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On some common coupled fixed point results in rectangular b-metric spaces

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Abstract

In this paper, by using the w-compatible conditions of mapping pair, we discuss the existence and uniqueness problem of the common coupled fixed point for mappings defined on a set equipped with two rectangular b-metrics. Some new common coupled fixed point theorems are obtained. We also provide illustrative examples in support of our new results. As application, we provide an existence and uniqueness theorem of common solution for a class of nonlinear integral equations by using the obtained new result. The results presented in this paper generalize the well-known comparable results in the literature.

Keywords: Rectangular b-metric space, coupled coincidence point, common coupled fixed point, w-compatible mapping pairs.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The concept of rectangular metric space was first introduced by Branciari [6] in 2000. The applications and theories in rectangular metric space have also been extensively studied by many scholars [2, 3, 5, 8, 10, 12, 14–16, 18, 19, 21] and they also have proved the common fixed point theorems under Banach contractive conditions and Kannan type contractive conditions in rectangular metric spaces.

In 2015, George et al. [11] introduced the concept of rectangular b-metric space. It is well-known that rectangular b-metric space is an important generalization of usual metric space, b-metric spaces [7], and rectangular metric spaces. Since then, Aydi et al. [4], Ding et al. [9], and Roshan et al. [20] obtained some fixed point and common fixed point results for mappings satisfying different contractive conditions in rectangular b-metric spaces. However, so far, no one discussed coupled fixed point problem in rectangular b-metric space.

The purpose of this paper is to prove some new coupled common fixed point theorems for mappings defined on a set equipped with two b-rectangular metrics. We also provide an existence and uniqueness theorem of solution for a class of nonlinear integral equations by using the obtained result.

Firstly, given some basic notions before introducing some main results.

Definition 1.1 ([7]). Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $b : X \times X \to [0, \infty)$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

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(b1) $b(x,y)=0$ if and only if $x=y$;
(b2) $b(x,y)=b(y,x)$;
(b3) $b(x,z)\leqslant s[b(x,y)+b(y,z)]$ (b-triangular inequality).

In this case, the pair (X,b) is called a b-metric space (metric type space).

Definition 1.2 ([6]). Let X be a nonempty set, and let $r: X\times X\to[0,\infty)$ be a mapping such that for all $x,y\in X$ and all distinct points $u,v\in X$, each distinct from x and y:

(R1) $r(x,y)=0$ if and only if $x=y$;
(R2) $r(x,y)=r(y,x)$;
(R3) $r(x,z)\leqslant r(x,u)+r(u,v)+r(v,z)$ (rectangular inequality).

Then (X,r) is called a rectangular or generalized metric space or Branciari's space.

For properties and definitions of notions in Branciari's spaces see [5].

Definition 1.3 ([11]). Let X be a nonempty set, $s\geqslant 1$ be a given real number and let $r_b: X\times X\to[0,\infty)$ be a mapping such that for all $x,y\in X$ and distinct points $u,v\in X$, each distinct from x and y:

(Rb1) $r_b(x,y)=0$ if and only if $x=y$;
(Rb2) $r_b(x,y)=r_b(y,x)$;
(Rb3) $r_b(x,z)\leqslant s[r_b(x,u)+r_b(u,v)+r_b(v,z)]$ (b-rectangular inequality).

Then (X,r_b) is called a rectangular b-metric space (RbMS) or a generalized b-metric space (b-g.m.s). Moreover, $s\geqslant 1$ is the coefficient of rectangular b-metric space X.

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s=1$). However the converse is not necessarily true ([11, Examples 2.4 and 2.5]). Also, every metric space is a b-metric space (metric type space) and every b-metric space is a rectangular b-metric space (not necessarily with the same coefficient).

Note also that every b-metric space with coefficient s is a rectangular b-metric space with coefficient s^2 but the converse is not necessarily true ([11, Example 2.7]).

Definition 1.4 ([11]). Let (X,r_b) be a rectangular b-metric space, $\{x_n\}$ be a sequence in X, and $x\in X$. Then

(a) the sequence $\{x_n\}$ is said to be convergent in X and converges to x, if for every $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that $r_b(x_n,x)<\epsilon$ for all $n>n_0$ and this fact is represented by $\lim_{n\to\infty}x_n=x$ or $x_n\to x$ as $n\to\infty$;
(b) the sequence $\{x_n\}$ is said to be Cauchy sequence in (X,r_b) if for every $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that $r_b(x_n,x_{n+p})<\epsilon$ for all $n>n_0$, $p>0$ or equivalently, if $\lim_{n\to\infty}r_b(x_n,x_{n+p})=0$ for all $p>0$;
(c) (X,r_b) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $x\in X$.

