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# On the generalized solutions of a certain fourth order Euler equations

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**Abstract**

In this paper, using Laplace transform technique, we propose the generalized solutions of the fourth order Euler differential equations

$$t^4y^{(4)}(t)+t^3y'''(t)+t^2y''(t)+ty'(t)+my(t)=0,$$

where $m$ is an integer and $t\in\mathbb{R}$. We find types of solutions depend on the values of $m$. Precisely, we have a distributional solution for $m=-k^4-5k^3-9k^2-4k$ and a weak solution for $m=-k^4+5k^3-9k^2+4k$, where $k\in\mathbb{N}$. 

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## 1. Introduction

A linear ordinary differential equation of order $n$ can be expressed in the form

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(t),$$

where $a_0(t), a_1(t),\cdots,a_n(t), f(t)$ are continuous real functions and $a_n(t)\neq 0$. We can write the above equation in the short from as

$$P(D)y=f, \tag{1.1}$$

where

$$P(D)=a_n(t)\frac{d^n}{dt^n}+a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}+\cdots+a_1(t)\frac{d}{dt}+a_0(t).$$

In searching for a solution $y$ of differential equation (1.1), we may have the following situations (see Kanwal [12]):

(i) The solution $y$ is a smooth function such that the operation can be performed as in the classical sense and the resulting equation is an identity. Then $y$ is a classical solution.

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(ii) The solution $y$ is not smooth enough, so that the operation cannot be performed but it satisfies as a distribution. Then $y$ is a weak solution.

(iii) The solution $y$ is a singular distribution. Then the solution is a distributional solution.

All these solutions are said to be generalized solutions.

For the solution $y$ in the first case, i.e., the classical solution, we consider the $n$-th-order Euler differential equation which can be written as

$$\alpha_n t^n y^{(n)}(t) + \alpha_{n-1} t^{n-1} y^{(n-1)}(t) + \cdots + \alpha_0 y(t) = 0,$$

where $\alpha_0, \alpha_1, \cdots, \alpha_n$ are real constants and $\alpha_n \neq 0$. Details for a method to construct solutions to the problem of solving the classical solution was explained in [2, 4, 5]. Moreover, Sabuwala and Leon [18] studied the particular solution for the most general $n$-th-order Euler differential equation when the non-homogeneity is a polynomial. They found a formula which can be used to compute the unknown coefficients in the form of the particular solution.

For the weak solution, Kananthai and Nonlaopon [10] studied the weak solution of the compound ultra-hyperbolic equation. Next, Sarikaya and Yildirim [19] studied the weak solution of the compound Bessel ultra-hyperbolic equation. Furthermore, several papers have also studied the weak solution in the field of theory of distributions; see [3, 11, 23].

The distributional solution of ordinary differential equations, particularly in the form of series of Dirac delta function, have been used in several areas of applied mathematics. Many important areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis have used the methods of the theory of distributions. We refer the readers to [6, 7, 16, 25, 27, 28] for more details.

For the generalized solutions, by using Laplace transform technique, Nonlaopon et al. [17] studied the generalized solutions of the differential equation $ty^{(n)}(t) + my^{(n-1)}(t) + ty(t) = 0$, where $m$ and $n$ are integers with $n \geqslant 2$, and $t \in \mathbb{R}$. See [26] for more details about generalized solutions of functional differential equations. For numerical solution, Singh et al. [21] studied the numerical solution of the damped Berger's equation by using the concept of an iterative method. Moreover, there are works on analytical and numerical techniques to solve differential equations, see [14, 15, 22, 24] for more details.

In 1999, Kananthai [8] studied the distributional solutions of the third order Euler differential equation

$$t^3 y'''(t) + t^2 y''(t) + ty'(t) + my(t) = 0, \tag{1.2}$$

where $m$ is an integer and $t \in \mathbb{R}$. He found that solutions of (1.2), which are either the distributional solutions or the classical solutions, depend on the values of $m$. Next, Kananthai [9] studied the distributional solutions of ordinary differential equation with polynomial coefficients. Furthermore, the third order Euler differential equation was studied by Akanbi [1], where Euler methods for solving initial value problems in ordinary differential equations were improved and the numerical solutions were obtained.

