C_2 = \frac{T_2^* u^2}{V_2 + W_2 u^3 - X_2 u^3 \frac{C_5}{C_2}}, \tag{3.49}$$

where,

$$T_0^* = T_0^*(u, \alpha) = (1 + \alpha u)(1 + 4\alpha u)(1 - 5\alpha u), \tag{3.50}$$

$$V_0^* = V_0^*(u, \alpha) = (1 + \alpha u)(1 + 4\alpha u)(1 - 5\alpha u)(1 - 2\alpha u), \tag{3.51}$$

$$W_0^* = W_0^*(u, \alpha) = 4(1 + 4\alpha u), \tag{3.52}$$

$$X_0^* = X_0^*(u, \alpha) = 24(1 + \alpha u), \tag{3.53}$$

$$T_1^* = T_1^*(u, \alpha) = (1 + 2\alpha u)(1 - 7\alpha u)(1 + 5\alpha u), \tag{3.54}$$

$$V_1^* = V_1^*(u, \alpha) = (1 + 2\alpha u)(1 - 7\alpha u)(1 + 5\alpha u)(1 - \alpha u)(1 - 4\alpha u), \tag{3.55}$$

$$W_1^* = W_1^*(u, \alpha) = 2(1 - 7\alpha u)(1 + 5\alpha u) + 18(1 - \alpha u)(1 + 5\alpha u), \tag{3.56}$$

$$X_1^* = X_1^*(u, \alpha) = 60(1 - \alpha u)(1 + 2\alpha u), \tag{3.57}$$

$$T_2^* = T_2^*(u, \alpha) = 2(1 + 3\alpha u)(1 - 9\alpha u)(1 + 6\alpha u). \tag{3.58}$$

Now equations (3.36), (3.37), (3.38) follow from respective equations (3.47), (3.48), (3.49) iterating with (3.42). Equations (3.39), (3.40), (3.41) are obtained by iterating continuously (3.42) and simplifying completes the proof. □

Remark 3.6. All the results in Theorem 3.5, equations (3.36), (3.37), (3.38), (3.39), (3.40), (3.41) are new with the existing literature.

Remark 3.7. NIT of DEF $sm^N(x, \alpha); N \geqslant 1$, $sm^N(x, \alpha)cm(x, \alpha); N \geqslant 0$ and $sm^N(x, \alpha)cm^2(x, \alpha); N \geqslant 0$ can be obtained from Theorems 3.1, 3.3 and 3.5 respectively by replacing $u = 1/\gamma$ and multiplying with $1/\gamma^2$ and simplifying yields the NIT of DEF.

4. Main results 2: Hankel determinants derivations

In [9] QACF is given for Laplace transform of DEF ($\alpha \neq 0$) and for Hankel determinants calculations to those QACF ([9, first query, pp 73–73]) are given as future directions. In this section Hankel determinants pertaining from Section 3 using Lemma 2.5 are calculated. Main results of this section are given in the following three theorems.

Theorem 4.1. Hankel determinants corresponding to Theorem 3.1 are given by,

(i)

$$H_1^{(1)}([sm(x, \alpha)]_{3\nu+1}) = (1 - 5\alpha u), \tag{4.1}$$

$$H_2^{(1)}([sm(x, \alpha)]_{3\nu+1}) = 6(1 - 5\alpha u)^2 E_4, \tag{4.2}$$

$$H_m^{(1)}([sm(x, \alpha)]_{3\nu+1}) = 6^{(m-1)}(1 - 5\alpha u)^m \prod_{j=0}^{m-1} E_{3j+4}^{(m-j-1)} \prod_{j=1}^{m-2} H_{3j+1}^{(m-j-1)}, \quad m \geqslant 3, \tag{4.3}$$

$$\frac{\chi_1([sm(x, \alpha)]_{3\nu+1})}{H_1^{(1)}([sm(x, \alpha)]_{3\nu+1})} = -4, \tag{4.4}$$

$$\frac{\chi_m([sm(x, \alpha)]_{3\nu+1})}{H_m^{(1)}([sm(x, \alpha)]_{3\nu+1})} = -\left[4 + \sum_{j=1}^{m-1} G_{3j+1}\right], \quad m \geqslant 2. \tag{4.5}$$