Note that limit of sequence in a rectangular b-metric space is not necessarily unique and also every rectangular b-metric convergent sequence in a rectangular b-metric space is not necessarily rectangular b-metric-Cauchy ([11, Example 2.7]).

Definition 1.5 ([13]). An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F: X\times X\to X$, if $F(x,y)=x, F(y,x)=y$.

Definition 1.6 ([17]). An element $(x,y)\in X\times X$ is called a coupled coincidence point of the mappings $F: X\times X\to X$ and $g: X\to X$, if $F(x,y)=gx, F(y,x)=gy$, and in this case, (gx,gy) is called a coupled point of coincidence.

Definition 1.7 ([17]). An element $(x,y)\in X\times X$ is called a common coupled fixed point of mappings $F: X\times X\to X$ and $g: X\to X$, if $F(x,y)=gx=x, F(y,x)=gy=y$.

Definition 1.8 ([1]). Let X be a nonempty set. A pair of mappings $F: X\times X\to X$ and $g: X\to X$ are called to be w-compatible, if $F(x,y)=gx$ and $F(y,x)=gy$, then we have $gF(x,y)=F(gx,gy)$.

2. Main results

Theorem 2.1. *Let X be a nonempty set and r_{b1}, r_{b2} are two rectangular b-metrics in X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$, for all $x,y \in X$. And the real number $s \geqslant 1$ is the coefficient of rectangular b-metric d_1. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings, and there exist k_1, k_2, and k_3 in $[0,1)$ with $0 \leqslant k_1 + k_2 + k_3 < 1$ and $0 \leqslant sk_3 < 1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) + r_{b1}(F(y,x),F(v,u)) \leqslant\ & k_1[r_{b2}(gx,gu) + r_{b2}(gy,gv)] \\ & + k_2[r_{b2}(gx,F(x,y)) + r_{b2}(gy,F(y,x))] \\ & + k_3[r_{b2}(gu,F(u,v)) + r_{b2}(gv,F(v,u))] \end{aligned} \tag{2.1}$$

holds for all $(x,y),(u,v) \in X \times X$. If $F(X \times X) \subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y) \in X \times X$, satisfying that $gx = F(x,y) = gy = F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Proof. Let $(x_0,y_0) \in X \times X$, by making the use of $F(X \times X) \subset g(X)$, then there exist $x_1, y_1 \in X$ such that $gx_1 = F(x_0,y_0)$, $gy_1 = F(y_0,x_0)$, and $gz_1 = F(z_0,x_0)$. By similar arguments as above, we can show that there exist $x_2, y_2 \in X$ such that $gx_2 = T(x_1,y_1)$ and $gy_2 = T(y_1,x_1), \cdots$. Repeating the above procedure, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$gx_{n+1} = F(x_n,y_n), \quad gy_{n+1} = F(y_n,x_n), \quad \forall n \geqslant 0.$$

By taking $(x,y) = (x_n,y_n)$ and $(u,v) = (x_{n+1},y_{n+1})$ in (2.1), we obtain

$$\begin{aligned} & r_{b1}(gx_{n+1},gx_{n+2}) + r_{b1}(gy_{n+1},gy_{n+2}) \\ &= r_{b1}(F(x_n,y_n),F(x_{n+1},y_{n+1})) + r_{b1}(F(y_n,x_n),F(y_{n+1},x_{n+1})) \\ &\leqslant k_1[r_{b2}(gx_n,gx_{n+1}) + r_{b2}(gy_n,gy_{n+1})] + k_2[r_{b2}(gx_n,F(x_n,y_n)) + r_{b2}(gy_n,F(y_n,x_n))] \\ &\quad + k_3[r_{b2}(gx_{n+1},F(x_{n+1},y_{n+1})) + r_{b2}(gy_{n+1},F(y_{n+1},x_{n+1}))] \\ &= k_1[r_{b2}(gx_n,gx_{n+1}) + r_{b2}(gy_n,gy_{n+1})] + k_2[r_{b2}(gx_n,gx_{n+1}) + r_{b2}(gy_n,gy_{n+1})] \\ &\quad + k_3[r_{b2}(gx_{n+1},gx_{n+2}) + r_{b2}(gy_{n+1},gy_{n+2})] \\ &\leqslant k_1[r_{b1}(gx_n,gx_{n+1}) + r_{b1}(gy_n,gy_{n+1})] + k_2[r_{b1}(gx_n,gx_{n+1}) + r_{b1}(gy_n,gy_{n+1})] \\ &\quad + k_3[r_{b1}(gx_{n+1},gx_{n+2}) + r_{b1}(gy_{n+1},gy_{n+2})]. \end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$\begin{aligned} r_{b1}(gx_{n+1},gx_{n+2}) + r_{b1}(gy_{n+1},gy_{n+2}) &\leqslant \left(\frac{k_1+k_2}{1-k_3}\right)[r_{b1}(gx_n,gx_{n+1}) + r_{b1}(gy_n,gy_{n+1})] \\ &= k[r_{b1}(gx_n,gx_{n+1}) + r_{b1}(gy_n,gy_{n+1})], \end{aligned} \tag{2.3}$$