Consider the fourth order Euler differential equation in the form

$$t^4 y^{(4)}(t) + t^3 y'''(t) + t^2 y''(t) + ty'(t) + my(t) = 0,$$

where $m$ is an integer and $t \in \mathbb{R}$. It is well-known that the solutions of the above equation are of the form $t^\lambda$, where $\lambda$ are real or complex numbers, that is, the solutions are in the space $\mathbb{C}(-\infty, \infty)$ of continuous functions, classical solution. The goal of this work is to determine the solutions of such equation in $\mathscr{D}'$, the space of distributions. To obtain such solutions, we use the Laplace transform of distribution to solve the equation. For more details on applications of the theory of distributions to differential equations, see Schwartz [20] and Zemanian [29].

The next section gives the necessary facts about Laplace transform. We use Laplace transform to obtain our main results in Section 3. At the end of Section 3, some examples as a consequence of our results are shown. Finally we give the conclusions in Section 4.

## 2. Preliminaries

Before proceeding to our main results, the following definitions and concepts are required.

**Definition 2.1.** A distribution $T$ is a continuous linear functional on the space $\mathscr{D}$ of the complex-valued functions with infinitely differentiable and bounded support. The space of all such distributions is denoted by $\mathscr{D}'$.

For every $T \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$, the value that $T$ acts on $\varphi$ is denoted by $\langle T, \varphi\rangle$. Note that $\langle T, \varphi\rangle \in \mathbb{C}$. Now $\varphi$ is called a test function in $\mathscr{D}$.

**Example 2.2.**

(i) The locally integrable function f is a distribution generated by the locally integrable function $f$. Then we define $\langle f, \varphi\rangle = \int_\Omega f(t)\varphi(t)dt$, where $\Omega$ is a support of $\varphi$ and $\varphi \in \mathscr{D}$.

(ii) The Dirac delta function is a distribution defined by $\langle \delta, \varphi\rangle = \varphi(0)$ and the support of $\delta$ is $\{0\}$.

A distribution $T$ generated by a locally integrable function is called a regular distribution, otherwise, it is called a singular distribution.

**Definition 2.3** (The Differentiation of Distribution)**.** The $k$-order derivative of a distribution $T$, denoted by $T^{(k)}$, is defined by $\left\langle T^{(k)}, \varphi\right\rangle = (-1)^k \left\langle T, \varphi^{(k)}\right\rangle$ for all $\varphi \in \mathscr{D}$.

**Example 2.4.**

(i) $\langle \delta', \varphi\rangle = -\langle \delta, \varphi'\rangle = -\varphi'(0)$.

(ii) $\left\langle \delta^{(k)}, \varphi\right\rangle = (-1)^k \left\langle \delta, \varphi^{(k)}\right\rangle = (-1)^k \varphi^{(k)}(0)$.

**Definition 2.5** (The Multiplication of a Distribution by Infinitely Differentiable Function)**.** Let $\alpha(t)$ be an infinitely differentiable function. We define the product of $\alpha(t)$ with any distribution $T$ in $\mathscr{D}'$ by $\langle \alpha T, \varphi\rangle = \langle T, \alpha\varphi\rangle$ for all $\varphi \in \mathscr{D}$.

**Definition 2.6.** Let $T \in \mathbb{R}$, and $f(t)$ be a locally integrable function satisfying the following conditions:

(i) $f(t) = 0$ for all $t < T$.

(ii) There exists a real number $c$ such that $e^{-ct}f(t)$ is absolutely integrable over $\mathbb{R}$.

The Laplace transform of $f(t)$ is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_T^\infty f(t)e^{-st}dt,$$

where $s$ is a complex variable.