(ii)

$$H_1^{(1)}([sm^2(x, \alpha)]_{3\nu+2}) = 2(1 - 7\alpha u), \tag{4.6}$$

$$H_m^{(1)}([sm^2(x, \alpha)]_{3\nu+2}) = 2^m(1 - 7\alpha u)^m \prod_{j=1}^{m-1} E_{3j+2}^{(m-j)} H_{3j-1}^{(m-j)}, \quad m \geqslant 2, \tag{4.7}$$

$$\frac{\chi_m([sm^2(x,\alpha)]_{3\nu+2})}{H_m^{(1)}([sm^2(x,\alpha)]_{3\nu+2})} = -\sum_{j=1}^m G_{3j-1}, \quad m \geqslant 1. \tag{4.8}$$

(iii) Let $N = 3, 6, 9, 12, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i}, \tag{4.9}$$

$$H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i}^m \prod_{j=1}^{m-1} E_{3j+3i}^{(m-j)} H_{3j+3i-3}^{(m-j)}, \quad m \geqslant 2, \tag{4.10}$$

$$\frac{\chi_m([sm^N(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^m G_{3j+3i-3}, \quad m \geqslant 1, \tag{4.11}$$

where $i = 1, 2, 3, \cdots, \frac{N}{3}$.

(iv) Let $N = 4, 7, 10, 13, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i+1},$$

$$H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i+1}^m \prod_{j=1}^{m-1} E_{3j+3i+1}^{(m-j)} H_{3j+3i-2}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^m G_{3j+3i-2}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-1}{3}$.

(v) Let $N = 5, 8, 11, 14, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i+2},$$

$$H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N}) = E_{3i+2}^m \prod_{j=1}^{m-1} E_{3j+3i+2}^{(m-j)} H_{3j+3i-1}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^m G_{3j+3i-1}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-2}{3}$.

Here $E_{(.)}$, $G_{(.)}$ and $H_{(.)}$ are given by (3.10), (3.12) and (3.13), respectively.

Proof. Maclaurin series of $sm^N(x,\alpha); N \geqslant 1$ is given by,

$$sm^N(x,\alpha) = \sum_{\nu=0}^\infty \frac{[sm^N(x,\alpha)]_{3\nu+N} x^{3\nu+N}}{(3\nu+N)!}, \quad N \geqslant 1,$$

where $[sm^N(x,\alpha)]_{3\nu+N}$ are Maclaurin series coefficients in polynomials of α. Next assume

$$(1-5\alpha u)(1-2\alpha u)(1+\alpha u) \equiv 1,$$

in (3.1), (3.4) and $F_N; (N \geqslant 2) \equiv 1$ in Theorem 3.1, so that it satisfies Lemma 2.5. Now applying coefficients of Theorem 3.1 in (2.6) of Lemma 2.5 gives the Hankel determinants for equations (3.1), (3.2), (3.3), (3.4), (3.5) of Theorem 3.1 by respective enumerates in Theorem 4.1. $\square$

Remark 4.2. Substituting $\alpha = 0$ in equations (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10), (4.11) for $N = 3$ in Theorem 4.1 gives the Hankel determinants in [9] ([9, Theorems 10–12, pp 48–50]), respectively.