where $k = \frac{k_1+k_2}{1-k_3}$, by the condition $0 \leqslant k_1 + k_2 + k_3 < 1$, then we have $0 \leqslant k < 1$. By taking $\delta_n = r_{b1}(gx_n,gx_{n+1}) + r_{b1}(gy_n,gy_{n+1})$ and repetition of the above inequality (2.3) $n+1$ times, we deduce that

$$\delta_{n+1} \leqslant k\delta_n \leqslant k^2\delta_{n-1} \leqslant \cdots \leqslant k^{n+1}\delta_0. \tag{2.4}$$

As $(x,y) = (x_n,y_n)$ and $(u,v) = (x_{n+2},y_{n+2})$ in (2.1), also with (2.4), we get

$$\begin{aligned} & r_{b1}(gx_{n+1},gx_{n+3}) + r_{b1}(gy_{n+1},gy_{n+3}) \\ &= r_{b1}(F(x_n,y_n),F(x_{n+2},y_{n+2})) + r_{b1}(F(y_n,x_n),F(y_{n+2},x_{n+2})) \\ &\leqslant k_1[r_{b2}(gx_n,gx_{n+2}) + r_{b2}(gy_n,gy_{n+2})] + k_2[r_{b2}(gx_n,F(x_n,y_n)) + r_{b2}(gy_n,F(y_n,x_n))] \\ &\quad + k_3[r_{b2}(gx_{n+2},F(x_{n+2},y_{n+2})) + r_{b2}(gy_{n+2},F(y_{n+2},x_{n+2}))] \end{aligned}$$

$$\begin{aligned}
&= k_1[r_{b2}(gx_n, gx_{n+2}) + r_{b2}(gy_n, gy_{n+2})] + k_2[r_{b2}(gx_n, gx_{n+1}) + r_{b2}(gy_n, gy_{n+1})] \\
&\quad + k_3[r_{b2}(gx_{n+2}, gx_{n+3}) + r_{b2}(gy_{n+2}, gy_{n+3})] \\
&\leqslant k_1[r_{b2}(gx_n, gx_{n+2}) + r_{b2}(gy_n, gy_{n+2})] + k_2[r_{b2}(gx_n, gx_{n+1}) + r_{b2}(gy_n, gy_{n+1})] \\
&\quad + k_3k^2[r_{b2}(gx_n, gx_{n+1}) + r_{b2}(gy_n, gy_{n+1})] \\
&= k_1[r_{b2}(gx_n, gx_{n+2}) + r_{b2}(gy_n, gy_{n+2})] + (k_2 + k_3k^2)[r_{b2}(gx_n, gx_{n+1}) + r_{b2}(gy_n, gy_{n+1})] \\
&\leqslant k_1[r_{b1}(gx_n, gx_{n+2}) + r_{b1}(gy_n, gy_{n+2})] + (k_2 + k_3k^2)[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gy_n, gy_{n+1})].
\end{aligned} \tag{2.5}$$

By taking

$$\delta_n^* = r_{b1}(gx_n, gx_{n+2}) + r_{b1}(gy_n, gy_{n+2})$$

and from $k = \frac{k_1+k_2}{1-k_3} \in [0, 1)$, we have

$$k_1 + k_2 + k_3k^2 \leqslant k_1 + k_2 + k_3k = k_1 + k_2 + k_3\frac{k_1 + k_2}{1 - k_3} = \frac{k_1 + k_2}{1 - k_3} = k.$$

Consequently, by the use of (2.4) and (2.5), we have

$$\delta_{n+1}^* \leqslant k_1\delta_n^* + (k_2 + k_3k^2)\delta_n \leqslant (k_1 + k_2 + k_3k^2)\max\{\delta_n, \delta_n^*\} \leqslant k\max\{\delta_n, \delta_n^*\}. \tag{2.6}$$

It follows from (2.4) and (2.6) that

$$\begin{aligned}
\delta_{n+1}^* &\leqslant k\max\{\delta_n, \delta_n^*\} \leqslant k\max\{k\delta_{n-1}, k\max\{\delta_{n-1}, \delta_{n-1}^*\}\} \\
&= k^2\max\{\delta_{n-1}, \delta_{n-1}^*\} \leqslant k^3\max\{\delta_{n-2}, \delta_{n-2}^*\} \leqslant \cdots \leqslant k^{n+1}\max\{\delta_0, \delta_0^*\}.
\end{aligned} \tag{2.7}$$

Next, we show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. For this, we consider $r_{b1}(x_n, x_{n+p})$ two cases.