It is well-known that if $f$ is continuous, then $F(s)$ is an analytic function on the half-plane $\Re(s) > \sigma_a$, where $\sigma_a$ is an abscissa of absolute convergence for $\mathscr{L}\{f(t)\}$.

**Definition 2.7.** Let $f(t)$ be a function satisfying the same conditions as in Definition 2.6, and $\mathscr{L}\{f(t)\} = F(s)$. The inverse Laplace transform of $F(s)$ is defined by

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{\omega\to\infty} \int_{c-i\omega}^{c+i\omega} F(s)e^{st}ds,$$

where $\Re(s) > \sigma_a$.

Recall that the Laplace transform $G(s)$ of a locally integrable function $g(t)$ satisfying the conditions of Definition 2.6, that is,

$$G(s) = \mathscr{L}\{g(t)\} = \int_T^\infty g(t)e^{-st}dt,$$

where $\Re(s) > \sigma_a$, can be written in the form $G(s) = \left\langle g(t), e^{-st}\right\rangle$.

**Definition 2.8.** Let $f(t)$ be a distribution satisfying the following properties:

(i) $f$ is a right-sided distribution, that is, $f \in \mathscr{D}'_R$.
(ii) There exists a real number $c$ such that $e^{-ct}f(t)$ is a tempered distribution.

The Laplace transform of a right-sided distribution $f(t)$ satisying (ii) is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \left\langle e^{-st}f(t), X(t)e^{-(s-c)t} \right\rangle, \tag{2.1}$$

where $X(t)$ is an infinitely differentiable function with support bounded on the left, which equals to 1 over the neighbourhood of the support of $f(t)$.

For $\Re(s) > \sigma_a$, the function $X(t)e^{-(s-c)t}$ is a testing function in the space $S$ of testing functions of rapid descent and that $e^{-ct}f(t)$ is in the space $S'$ of tempered distributions. Equation (2.1) can be deduced to

$$F(s) = \mathscr{L}\{f(t)\} = \left\langle f(t), e^{-st} \right\rangle,$$

possessing the sense given by the right-hand side of (2.1). Now, $F(s)$ is a function of $s$ defined over the right half-plane $\Re(s) > c$. Zemanian [29] proved that $F(s)$ is an analytic function in the region of convergence $\Re(s) > \sigma_1$, where $\sigma_1$ is the abscissa of convergence and $e^{-ct}f(t) \in S'$ for some real $c > \sigma_1$.

**Example 2.9.** Let $\delta(t)$ be the Dirac delta function, $H(t)$ be the Heaviside function (which is given by $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$), and $f(t)$ be a Laplace-transformable distribution in $\mathscr{D}'_R$ (so that $F(s) = \mathscr{L}\{f(t)\}$ for $\Re(s) > \sigma_1$). If $k$ is a positive integer, then

(i) $\mathscr{L}\{(t^k H(t))/k!\} = 1/s^{k+1}, \quad \Re(s) > \sigma_1$.
(ii) $\mathscr{L}\{\delta\} = 1, \quad -\infty < \Re(s) < \infty$.
(iii) $\mathscr{L}\left\{\delta^{(k)}\right\} = s^k, \quad -\infty < \Re(s) < \infty$.
(iv) $\mathscr{L}\left\{t^k f(t)\right\} = (-1)^k F^{(k)}(s), \quad \Re(s) > \sigma_1$.
(v) $\mathscr{L}\left\{f^{(k)}(t)\right\} = s^k F(s), \quad \Re(s) > \sigma_1$.

**Lemma 2.10.** *If the equation*

$$\sum_{i=0}^{n} a_i(t)t^i y^{(i)}(t) = 0,$$

*with infinitely differentiable coefficients* $a_i(t)$ *and* $a_n(0) \neq 0$ *has a solution*

$$y(t) = \sum_{i=0}^{h} a_i \delta^{(i)}(t), \quad a_h \neq 0, \tag{2.2}$$

*of order* $h$*, then we obtain the relation*

$$\sum_{i=0}^{n} (-1)^i a_i(0)(h+i)! = 0. \tag{2.3}$$

*Conversely, if* $h$ *is the smallest non-negative integer root of* (2.3)*, then there exists an* $h$*-th order solution of* (2.2) *at* $t = 0$.

The proof of this lemma is given in [25].

**Lemma 2.11.** *Let* $\psi(t)$ *be an infinitely differentiable function. Then*

$$\begin{aligned} \psi(t)\delta^{(m)} &= (-1)^m \psi^{(m)}(0)\delta(t) + (-1)^{m-1} m \psi^{(m-1)}(0)\delta'(t) \\ &\quad + (-1)^{m-2}\frac{m(m-1)}{2!}\psi^{(m-1)}(0)\delta''(t) + \cdots + \psi(0)\delta^{(m)}(t). \end{aligned} \tag{2.4}$$

The proof of this lemma is given in [12].

A useful formula that follows from (2.4), for any monomial $\psi(t) = t^n$, is

$$t^n \delta^{(m)}(t) = \begin{cases} 0, & \text{for } m < n, \\ (-1)^n \frac{m!}{(m-n)!}\delta^{(m-n)}(t), & \text{for } m \geqslant n. \end{cases} \tag{2.5}$$

## 3. Main results

In this section, we will state our main results and give their proofs.

**Theorem 3.1.** *Consider the fourth order Euler differential equations of the form*

$$t^4y^{(4)}(t)+t^3y'''(t)+t^2y''(t)+ty'(t)+my(t)=0, \tag{3.1}$$

*where $m$ is an integer and $t \in \mathbb{R}$. Then the solutions of (3.1) depend on the value of $m$ and is given by the following cases:*

(i) *If*

$$m=-k^4-5k^3-9k^2-4k, \quad k=1,2,3,\cdots, \tag{3.2}$$

*then there exists the distributional solutions of (3.1), which are singular distribution of Dirac delta function and its derivatives.*

(ii) *If*

$$m=-k^4+5k^3-9k^2+4k, \quad k=1,2,3,\cdots,$$

*then there exists the weak solutions of (3.1), which are continuous functions.*

*Proof.* From (3.1), taking the Laplace transform and using Example 2.9 (iv), (v), we obtain

$$\frac{d^4}{ds^4}\left[s^4Y(s)\right]-\frac{d^3}{ds^3}\left[s^3Y(s)\right]+\frac{d^2}{ds^2}\left[s^2Y(s)\right]-\frac{d}{ds}\left[sY(s)\right]+mY(s)=0.$$

Then we have the fourth order Euler differential equation

$$s^4Y^{(4)}(s)+15s^3Y'''(s)+64s^2Y''(s)+81sY'(s)+(19+m)Y(s)=0. \tag{3.3}$$

Let a solution of (3.3) be in the form $Y(s)=s^r$, where $r$ is a real constant. Substituting $Y(s), Y'(s), Y''(s)$, $Y'''(s)$ and $Y^{(4)}(s)$ into (3.3), we obtain

$$\left[r(r-1)(r-2)(r-3)+15r(r-1)(r-2)+64r(r-1)+81r+19+m\right]s^r=0.$$

Since $s^r \neq 0$, we have

$$r(r-1)(r-2)(r-3)+15r(r-1)(r-2)+64r(r-1)+81r+19+m=0,$$

or equivalently,

$$r^4+9r^3+30r^2+41r+19+m=0. \tag{3.4}$$

Case (i). If $m=-19,-100,-309,\cdots,-k^4-5k^3-9k^2-4k,\cdots$, then by (3.4), we have the real roots

$$r=0,1,2,\ldots,k-1,\cdots,$$

respectively, and the solutions of (3.3) are

$$Y(s)=1,s,s^2,\cdots,s^{k-1},\cdots,$$

respectively.

Now $Y(s)$ are analytic functions over the entire $s$-plane. Taking the inverse Laplace transform to $Y(s)$ and by Example 2.9, we obtain the solutions of (3.1), which are the singular distributions

$$y(t)=\delta,\delta',\delta'',\ldots,\delta^{(k-1)},\ldots,$$

respectively.

Case (ii). If $m = -1, -4, -15, \cdots, -k^4 + 5k^3 - 9k^2 + 4k, \cdots$, then we have the real roots

$$r = -2, -3, -4, \cdots, -k-1, \cdots,$$

respectively, and the solutions of (3.3) are

$$Y(s) = \frac{1}{s^2}, \frac{1}{s^3}, \frac{1}{s^4}, \cdots, \frac{1}{s^{k+1}}, \cdots,$$

respectively.

Now $Y(s)$ are analytic functions over the entire $s$-plane. Taking the inverse Laplace transform to $Y(s)$ and by Example 2.9, we obtain the solutions of (3.1), which are the weak solutions

$$y(t) = H(t)t, H(t)\frac{t^2}{2!}, H(t)\frac{t^3}{3!}, \cdots, H(t)\frac{t^k}{k!}, \cdots,$$

respectively. □

**Theorem 3.2.** *The distributional solution of the fourth order Euler differential equation*

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0,$$

*where* $m$ *is an integer and* $t \in \mathbb{R}$, *depends on the values of* $m$ *of the form*

$$m = -k^4 - 9k^3 - 30k^2 - 41k - 19, \tag{3.5}$$

*where* $k = 0, 1, 2, \cdots$ *is the order of distribution.*

*Proof.* By Lemma 2.10, substituting $n = 4$, $a_i(0) = 1$ for $i = 1, 2, 3, 4$, and $a_0(0) = m$ into (2.3), we have

$$m \cdot h! + \sum_{i=1}^{4} (-1)^i (h+i)! = 0.$$

Thus, we obtain (3.5) as required. □

*Remark* 3.3. The values $m$ in (3.2) and the values $m$ in (3.5) are identical.

*Remark* 3.4. On finding the distributional solution of (3.1) by the methods of Krall et al. [13], and Littlejohn and Kanwal [16], we find that the conditions of $m$ in their methods are identical to ours.

**Example 3.5.**

(i) For $m = -100$, equation (3.1) becomes

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) - 100y(t) = 0. \tag{3.6}$$

It follows that the distributional solution of (3.6) is

$$y(t) = \delta'(t). \tag{3.7}$$

(ii) For $m = -309$, equation (3.1) becomes

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) - 309y(t) = 0. \tag{3.8}$$

It follows that the distributional solution of (3.8) is

$$y(t) = \delta''(t). \tag{3.9}$$

By applying (2.5), it is easy to verify that the distributional solutions (3.7) and (3.9) satisfy (3.6) and (3.8), respectively.

**Example 3.6.** For $m = -15$, equation (3.1) becomes

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) - 15y(t) = 0. \tag{3.10}$$

It follows that the weak solution of (3.10) is $y(t) = H(t)\frac{t^3}{3!}$.

## 4. Conclusions

We use the Laplace transform technique to find generalized solutions in the space of distributions for the fourth Euler differential equations

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0.$$

It is found that if $m = -k^4 - 5k^3 - 9k^2 - 4k$, then there exists the distributional solution of such equation, which is singular distribution of Dirac delta function and its derivatives, and if $m = -k^4 + 5k^3 - 9k^2 + 4k$, then there exists the weak solution of such equation for any $k \in \mathbb{N}$. It should be noted here that there are solutions to the considered equation which cannot be obtained by Laplace transform as shown in the example. Furthermore, in our study we concern only the solution in the space of distribution which is the one widely used in application. However, it would be of interest to find the generalized solutions in the space of distributions having unbounded supports.

In the future, we will devote our attention to the generalization of the fourth order Euler differential equation with constant coefficients of the form $a_1t^4y^{(4)}(t) + a_2t^3y'''(t) + a_3t^2y''(t) + a_4ty'(t) + my(t) = 0$, where $m$ depends on the values of $a_i$, $i = 1, 2, 3, 4$.

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