Theorem 4.3. Hankel determinants corresponding to Theorem 3.3 are given by,

(i)

$$H_1^{(1)}([cm(x,\alpha)]_{3\nu}) = P_0^*, \tag{4.12}$$

$$H_2^{(1)}([cm(x,\alpha)]_{3\nu}) = 6(P_0^*)^2 P_3, \tag{4.13}$$

$$H_m^{(1)}([cm(x,\alpha)]_{3\nu}) = 6^{(m-1)}(P_0^*)^m P_3^{(m-1)} \prod_{j=1}^{m-2} S_{3j}^{(m-j-1)} P_{3j+3}^{(m-j-1)}, \quad m \geqslant 3, \tag{4.14}$$

$$\frac{\chi_1([cm(x,\alpha)]_{3\nu})}{H_1^{(1)}([cm(x,\alpha)]_{3\nu})} = -2, \tag{4.15}$$

$$\frac{\chi_m([cm(x,\alpha)]_{3\nu})}{H_m^{(1)}([cm(x,\alpha)]_{3\nu})} = -\left[2 + \sum_{j=1}^{m-1} R_{3j}\right], \quad m \geqslant 2. \tag{4.16}$$

(ii)

$$H_1^{(1)}([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1}) = P_1^*, \tag{4.17}$$

$$H_m^{(1)}([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1}) = (P_1^*)^m \prod_{j=1}^{m-1} S_{3j-2}^{(m-j)} P_{3j+1}^{(m-j)}, \quad m \geqslant 2, \tag{4.18}$$

$$\frac{\chi_1([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1})}{H_1^{(1)}([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1})} = -12, \tag{4.19}$$

$$\frac{\chi_m([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1})}{H_m^{(1)}([sm(x,\alpha)cm(x,\alpha)]_{3\nu+1})} = -\left[12 + \sum_{j=1}^{m-1} R_{3j+1}\right], \quad m \geqslant 2. \tag{4.20}$$

(iii)

$$H_1^{(1)}([sm^2(x,\alpha)cm(x,\alpha)]_{3\nu+2}) = 2P_2^*, \tag{4.21}$$

$$H_m^{(1)}([sm^2(x,\alpha)cm(x,\alpha)]_{3\nu+2}) = 2^m (P_2^*)^m \prod_{j=1}^{m-1} S_{3j-1}^{(m-j)} P_{3j+2}^{(m-j)}, \quad m \geqslant 2, \tag{4.22}$$

$$\frac{\chi_m([sm^2(x,\alpha)cm(x,\alpha)]_{3\nu+2})}{H_m^{(1)}([sm^2(x,\alpha)cm(x,\alpha)]_{3\nu+2})} = -\sum_{j=1}^{m} R_{3j-1}, \quad m \geqslant 1. \tag{4.23}$$

(iv) Let $N = 3, 6, 9, 12, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)cm(x,\alpha)]_{3\nu+N}) = P_{3i},$$

$$H_m^{(1)}([sm^N(x,\alpha)cm(x,\alpha)]_{3\nu+N}) = (P_{3i})^m \prod_{j=1}^{m-1} S_{3j+3i-3}^{(m-j)} P_{3j+3i}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)cm(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)cm(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^{m} R_{3j+3i-3}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N}{3}$.

(v) Let $N = 4, 7, 10, 13, \cdots$

$$H_1^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N}) = P_{3i+1},$$

$$H_m^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N}) = (P_{3i+1})^m \prod_{j=1}^{m-1} S_{3j+3i-2}^{(m-j)} P_{3j+3i+1}^{(m-j)} \; ; \; m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N})} = - \sum_{j=1}^m R_{3j+3i-2}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-1}{3}$.

(vi) Let $N = 5, 8, 11, 14, \cdots$

$$H_1^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N}) = P_{3i+2},$$

$$H_m^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N}) = (P_{3i+2})^m \prod_{j=1}^{m-1} S_{3j+3i-1}^{(m-j)} P_{3j+3i+2}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N})} = - \sum_{j=1}^m R_{3j+3i-1}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-2}{3}$.

Here $P_{(.)}$, $R_{(.)}$, $S_{(.)}$, P_0^*, P_1^* and P_2^* are given by respective equations (3.24), (3.26), (3.27), (3.31), (3.33) and (3.35).