Case 1. p is an odd number, assume that $p = 2m + 1$, then using (Rb3) we obtain

$$\begin{aligned}
r_{b1}(gx_n, gx_{n+p}) &= r_{b1}(gx_n, gx_{n+2m+1}) \\
&\leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2}) + r_{b1}(gx_{n+2}, gx_{n+2m+1})] \\
&\leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] \\
&\quad + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4}) + r_{b1}(gx_{n+4}, gx_{n+2m+1})] \\
&\leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4})] \\
&\quad + s^3[r_{b1}(gx_{n+4}, gx_{n+5}) + r_{b1}(gx_{n+5}, gx_{n+6}) + r_{b1}(gx_{n+6}, gx_{n+2m+1})] \\
&\leqslant \cdots \\
&\leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4})] \\
&\quad + s^3[r_{b1}(gx_{n+4}, gx_{n+5}) + r_{b1}(gx_{n+5}, gx_{n+6})] + \cdots \\
&\quad + s^m[r_{b1}(gx_{n+2m-2}, gx_{n+2m-1}) + r_{b1}(gx_{n+2m-1}, gx_{n+2m}) + r_{b1}(gx_{n+2m}, gx_{n+2m+1})].
\end{aligned}$$

That is

$$\begin{aligned}
&r_{b1}(gx_n, gx_{n+p}) \\
&\quad = r_{b1}(gx_n, gx_{n+2m+1}) \\
&\quad \leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4})] \\
&\quad\quad + s^3[r_{b1}(gx_{n+4}, gx_{n+5}) + r_{b1}(gx_{n+5}, gx_{n+6})] + \cdots \\
&\quad\quad + s^m[r_{b1}(gx_{n+2m-2}, gx_{n+2m-1}) + r_{b1}(gx_{n+2m-1}, gx_{n+2m})] + s^m r_{b1}(gx_{n+2m}, gx_{n+2m+1}).
\end{aligned} \tag{2.8}$$

We can similarly prove the following result

$$\begin{aligned}
&r_{b1}(gy_n, gy_{n+p}) \\
&\quad = r_{b1}(gy_n, gy_{n+2m+1}) \\
&\quad \leqslant s[r_{b1}(gy_n, gy_{n+1}) + r_{b1}(gy_{n+1}, gy_{n+2})] + s^2[r_{b1}(gy_{n+2}, gy_{n+3}) + r_{b1}(gy_{n+3}, gy_{n+4})] \\
&\qquad + s^3[r_{b1}(gy_{n+4}, gy_{n+5}) + r_{b1}(gy_{n+5}, gy_{n+6})] + \cdots \\
&\qquad + s^m[r_{b1}(gy_{n+2m-2}, gy_{n+2m-1}) + r_{b1}(gy_{n+2m-1}, gy_{n+2m})] + s^m r_{b1}(gy_{n+2m}, gy_{n+2m+1}).
\end{aligned} \tag{2.9}$$

Combining (2.4), (2.8), and (2.9), we have

$$\begin{aligned}
&r_{b1}(gx_n, gx_{n+p}) + r_{b1}(gy_n, gy_{n+p}) \\
&\quad = r_{b1}(gx_n, gx_{n+2m+1}) + r_{b1}(gy_n, gy_{n+2m+1}) \\
&\quad \leqslant s(\delta_n + \delta_{n+1}) + s^2(\delta_{n+2} + \delta_{n+3}) + \cdots + s^m(\delta_{n+2m-2} + \delta_{n+2m-1}) + s^m\delta_{n+2m} \\
&\quad \leqslant s\left(k^n + k^{n+1}\right)\delta_0 + s^2\left(k^{n+2} + k^{n+3}\right)\delta_0 + \cdots + s^m\left(k^{n+2m-2} + k^{n+2m-1}\right)\delta_0 + s^m k^{n+2m}\delta_0 \\
&\quad = sk^n(1+k)\left[1 + sk^2 + \left(sk^2\right)^2 + \cdots + \left(sk^2\right)^{m-1}\right]\delta_0 + s^m k^{n+2m}\delta_0 \\
&\quad = \begin{cases} \left[sk^n(1+k)\cdot m + s^m k^{n+2m}\right]\delta_0, & sk^2 = 1, \\ \left(sk^n(1+k)\cdot \frac{1-\left(sk^2\right)^m}{1-sk^2} + s^m k^{n+2m}\right)\delta_0, & sk^2 \neq 1, \end{cases} \\
&\quad \leqslant \begin{cases} \left[sk^n(1+k)\cdot m + s^m k^{n+2m}\right]\delta_0, & sk^2 = 1, \\ \left(\frac{sk^n(1+k)}{1-sk^2} + s^m k^{n+2m}\right)\delta_0, & sk^2 \neq 1. \end{cases}
\end{aligned} \tag{2.10}$$

Case 2. p is an even number, assume that $p = 2m$, then using (Rb3) we obtain

$$\begin{aligned}
&r_{b1}(gx_n, gx_{n+p}) \\
&\quad = r_{b1}(gx_n, gx_{n+2m}) \\
&\quad \leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2}) + r_{b1}(gx_{n+2}, gx_{n+2m})] \\
&\quad \leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] \\
&\qquad + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4}) + r_{b1}(gx_{n+4}, gx_{n+2m})] \\
&\quad \leqslant \cdots \\
&\quad \leqslant s[r_{b1}(gx_n, gx_{n+1}) + r_{b1}(gx_{n+1}, gx_{n+2})] + s^2[r_{b1}(gx_{n+2}, gx_{n+3}) + r_{b1}(gx_{n+3}, gx_{n+4})] \\
&\qquad + \cdots + s^{m-1}[r_{b1}(gx_{n+2m-4}, gx_{n+2m-3}) + r_{b1}(gx_{n+2m-3}, gx_{n+2m-2})] \\
&\qquad + s^{m-1}r_{b1}(gx_{n+2m-2}, gx_{n+2m}).
\end{aligned} \tag{2.11}$$

By similar arguments as above,

$$\begin{aligned}
&r_{b1}(gy_n, gy_{n+p}) \\
&\quad = r_{b1}(gy_n, gy_{n+2m}) \\
&\quad \leqslant s[r_{b1}(gy_n, gy_{n+1}) + r_{b1}(gy_{n+1}, gy_{n+2})] + s^2[r_{b1}(gy_{n+2}, gy_{n+3}) + r_{b1}(gy_{n+3}, gy_{n+4})] \\
&\qquad + \cdots + s^{m-1}[r_{b1}(gy_{n+2m-4}, gy_{n+2m-3}) + r_{b1}(gy_{n+2m-3}, gx_{n+2m-2})] \\
&\qquad + s^{m-1}[r_{b1}(gy_{n+2m-2}, gy_{n+2m})].
\end{aligned} \tag{2.12}$$

Combining (2.4), (2.7), (2.11) and (2.12), we have

$$\begin{aligned}
&r_{b1}(gx_n, gx_{n+p}) + r_{b1}(gy_n, gy_{n+p}) \\
&\quad = r_{b1}(gx_n, gx_{n+2m}) + r_{b1}(gy_n, gy_{n+2m})
\end{aligned}$$

$$
\begin{aligned}
&\leqslant s(\delta_n+\delta_{n+1})+s^2(\delta_{n+2}+\delta_{n+3})+\cdots+s^{m-1}(\delta_{n+2m-4}+\delta_{n+2m-3})+s^{m-1}\delta^*_{n+2m-2}\\
&\leqslant s\left(k^n+k^{n+1}\right)\delta_0+s^2\left(k^{n+2}+k^{n+3}\right)\delta_0\\
&\quad+\cdots+s^{m-1}\left(k^{n+2m-4}+k^{n+2m-3}\right)\delta_0+s^{m-1}k^{n+2m-2}\max\{\delta_0,d^*_0\}\\
&=sk^n(1+k)\left[1+sk^2+\left(sk^2\right)^2+\cdots+\left(sk^2\right)^{m-2}\right]\delta_0+s^{m-1}k^{n+2m-2}\max\{\delta_0,\delta^*_0\}\\
&=\begin{cases}
sk^n(1+k)(m-1)\delta_0+s^{m-1}k^{n+2m-2}\max\{\delta_0,\delta^*_0\}, & sk^2=1,\\
sk^n(1+k)\cdot\frac{1-\left(sk^2\right)^{m-1}}{1-sk^2}\cdot\delta_0+s^{m-1}k^{n+2m-2}\max\{d_0,d^*_0\}, & sk^2\neq 1.
\end{cases}\\
&\leqslant\begin{cases}
sk^n(1+k)(m-1)\delta_0+s^{m-1}k^{n+2m-2}\max\{\delta_0,\delta^*_0\}, & sk^2=1,\\
\frac{sk^n(1+k)}{1-sk^2}\cdot\delta_0+s^{m-1}k^{n+2m-2}\max\{\delta_0,\delta^*_0\}, & sk^2\neq 1.
\end{cases}
\end{aligned}
\tag{2.13}
$$

Since $k\in[0,1)$, so $k^n\to 0$ as $n\to\infty$. Taking limit as $n\to\infty$ in (2.10) and (2.13), we get

$$\lim_{n\to\infty}[r_{b1}(gx_n,gx_{n+p})+r_{b1}(gy_n,gy_{n+p})]=0,$$

which implies that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, then there exist $x,y\in X$ such that

$$\lim_{n\to\infty}gx_n=gx \quad\text{and}\quad \lim_{n\to\infty}gy_n=gy.$$

It follows from (2.1) and (2.4) that

$$
\begin{aligned}
&r_{b1}(gx_{n+1},F(x,y))+r_{b1}(gy_{n+1},F(y,x))\\
&\quad=r_{b1}(F(x_n,y_n),F(x,y))+r_{b1}(F(y_n,x_n),F(y,x))\\
&\quad\leqslant k_1[r_{b2}(gx_n,gx)+r_{b2}(gy_n,gy)]+k_2[r_{b2}(gx_n,F(x_n,y_n))+r_{b2}(gy_n,F(y_n,x_n))]\\
&\qquad+k_3[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))]\\
&\quad=k_1[r_{b2}(gx_n,gx)+r_{b2}(gy_n,gy)]+k_2[r_{b2}(gx_n,gx_{n+1})+r_{b2}(gy_n,gy_{n+1})]\\
&\qquad+k_3[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))]\\
&\quad\leqslant k_1[r_{b1}(gx_n,gx)+r_{b1}(gy_n,gy)]+k_2[r_{b1}(gx_n,gx_{n+1})+r_{b1}(gy_n,gy_{n+1})]\\
&\qquad+k_3[r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))]\\
&\quad=k_1[r_{b1}(gx_n,gx)+r_{b1}(gy_n,gy)]+k_2\delta_n+k_3[r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))]\\
&\quad\leqslant k_1[r_{b1}(gx_n,gx)+r_{b1}(gy_n,gy)]+k_2k^n\delta_0+k_3[r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))].
\end{aligned}
\tag{2.14}
$$

Applying (Rb3), (2.14), and (2.4) we have

$$
\begin{aligned}
r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))&\leqslant s[r_{b1}(gx,gx_n)+r_{b1}(gx_n,gx_{n+1})+r_{b1}(gx_{n+1},F(x,y))]\\
&\quad+s[r_{b1}(gy,gy_n)+r_{b1}(gy_n,gy_{n+1})+r_{b1}(gy_{n+1},F(y,x))]\\
&=s[r_{b1}(gx,gx_n)+r_{b1}(gy,gy_n)]+s\delta_n\\
&\quad+s[r_{b1}(gx_{n+1},F(x,y))+r_{b1}(gy_{n+1},F(y,x))]\\
&\leqslant s(1+k_1)[r_{b1}(gx_n,gx)+r_{b1}(gy_n,gy)]+s(1+k_2)k^n\delta_0\\
&\quad+sk_3[r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))].
\end{aligned}
\tag{2.15}
$$

By taking $n\to\infty$ in the above inequality (2.15), we have

$$r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))\leqslant sk_3[r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))]. \tag{2.16}$$

By condition $0\leqslant sk_3<1$ and (2.16), we can easily obtain that

$$r_{b1}(gx,F(x,y))+r_{b1}(gy,F(y,x))=0,$$

which implies that

$$gx = F(x,y) \text{ and } gy = F(y,x).$$

Therefore, we conclude that (x,y) is the coupled coincidence point of F and g.

Next, we show the uniqueness of the triple point of coincidence of F and g. Assume that (x^*,y^*) is another tripled coincidence point of mappings F and g. By (2.1), we derive

$$\begin{aligned} r_{b1}(gx,gx^*)+r_{b1}(gy,gy^*) &= r_{b1}(F(x,y),F(x^*,y^*))+r_{b1}(F(y,x),F(y^*,x^*)) \\ &\leqslant k_1[r_{b2}(gx,gx^*)+r_{b2}(gy,gy^*)]+k_2[r_{b2}(gx,F(x,y)) \\ &\quad +r_{b2}(gy,F(y,x))]+k_3[r_{b2}(gx^*,F(x^*,y^*))+r_{b2}(gy^*,F(y^*,x^*))] \\ &= k_1[r_{b2}(gx,gx^*)+r_{b2}(gy,gy^*)] \\ &\leqslant k_1[r_{b1}(gx,gx^*)+r_{b1}(gy,gy^*)]. \end{aligned} \tag{2.17}$$

By virtue of $0 \leqslant k_1 \leqslant k_1+k_2+k_3 < 1$ and (2.17), we deduce

$$r_{b1}(gx,gx^*)+r_{b1}(gy,gy^*) = 0.$$

This implies that $gx = gx^*$ and $gy = gy^*$. So that the couple point of coincidence of F and g is unique.

Next, we show that $gx = gy$. In fact, it follows from (2.1) that

$$\begin{aligned} r_{b1}(gx,gy)+r_{b1}(gy,gx) &= r_{b1}(F(x,y),F(y,x))+r_{b1}(F(y,x),F(x,y)) \\ &\leqslant k_1[r_{b2}(gx,gy)+r_{b2}(gy,gx)]+k_2[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))] \\ &\quad +k_3[r_{b2}(gy,F(y,x))+r_{b2}(gx,F(x,y))] \\ &= k_1[r_{b2}(gx,gy)+r_{b2}(gy,gx)] \\ &\leqslant k_1[r_{b1}(gx,gy)+r_{b1}(gy,gx)]. \end{aligned}$$

By making use of $0 \leqslant k_1 \leqslant k_1+k_2+k_3 < 1$ and (2.17), we deduce

$$r_{b1}(gx,gy)+r_{b1}(gy,gx) = 0.$$

This means that $gx = gy$.

Finally, if F and g are w-compatible, then we have $g(F(x,y)) = F(gx,gy)$. Therefore, by taking $u = gx$, we have $u = gx = F(x,y) = gy = F(y,x)$, hence we have

$$gu = ggx = g(F(x,y)) = F(gx,gy) = F(u,u).$$

Consequently, (gu,gu) is a coupled point of coincidence of g and F, and by its uniqueness, we get $gu = gx$. Thus, we obtain $F(u,u) = gu = u$. Therefore, (u,u) is the unique common coupled fixed point of g and F. This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, (X,r_{b1}) with coefficient $s \geqslant 1$, and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that there exist k_1,k_2, and k_3 in $[0,1)$ with $0 \leqslant 2(k_1+k_2+k_3) < 1$ and $0 \leqslant 2sk_3 < 1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant\ & k_1[r_{b2}(gx,gu)+r_{b2}(gy,gv)]+k_2[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))] \\ & +k_3[r_{b2}(gu,F(u,v))+r_{b2}(gv,F(v,u))] \end{aligned} \tag{2.18}$$

holds for all $(x,y),(u,v) \in X \times X$. If $F(X \times X) \subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y) \in X \times X$, which satisfies that $gx = F(x,y) = gy = F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Proof. It follows from (2.18) that

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant\ & k_1[r_{b2}(gx,gu)+r_{b2}(gy,gv)]+k_2[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))] \\ & +k_3[r_{b2}(gu,F(u,v))+r_{b2}(gv,F(v,u))], \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} r_{b1}(F(y,x),F(v,u)) \leqslant\ & k_1[r_{b2}(gy,gv)+r_{b2}(gx,gu)]+k_2[r_{b2}(gy,F(y,x))+r_{b2}(gx,F(x,y))] \\ & +k_3[r_{b2}(gv,F(v,u))+r_{b2}(gu,F(u,v))]. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we have

$$\begin{aligned} r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u)) \leqslant\ & 2k_1[r_{b2}(gx,gu)+r_{b2}(gy,gv)]+2k_2[r_{b2}(gx,F(x,y)) \\ & +r_{b2}(gy,F(y,z))]+2k_3[r_{b2}(gu,F(u,v))+r_{b2}(gv,F(v,u))]. \end{aligned}$$

By Theorem 2.1, we can get the conclusion. □

In Theorem 2.1, if we take $r_{b1}(x,y)=r_{b2}(x,y)=r_b(x,y)$ for all $x,y\in X$, then we get the following corollary.

Corollary 2.3. *Let (X,r_b) be a r_b-complete rectangular b-metric space with coefficient $s\geqslant 1$, $F:X\times X\to X$ and $g:X\to X$ be two mappings. Suppose that there exist k_1,k_2, and k_3 in $[0,1)$ with $0\leqslant k_1+k_2+k_3<1$ and $0\leqslant sk_3<1$ such that the condition*

$$\begin{aligned} & r_b(F(x,y),F(u,v))+r_b(F(y,x),F(v,u)) \\ & \leqslant k_1[r_b(gx,gu)+r_b(gy,gv)]+k_2[r_b(gx,F(x,y))+r_b(gy,F(y,x))]+k_3[r_b(gu,F(u,v))+r_b(gv,F(v,u))] \end{aligned}$$

holds for all $(x,y),(u,v)\in X\times X$. If $F(X\times X)\subset g(X)$, then F and g have a coupled coincidence point $(x,y)\in X\times X$, satisfying that $gx=F(x,y)=gy=F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u=gu=F(u,u)$.

Corollary 2.4. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y)\leqslant r_{b1}(x,y)$ for all $x,y\in X$, (X,r_{b1}) with coefficient $s\geqslant 1$, and $F:X\times X\to X$ and $g:X\to X$ be two mappings. Suppose that there exist $a_i\in[0,1)$ $(i=1,2,3,\cdots,6)$ with $0\leqslant a_1+a_2+a_3+\cdots+a_6<1$ and $0\leqslant s(a_5+a_6)<1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant\ & a_1r_{b2}(gx,gu)+a_2r_{b2}(gy,gv)+a_3r_{b2}(gx,F(x,y))+a_4r_{b2}(gy,F(y,x)) \\ & +a_5r_{b2}(gu,F(u,v))+a_6r_{b2}(gv,F(v,u)) \end{aligned} \tag{2.21}$$

holds for all $(x,y),(u,v)\in X\times X$. If $F(X\times X)\subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y)\in X\times X$, which satisfies that $gx=F(x,y)=gy=F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u=gu=F(u,u)$.

Proof. Given $(x,y),(u,v)\in X\times X$, it follows from (2.21) that

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant\ & a_1r_{b2}(gx,gu)+a_2r_{b2}(gy,gv)+a_3r_{b2}(gx,F(x,y))+a_4r_{b2}(gy,F(y,x)) \\ & +a_5r_{b2}(gu,F(u,v))+a_6r_{b2}(gv,F(v,u)) \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} r_{b1}(F(y,x),F(v,u)) \leqslant\ & a_1r_{b2}(gy,gv)+a_2r_{b2}(gx,gu)+a_3r_{b2}(gy,F(y,x))+a_4r_{b2}(gx,F(x,y)) \\ & +a_5r_{b2}(gv,F(v,u))+a_6r_{b2}(gu,F(u,v)). \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we have

$$\begin{aligned} r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u)) \leqslant{}& (a_1+a_2)[r_{b2}(gx,gu)+r_{b2}(gy,gv)] \\ &+(a_3+a_4)[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))] \\ &+(a_5+a_6)[r_{b2}(gu,F(u,v))+r_{b2}(gv,F(v,u))]. \end{aligned}$$

Therefore, the result follows from Theorem 2.1. □

Remark 2.5. If we take $r_{b1}(x,y)=r_{b2}(x,y)=r_b(x,y)$ for all $x,y\in X$, where r_b is a rectangular b-metric on X, then Corollary 2.4 will be reduced to a new result.

The following corollary can be obtained from Theorem 2.1 immediately.

Corollary 2.6. *Let r_{b1} and r_{b2} be two rectangular* b*-metrics on X such that $r_{b2}(x,y)\leqslant r_{b1}(x,y)$ for all $x,y\in X$, and $F:X\times X\to X$ and $g:X\to X$ be two mappings. Suppose that there exists $k\in[0,1)$ such that the condition*

$$r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u))\leqslant k[r_{b2}(gx,gu)+r_{b2}(gy,gv)]$$

holds for all $(x,y),(u,v)\in X\times X$. If $F(X\times X)\subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y)\in X\times X$, which satisfies that $gx=F(x,y)=gy=F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u=gu=F(u,u)$.

Corollary 2.7. *Let r_{b1} and r_{b2} be two rectangular* b*-metrics on X such that $r_{b2}(x,y)\leqslant r_{b1}(x,y)$ for all $x,y\in X$, and $F:X\times X\to X$ and $g:X\to X$ be two mappings. Suppose that there exists $k\in[0,1)$ such that the condition*

$$r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u))\leqslant k[r_{b2}(gx,F(x,y))+r_{b2}(gy,F(y,x))]$$

holds for all $(x,y),(u,v)\in X\times X$. If $F(X\times X)\subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y)\in X\times X$, which satisfies that $gx=F(x,y)=gy=F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u=gu=F(u,u)$.

Corollary 2.8. *Let r_{b1} and r_{b2} be two rectangular* b*-metrics on X such that $r_{b2}(x,y)\leqslant r_{b1}(x,y)$ for all $x,y\in X$, (X,r_{b1}) with coefficient $s\geqslant 1$, and $F:X\times X\to X$ and $g:X\to X$ be two mappings. