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Some fixed point results via measure of noncompactness

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Abstract

In this paper, by using the measure of noncompactness and Meir-Keeler type mappings, we prove some new fixed point theorems for some certain mappings, namely, the weaker φ-Meir-Keeler type contractions, asymptotic weaker φ-Meir-Keeler type contractions, asymptotic sequence $\{\phi_i\}$-Meir-Keeler type contraction, ξ-generalized comparison type contraction, and $\mathcal{R}$-functional type ψ-contractions. Our results improve and hence cover the well-known Darbo's fixed point theorem, and several related recent fixed point results. ©2017 All rights reserved.

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1. Introduction and preliminaries

In the frame of topology, initial fixed point theorems were given by Brouwer and Schauder. In both theorems the notion of the compactness plays an essential role. On the other hand, the notion of a measure of noncompactness appears in various contexts and played an important role in several branch of mathematics, in particular, nonlinear functional analysis (see, for instance, [3, 5, 11, 13, 14]). Consequently, a natural question was appeared: "is it possible to get a fixed point if the compactness is dropped in well-known Schauder fixed point theorem". An affirmative answer was given by Darbo [8] in 1955. He successively extended the Schauder fixed point theorem to the setting of noncompact operators, by introducing the notion of k-set contraction.

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_0^+$ denote the set of all natural numbers, real numbers, and non-negative real numbers, respectively. Let $(E, \|\cdot\|)$ be a Banach space. Moreover, closed convex hull of a subset X of E will be denoted by $co(X)$ and the closure of X will be represented by $\overline{X}$. Furthermore, $B(E)$, $CB(E)$, and $K(E)$ denote the family of all nonempty and bounded subsets of E, the family of all nonempty, bounded and closed subsets of E, and the family of all nonempty and relatively compact subsets of E, respectively.

For the sake of completeness, we first recall the well-known fixed point theorem of Schauder.

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Theorem 1.1 (Schauder fixed point theorem)**.** *Let* X *be a nonempty, closed, and convex subset of a Banach space* $(E, \|\cdot\|)$*. Then, every compact and continuous map* $T : X \to X$ *has at least one fixed point in* X*.*

Notice that the notion of measure of noncompactness was first introduced and studied by Kuratowski [10]. For a bounded subset X of a metric space (M, d), the Kuratowski concept of measure of noncompactness is defined as follows:

$$\mu(X) = \inf\{\delta > 0 : \ X = \bigcup_{i=1}^{n} X_i \ \text{ for some } \ X_i \ \text{ with } \ \operatorname{diam}(X_i) \leqslant \delta \ \text{ for } \ 1 \leqslant n < \infty\}.$$

Here, $\operatorname{diam}(S)$ denotes the diameter of a set $S \subset M$, that is,

$$\operatorname{diam}(S) = \sup\{d(x, y) : x, y \in S\}.$$

Now, we shall list some basic definitions and essential results.

Definition 1.2 ([6])**.** A mapping $\sigma : B(E) \to \mathbb{R}$ is said to be a measure of noncompactness in E if it satisfies the following conditions: for all $X, Y \in B(E)$ and $\lambda \in [0, 1]$,

(1) the family $\operatorname{Ker}(\sigma)$ is nonempty and $\operatorname{Ker}(\sigma) \subset B(E)$, where

$$\operatorname{Ker}(\sigma) = \{X \in B(E) : \ \sigma(X) = 0\};$$

(2) $X \subset Y \implies \sigma(X) \leqslant \sigma(Y)$;
(3) $\sigma(\overline{X}) \leqslant \sigma(X)$;
(4) $\sigma(co(X)) \leqslant \sigma(X)$;
(5) $\sigma(\lambda X + (1-\lambda)Y)) \leqslant \lambda\sigma(X) + (1-\lambda)\sigma(Y)$;
(6) if $\{X_n\}$ is a sequence of bounded subsets of $B(E)$ such that $X_{n+1} \subset X_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} \sigma(X_n) = 0$, then $X_\infty = \cap_{n\in\mathbb{N}} X_n \neq \phi$.

Darbo [8] used measure of noncompactness to generalize Schauder theorem to wide class of operators, called k-set contractive operators, which satisfy the following condition:

$$\sigma(T(X)) \leqslant k\sigma(X)$$

for all $X \in B(E)$ and for some $k \in [0, 1)$. Notice that this observation extends both the classical Schauder fixed point theorem and the Banach contraction principle.

Theorem 1.3 (Darbo fixed point theorem)**.** *Let* X *be a nonempty, closed, and convex subset of a Banach space* $(E, \|\cdot\|)$ *and let* $T : X \to X$ *be a continuous mapping such that there exists a constant* $k \in [0, 1)$ *with the property* $\sigma(T(A)) \leqslant k\sigma(A)$ *for any nonempty subset* A *of* X*. Then* T *has at least one fixed point in* X*.*

Later, Aghajani et al. [2] introduced the concept of a Meir-Keleer condensing operator and proved some fixed point results.

Theorem 1.4 ([2])**.** *Let* X *be a nonempty, closed, and convex subset of a Banach space* $(E, \|\cdot\|)$ *and let* $T : X \to X$ *be a continuous and Meir-Keeler condensing operator. Then* T *has at least one fixed point in* X*.*

In this paper by using the measure of noncompactness and Meir-Keeler type mappings, we prove some new fixed point theorems for the weaker φ-Meir-Keeler type contractions, asymptotic weaker φ-Meir-Keeler type contractions, asymptotic sequence $\{\phi_i\}$-Meir-Keeler type contraction, ξ-generalized comparison type contraction and $\mathcal{R}$-functional type ψ-contractions. Our results provide the generalization of Darbo's fixed point theorem, and improve many recent fixed point results via measure of noncompactness.

2. Main results

In this section, before giving the main results, first, we recollect the notions of Meir-Keeler mapping (see [12]) and weaker Meir-Keeler mapping (see [7]) as follows:

Definition 2.1. A function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a Meir-Keeler mapping, if ϕ satisfies the following condition:

$$\forall \eta > 0 \;\; \exists \delta > 0 \;\; \forall t \in \mathbb{R} \;\; (\eta \leqslant t < \eta + \delta \Rightarrow \phi(t) < \eta).$$

Remark 2.2. It is clear that if $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Meir-Keeler mapping, then we have

$$\phi(t) < t \;\text{ for all }\; t > 0.$$

Example 2.3. Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined by

$$\phi(t) = \begin{cases} \frac{t}{3}, & \text{if } t \in (1, \infty), \\ \frac{t}{1+2t}, & \text{if } t \in [0, 1]. \end{cases}$$

Then, ϕ is a Meir-Keeler mapping.

Definition 2.4. A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a weaker Meir-Keeler mapping, if φ satisfies the following condition:

$$\forall \eta > 0 \;\; \exists \delta > 0 \;\; \forall t \in \mathbb{R} \;\; (\eta \leqslant t < \eta + \delta \Rightarrow \exists n_0 \in \mathbb{N}, \;\; \varphi^{n_0}(t) < \eta).$$

Remark 2.5. It is clear that if $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a weaker Meir-Keeler mapping, then there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^{n_0}(t) < t \;\text{ for all }\; t > 0.$$

Example 2.6. Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be denoted by

$$\varphi(t) = \begin{cases} \frac{t}{t+1}, & \text{if } t \leqslant 1, \\ 2t, & \text{if } 1 < t < 2, \\ \frac{t}{4}, & \text{if } t \geqslant 2. \end{cases}$$

Then, φ is a weaker Meir-Keeler mapping.

Inspired from the weaker Meir-Keeler mapping and the measure of noncompactness, we introduce the notion of weaker φ-Meir-Keeler type contraction as follows:

Definition 2.7. Let X be a nonempty subset of a Banach space E and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. A self-mapping $T : X \to X$ is said to be a weaker φ-Meir-Keeler type contraction with respect to the measure σ if the following conditions hold:

1. $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a nondecreasing weaker Meir-Keeler mapping,
2. for each bounded subset A of X, we have

$$\sigma(T(A)) \leqslant \varphi(\sigma(A)).$$

The following is the main fixed point result for the weaker φ-Meir-Keeler type contraction with respect to the measure σ.

Theorem 2.8. *Let X be a nonempty, bounded, closed, and convex subset of a Banach space E, and $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a mapping. Suppose that $T : X \to X$ is continuous and a weaker φ-Meir-Keeler type contraction with respect to the measure σ. Suppose also that $\{\varphi^n(\sigma(X))\}_{n \in \mathbb{N}}$ is decreasing. Then, T has a fixed point in X.*

Proof. Define the sequence $\{X_n\}$ of sets as follows:

$$X_0 = X \text{ and } X_{n+1} = co(T(X_n)) \text{ for all } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Then we have that for all $n \in \mathbb{N}_0$,

(1) X_n is a convex subset of X;
(2) $X_{n+1} \subset X_n$;
Indeed, it is clear that $X_1 \subset X_0$, and

$$X_2 = co(T(X_1)) \subset co(T(X_0)) = X_1.$$

Iteratively, one can observe that

$$X_{n+1} = co(T(X_n)) \subset co(T(X_{n-1})) = X_n.$$

(3) $T(X_n) \subset co(T(X_n)) = X_{n+1}$.

By the properties of measure of noncompactness, we obtain that

$$\sigma(X_{n+1}) \leqslant \sigma(X_n)$$

for each $n \in \mathbb{N}_0$, that is, $\{\sigma(X_n)\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$.

Now, if there exists an $N \in \mathbb{N}$ such that $\sigma(X_N) = 0$, then X_N is relatively compact. By (2) and (3), we also have that $T(X_N) \subset X_N$. Thus, by Schauder fixed point theorem, we deduce that T has a fixed point. So we assume that $\sigma(X_n) > 0$ for each $n \in \mathbb{N}_0$. By the properties of weaker Meir-Keeler mapping and since φ is nondecreasing, we obtain that for each $n \in \mathbb{N}_0$

$$\sigma(X_{n+1}) = \sigma(co(T(X_n))) = \sigma(T(X_n)) \leqslant \varphi(\sigma(X_n)).$$

Recursively, we obtain that

$$\sigma(X_{n+1}) \leqslant \varphi(\sigma(X_n)) \leqslant \varphi^2(\sigma(X_{n-1})) \leqslant \cdots \leqslant \varphi^n(\sigma(X_0)).$$

Since $\{\varphi^n(\sigma(X_0))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\gamma \geqslant 0$, that is, $\lim_{n\to\infty} \varphi^n(\sigma(X_0)) = \gamma$. We claim that $\gamma = 0$. Suppose to the contrary, that $\gamma > 0$. Since φ is a weaker Meir-Keeler mapping, there exists $\delta > 0$ such that for each $X_0 \subset X$ with $\gamma \leqslant \sigma(X_0) < \gamma + \delta$, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^{n_0}(\sigma(X_0)) < \gamma.$$

Since $\lim_{n\to\infty} \varphi^n(\sigma(X_0)) = \gamma$, corresponding to γ and δ, there exists $m_0 \in \mathbb{N}$ such that

$$\gamma \leqslant \varphi^m(\sigma(X_0)) < \gamma + \delta \text{ for all } m \geqslant m_0.$$

From the argument above, we easily get

$$\varphi^{n_0+m_0}(\sigma(X_0)) < \gamma,$$

a contradiction. Thus, we have

$$\lim_{n\to\infty} \varphi^n(\sigma(X_0)) = 0.$$

Hence, we deduce that

$$\lim_{n\to\infty} \sigma(X_n) = 0.$$

From condition (6) of Definition 1.2, $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$ is a nonempty closed and convex subset of X, and X_∞ is invariant under T and $X_\infty \in Ker(\sigma)$. Applying the Schauder fixed point theorem, we conclude that T has a fixed point. □

In what follows, we recollect the notion of contractive:

Definition 2.9. A mapping $T : X \to X$ is said to be contractive if $d(Tx, Ty) < d(X, y)$ for all $x, y \in X$ and $x \neq y$.

The following theorem was proposed by Edelstin [1] which infers our next main result.

Theorem 2.10 ([1])**.** *Let (M, d) be a nonempty and compact metric space. Let T be a contractive map on M, then there exists a unique fixed point in M.*

Definition 2.11. Let X be a nonempty subset of a Banach space E and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. We call $T : X \to X$ an asymptotic weaker φ-Meir-Keeler type contraction with respect to the measure σ if the following conditions hold:

1. $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a nondecreasing weaker Meir-Keeler mapping;
2. $\lim_{n\to\infty} \varphi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$;
3. for each $i \in \mathbb{N}$, we have
$$\sigma(T^i(X)) \leqslant \varphi^i(\sigma(X)).$$

The following is our main fixed point result for the asymptotic weaker φ-Meir-Keeler type contraction (the domain X is not necessarily convex).

Theorem 2.12. *Let X be a nonempty, bounded and closed subset of a Banach space E, let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, and let $T : X \to X$ be a contractive operator. Suppose $T : X \to X$ is a asymptotic weaker φ-Meir-Keeler type contraction with respect to the measure σ. Then T has a fixed point in X.*

Proof. Define the sequence $\{X_n\}$ of sets as follows:
$$X_0 = X \text{ and } X_n = \overline{T^n(X_0)} \text{ for all } n \in \mathbb{N}_0.$$
Since T is a contractive operator, T is continuous and $T(\overline{A}) \subset \overline{T(A)}$. From above argument, we conclude that for all $n \in \mathbb{N}_0$,

(1) $T^{n+1}(X) \subset T^n(X)$;
(2) $X_{n+1} \subset X_n$;
(3) $T(X_n) \subset X_n$.

By the properties of measure of noncompactness, we obtain that $\sigma(X_{n+1}) \leqslant \sigma(X_n)$ for each $n \in \mathbb{N}_0$, that is, $\{\sigma(X_n)\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$. Since φ is nondecreasing, we also conclude that $\{\varphi(\sigma(X_n))\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$.

Now if there exists an $N \in \mathbb{N}$ such that $\sigma(X_N) = 0$, then X_N is relatively compact. By (2) and (3), we also have that $T(X_N) \subset X_N$. Thus, by Schauder fixed point theorem, we deduce that T has a fixed point. So we assume that $\sigma(X_n) > 0$ for each $n \in \mathbb{N}_0$. Since $\{\varphi(\sigma(X_n))\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$, it must converge to some γ. Notice that
$$\gamma = \inf\{\varphi(\sigma(X_n)) : n \in \mathbb{N}_0\}.$$
We now claim that $\gamma = 0$. Suppose, on the the contrary, that $\gamma > 0$. Since φ is a weaker Meir-Keeler mapping, there exists $\delta > 0$ such that for each $X_0 \subset X$ with $\gamma \leqslant \sigma(X_0) < \gamma + \delta$, there exists $n_0 \in \mathbb{N}$ such that
$$\varphi^{n_0}(\sigma(X_0)) < \gamma.$$
And, corresponding to this δ, there exists $m_o \in \mathbb{N}$ such that
$$\gamma \leqslant \varphi^m(\sigma(X_0)) < \gamma + \delta$$
for all $m \geqslant m_0$.

By the condition (3) of Definition 2.11 and from above argument, we have

$$\begin{aligned}\varphi(\sigma(X_{n_0+m_0-1})) &= \varphi(\sigma(T^{n_0+m_0-1})(\sigma(X_0))) \leqslant \varphi(\varphi^{n_0+m_0-1}(\sigma(X_0))) \leqslant \varphi^{n_0+m_0}(\sigma(X_0)) \\ &= \varphi^{n_0}(\varphi^{m_0}(\sigma(X_0)) < \gamma,\end{aligned}$$

which is a contradiction. Thus, we obtain that $\gamma = 0$, and we have

$$\lim_{n\to\infty} \varphi(\sigma(X_n)) = 0.$$

By the condition (2) of Definition 2.11, we also have

$$\lim_{n\to\infty} \sigma(X_n) = 0.$$

From the condition (6) of Definition 1.2, $X_\infty = \cap_{n\in\mathbb{N}} X_n$ is a nonempty closed and convex subset of X, and X_∞ is invariant under T and $X_\infty \in \operatorname{Ker}(\sigma)$. By Theorem 2.10, we deduce that T has a fixed point in X_∞ and as $F_T = \{x \in X : T(x) = x\} \subset X_n$ for all $n \in \mathbb{N}$. Thus, $F_T \subset X_\infty$ and T has a fixed point in X. □

Definition 2.13. Let X be a nonempty subset of a Banach space E and let $\phi_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ for all $i \in \mathbb{N}$. We call $T : X \to X$ an asymptotic sequence $\{\phi_i\}$-Meir-Keeler type contraction with respect to the measure σ if the following conditions hold:

1. $\{\phi_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+\}$ is a sequence of Meir-Keeler mappings;
2. for each $i \in \mathbb{N}_0$, we have

$$\begin{cases} \sigma(T^{2i}(X)) < \phi_i(\sigma(T^i(X))), \\ \sigma(T^{2i+1}(X)) < \phi_{i+1}(\sigma(T^i(X))). \end{cases}$$

Theorem 2.14. *Let X be a nonempty, bounded and closed subset of a Banach space E, let $\phi_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ for all $i \in \mathbb{N}$, and let $T : X \to X$ be a contractive operator. Suppose $T : X \to X$ is an asymptotic sequence $\{\phi_i\}$-Meir-Keeler type contraction with respect to the measure σ. Then T has a fixed point in X.*

Proof. Define the sequence $\{X_n\}$ of sets as follows:

$$X_0 = X \;\text{ and }\; X_n = \overline{T^n(X_0)} \;\text{ for all }\; n \in \mathbb{N}_0.$$

Since T is a contractive operator, T is continuous and $T(\overline{A}) \subset \overline{T(A)}$. From above argument, we conclude that for all $n \in \mathbb{N}_0$,

(1) $T^{n+1}(X) \subset T^n(X)$;
(2) $X_{n+1} \subset X_n$;
(3) $T(X_n) \subset X_n$.

By the properties of measure of noncompactness, we obtain that $\sigma(X_{n+1}) \leqslant \sigma(X_n)$ for each $n \in \mathbb{N}_0$, that is, $\{\sigma(X_n)\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$.

Now if there exists an $N \in \mathbb{N}$ such that $\sigma(X_N) = 0$, then X_N is relatively compact. By (2) and (3), we also have that $T(X_N) \subset X_N$. Thus, by Schauder fixed point theorem, we deduce that T has a fixed point. So we assume that $\sigma(X_n) > 0$ for each $n \in \mathbb{N}_0$. Since $\{\sigma(X_n)\}_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathbb{R}_0^+$, it must converge to some γ. Notice that

$$\gamma = \inf\{\sigma(X_n) : n \in \mathbb{N}_0\}.$$

We now claim that $\gamma = 0$. Suppose, on the the contrary, that $\gamma > 0$. Since ϕ is a Meir-Keeler mapping, corresponding to γ, there exist a δ and a natural number k_0 such that

$$\gamma \leqslant \sigma(X_k) < \gamma + \delta \;\text{ for all }\; k \geqslant k_0.$$

Since T is an asymptotic sequence $\{\phi_i\}$-Meir-Keeler type contraction with respect to the measure σ, by the condition (2) of Definition 2.13, we have that

$$\begin{cases} \sigma(T^{2k_0}(X)) < \phi_{k_0}(\sigma(T^{k_0}(X))) \leqslant \gamma, \\ \sigma(T^{2k_0+1}(X)) < \phi_{k_0+1}(\sigma(T^{k_0}(X))) \leqslant \gamma, \end{cases}$$

since ϕ_{k_0}, ϕ_{k_0+1} are Meir-Keeler mappings. This implies a contradiction. Thus, we get $\gamma = 0$, and we have

$$\lim_{n\to\infty} \sigma(X_n) = 0.$$

From the property (6) of measure of noncompactness, $X_\infty = \cap_{n\in\mathbb{N}} X_n$ is a nonempty closed and convex subset of X, and X_∞ is invariant under T and $X_\infty \in \operatorname{Ker}(\sigma)$. By Theorem 2.10, we deduce that T has a fixed point in X_∞ and as $F_T = \{x \in X : T(x) = x\} \subset X_n$ for all $n \in \mathbb{N}$. Thus, $F_T \subset X_\infty$ and T has a fixed point in X. □

A function $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be upper semicontinuous, if for each $t_0 \in \mathbb{R}_0^+$, $\lim_{t\to t_0} \sup \xi(t) \leqslant \xi(t_0)$. We reall that $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a generalized comparison function, if ξ is increasing and upper semicontinuous with $\xi(0) = 0$ and $\xi(t) < t$ for all $t > 0$. The following proposition and remark are proved in [7].

Proposition 2.15. *If $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a generalized comparison function, then there exists a strictly increasing, continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that*

$$\begin{cases} \xi(t) \leqslant \alpha(t) < t \quad \textit{for all } t > 0, \\ \lim_{t\to\infty} \alpha(t) = \infty. \end{cases}$$

Remark 2.16. In the above case, the function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is invertible. If for each $t > 0$ we let $\alpha^0(t) = t$ and $\alpha^{-n}(t) = \alpha^{-1}(\alpha^{-n+1}(t))$ for all $n \in \mathbb{N}$, then we have that

$$\lim_{n\to\infty} \alpha^{-n}(t) = \infty,$$

that is,

$$\lim_{n\to\infty} \alpha^{n}(t) = 0.$$

Remark 2.17. From Proposition 2.15 and Remark 2.16, it is clear that if $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a generalized comparison function, then

$$\lim_{n\to\infty} \xi^{n}(t) = 0.$$

Definition 2.18. Let X be a nonempty subset of a Banach space E, and let $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a generalized comparison function. We call $T : X \to X$ a ξ-generalized comparison type contraction with respect to the measure σ if, for each bounded subset A of X,

$$\sigma(T(A)) \leqslant \xi(\sigma(A)).$$

Applying Theorem 2.8 and Remark 2.17, we easily get the following theorem.

Theorem 2.19. *Let X be a nonempty, bounded, closed and convex subset of a Banach space E, and let $T : X \to X$ be a continuous operator. Suppose $T : X \to X$ is a ξ-generalized comparison type contraction with respect to the measure σ. Then, T has a fixed point in X.*

Definition 2.20. A function $\psi : \mathbb{R}_0^+ \to [0, 1)$ is said to be a Reich's function ($\mathcal{R}$-function), if

$$\lim_{s\to t^+} \sup \psi(t) = \inf_{\alpha>0} \sup_{0<s-t<\alpha} \psi(t) < 1 \quad \text{for all} \quad t \in \mathbb{R}_0^+.$$

Remark 2.21. It is obvious that if $\psi : \mathbb{R}_0^+ \to [0, 1)$ is non-increasing or non-decreasing, then ψ is a $\mathcal{R}$-function.

In 2012, Du [9] proved the following theorem.

Theorem 2.22 ([9])**.** *Let $\psi : \mathbb{R}_0^+ \to [0,1)$ be a function. Then the following two statements are equivalent.*

(a) *ψ is a $\mathcal{R}$-function.*

(b) *For any non-increasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_0^+$, we have*

$$0 \leqslant \sup_{n\in\mathbb{N}} \psi(\gamma_n) < 1.$$

Definition 2.23. Let X be a nonempty subset of a Banach space E, and let $\psi : \mathbb{R}_0^+ \to [0,1)$ be a $\mathcal{R}$-function. We call $T : X \to X$ a $\mathcal{R}$-functional type ψ-contraction with respect to the measure σ, if, for each bounded subset A of X,

$$\sigma(T(A)) \leqslant \psi(\sigma(A))\sigma(A).$$

Theorem 2.24. *Let X be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : X \to X$ be a continuous operator, and let $\psi : \mathbb{R}_0^+ \to [0,1)$ be a $\mathcal{R}$-function. Suppose $T : X \to X$ is a $\mathcal{R}$-functional type ψ-contraction with respect to the measure σ. Then T has a fixed point in X.*

Proof. The skeleton of the proof is the same as the proofs of our previous results. Set the sequence $\{X_n\}$ of sets as follows:

$$X_0 = X \text{ and } X_{n+1} = \text{co}(T(X_n)) \text{ for all } n \in \mathbb{N}_0.$$

Then, we have that for all $n \in \mathbb{N}_0$,

(1) X_n is a convex subset of X;

(2) $X_{n+1} \subset X_n$; Clearly we have $X_1 \subset X_0$, and

$$X_2 = \text{co}(T(X_1)) \subset \text{co}(T(X_0)) = X_1,$$

by iteration, we can easily obtain that

$$X_{n+1} = \text{co}(T(X_n)) \subset \text{co}(T(X_{n-1})) = X_n;$$

(3) $T(X_n) \subset \text{co}(T(X_n)) = X_{n+1}$.

By the properties of measure of noncompactness, we obtain that $\sigma(X_{n+1}) \leqslant \sigma(X_n)$ for each $n \in \mathbb{N}_0$, that is, $\{\sigma(X_n)\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}_0^+$. Because ψ is an $\mathcal{R}$-function, by Theorem 2.22, we obtain

$$0 \leqslant \sup_{n\in\mathbb{N}} \psi(\sigma(X_n)) < 1.$$

Let

$$\lambda = \sup_{n\in\mathbb{N}} \psi(\sigma(X_n)).$$

Then

$$0 \leqslant \psi(\sigma(X_n)) \leqslant \lambda < 1 \text{ for all } n \in \mathbb{N}.$$

Thus, we conclude that for each $n \in \mathbb{N}_0$

$$\sigma(X_{n+1}) = \sigma(\text{co}(T(X_n))) = \sigma(T(X_n)) \leqslant \psi(\sigma(X_n))\sigma(X_n) \leqslant \lambda\sigma(X_n).$$

Thus, we also conclude that

$$\sigma(X_{n+1}) \leqslant \lambda\sigma(X_n) \leqslant \lambda^2\sigma(X_{n-1}) \leqslant \cdots \leqslant \lambda^{n+1}\sigma(X_0).$$

Since $\lambda < 1$, $\lim_{n\to\infty} \lambda^n = 0$, and we also get that

$$\lim_{n\to\infty} \sigma(X_n) = 0.$$

From the condition (6) of Definition 1.2, $X_\infty = \cap_{n\in\mathbb{N}} X_n$ is a nonempty closed and convex subset of X, and X_∞ is invariant under T and $X_\infty \in \text{Ker}(\sigma)$. Applying the Schauder fixed point theorem, we deduce that T has a fixed point. □

In what follows, $\mathcal{BC}(\mathbb{R}_0^+)$ will denote the family of all real functions defined, bounded and continuous on $\mathbb{R}_0^+$, and we define the norm in $\mathcal{BC}(\mathbb{R}_0^+)$ as the standard supreme norm, i.e.,

$$\|x\| = \sup\{| x(t) |: t \geqslant 0\}.$$

We will use a measure of noncompactness in the space $\mathcal{BC}(\mathbb{R}_0^+)$ that was introduced by Banaś [4]. Let X be nonempty bounded subset of $\mathcal{BC}(\mathbb{R}_0^+)$, and a positive number $T > 0$. For $x \in X$ and $\varepsilon \geqslant 0$, we denote by $\omega^T(x, \varepsilon)$, the modulus of continuity of x on the interval $[0, T]$, that is,

$$\omega^T(x, \varepsilon) = \sup\{| x(t) - x(s) |: t, s \in [0, T],\ | t - s | \leqslant \varepsilon\}.$$

Moreover, we also put

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\}, \quad \omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon), \quad \omega_0(X) = \lim_{T \to \infty} \omega_0^T(X),$$

and, let $t \in \mathbb{R}_0^+$, we denote by

$$X(t) = \{x(t) : x \in X\},$$

and

$$\operatorname{diam}(X(t)) = \sup\{| x(t) - y(t) |: x, y \in X\}.$$

Finally, we consider the function σ defined on $B(\mathcal{BC}(\mathbb{R}_0^+))$ by the formula

$$\sigma(X) = \omega_0(X) + \limsup_{t \to \infty} \operatorname{diam}(X(t)).$$

It can be shown [4] that the function σ was a ball measure of noncompactness in the space $\mathcal{BC}(\mathbb{R}_0^+)$.

Example 2.25. From above accepted definition, let X be a nonempty, bounded, closed, and convex subset of $\mathcal{BC}(\mathbb{R}_0^+)$, we consider the function $T : X \to X$ defined by

$$T(x) = \frac{x}{2x + 1} \quad \text{for all} \quad x \in X.$$

Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be denoted by

$$\varphi(t) = \begin{cases} \frac{t}{4}, & \text{if } t \leqslant 1, \\ 2t, & \text{if } 1 < t < 2, \\ \frac{t}{2}, & \text{if } t \geqslant 2. \end{cases}$$

Then the hypotheses of Theorem 2.8 are valid, and T has a fixed point $x = 0$.

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