Proof. Maclaurin series of $sm^N(x, \alpha)cm(x, \alpha)$; $N \geqslant 0$ is given by,

$$sm^N(x, \alpha)cm(x, \alpha) = \sum_{\nu=0}^{\infty} \frac{[sm^N(x, 0)cm(x, 0)]_{3\nu+N} x^{3\nu+N}}{(3\nu + N)!}, \quad N \geqslant 0,$$

where $[sm^N(x, \alpha)cm(x, \alpha)]_{3\nu+N}$; $N \geqslant 0$ denotes Maclaurin series coefficients in polynomials of α. Now assume $(1 - 4\alpha u)(1 + 2\alpha u)(1 - \alpha u) \equiv 1$ in (3.17) and (3.20), $(1 - 6\alpha u)(1 + 3\alpha u)(1 - 3\alpha u) \equiv 1$ in (3.18) and (3.21), Q_N; $N \geqslant 2 \equiv 1$ in Theorem 3.3 so that it satisfies Lemma 2.5. Applying coefficients of Theorem 3.3 in (2.6) of Lemma 2.5 gives the Hankel determinants for equations (3.17), (3.18), (3.19), (3.20), (3.21), (3.22) by respective enumerates of Theorem 4.3. □

Remark 4.4. Substituting $\alpha = 0$ in equations (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23), Theorem 4.3 gives the results in [9] ([9, Theorems 13–15, pp 51–53]), respectively.

Theorem 4.5. Hankel determinants corresponding to Theorem 3.5 are given by,

(i)

$$H_1^{(1)}([cm^2(x, \alpha)]_{3\nu}) = T_0^*,$$
$$H_2^{(1)}([cm^2(x, \alpha)]_{3\nu}) = (T_0^*)^2 X_0^* T_3,$$
$$H_m^{(1)}([cm^2(x, \alpha)]_{3\nu}) = (T_0^*)^m (X_0^*)^{m-1} (T_3)^{m-1} \prod_{j=1}^{m-2} X_{3j}^{(m-j-1)} T_{3j+3}^{(m-j-1)}, \quad m \geqslant 3,$$

$$\frac{\chi_1([cm^2(x, \alpha)]_{3\nu})}{H_1^{(1)}([cm^2(x, \alpha)]_{3\nu})} = -W_0^*,$$

$$\frac{\chi_m([cm^2(x, \alpha)]_{3\nu})}{H_m^{(1)}([cm^2(x, \alpha)]_{3\nu})} = - \left[W_0^* + \sum_{j=1}^{m-1} W_{3j} \right], \quad m \geqslant 2.$$

(ii)

$$H_1^{(1)}([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1}) = T_1^*,$$

$$H_2^{(1)}([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1}) = (T_1^*)^2 X_1^* T_4,$$

$$H_m^{(1)}([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1}) = (T_1^*)^m (X_1^*)^{m-1}(T_4)^{m-1} \prod_{j=1}^{m-2} X_{3j+1}^{(m-j-1)} T_{3j+4}^{(m-j-1)}, \quad m \geqslant 3,$$

$$\frac{\chi_1([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1})}{H_1^{(1)}([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1})} = -W_1^*,$$

$$\frac{\chi_m([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1})}{H_m^{(1)}([sm(x,\alpha)cm^2(x,\alpha)]_{3\nu+1})} = -\left[W_1^* + \sum_{j=1}^{m-1} W_{3j+1}\right], \quad m \geqslant 2.$$

(iii)

$$H_1^{(1)}([sm^2(x,\alpha)cm^2(x,\alpha)]_{3\nu+2}) = T_2^*,$$

$$H_m^{(1)}([sm^2(x,\alpha)cm^2(x,\alpha)]_{3\nu+2}) = (T_2^*)^m \prod_{j=1}^{m-1} X_{3j-1}^{(m-j)} T_{3j+2}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^2(x,\alpha)cm^2(x,\alpha)]_{3\nu+2})}{H_m^{(1)}([sm^2(x,\alpha)cm^2(x,\alpha)]_{3\nu+2})} = -\sum_{j=1}^{m} W_{3j-1}, \quad m \geqslant 1.$$

(iv) Let $N = 3, 6, 9, 12, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = T_{3i},$$

$$H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = (T_{3i})^m \prod_{j=1}^{m-1} X_{3j+3i-3}^{(m-j)} T_{3j+3i}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^{m} W_{3j+3i-3}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N}{3}$.

(v) Let $N = 4, 7, 10, 13, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = T_{3i+1},$$

$$H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = (T_{3i+1})^m \prod_{j=1}^{m-1} X_{3j+3i-2}^{(m-j)} T_{3j+3i+1}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^{m} W_{3j+3i-2}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-1}{3}$.

(vi) Let $N = 5, 8, 11, 14, \cdots$

$$H_1^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = T_{3i+2},$$

$$H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}) = (T_{3i+2})^m \prod_{j=1}^{m-1} X_{3j+3i-1}^{(m-j)} T_{3j+3i+2}^{(m-j)}, \quad m \geqslant 2,$$

$$\frac{\chi_m([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})}{H_m^{(1)}([sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N})} = -\sum_{j=1}^{m} W_{3j+3i-1}, \quad m \geqslant 1,$$

where $i = 1, 2, 3, \cdots, \frac{N-2}{3}$.

Here $T_{(.)}$, $W_{(.)}$, $X_{(.)}$, T_0^*, W_0^*, X_0^*, T_1^*, W_1^*, X_1^* and T_2^* are given by respective equations (3.43), (3.45), (3.46), (3.50), (3.52), (3.53), (3.54), (3.56), (3.57) and (3.58).

Proof. Maclaurin series of $sm^N(x,\alpha)cm^2(x,\alpha); N \geqslant 0$ is given by,

$$sm^N(x,\alpha)cm^2(x,\alpha) = \sum_{\nu=0}^{\infty} \frac{[sm^N(x,0)cm(x,0)]_{3\nu+N} x^{3\nu+N}}{(3\nu+N)!}, \quad N \geqslant 0,$$

where $[sm^N(x,\alpha)cm^2(x,\alpha)]_{3\nu+N}; N \geqslant 0$ denotes Maclaurin series coefficients in polynomials of α. Now assume $V_0^* \equiv 1, V_1^* \equiv 1$ and $V_N; N \geqslant 2 \equiv 1$ in Theorem 3.5 so that it satisfies Lemma 2.5. Applying coefficients of Theorem 3.5 in (2.6) of Lemma 2.5 gives the Hankel determinants for equations (3.36), (3.37), (3.38), (3.39), (3.40), (3.41) by respective enumerates of Theorem 4.5.   □

Remark 4.6. Theorem 4.5 appears for first time in this work.

5. Conclusion

QCAF expansions of $sm^N(x,\alpha); N \geqslant 4$, $sm^N(x,\alpha)cm(x,\alpha); N \geqslant 3$ and $sm^N(x,\alpha)cm^2(x,\alpha); N \geqslant 0$ were appears for the first time in this work to the literature reviewed. Hankel determinants results were checked with Maple and confirmed that the results were exact as well as coincides with the results of lower order associated continued fractions in literature. Remarks 4.2 and 4.4 ensures that the assumption of c_0 and $c_n; n \geqslant 1$ in the denominator of QACF to 1 is correct in Hankel determinants calculations. Also the compact form of Hankel determinants results in this work will ease the implementation in any of the Computer Algebra Software (CAS). For the future study it remains the open query to find Sumudu transform of $cm^3(x,\alpha)$ and other higher powers which leads to the four term recurrences. Next Hankel determinants calculations of QACF by restricting the assumptions made in this paper remains open query for further study.

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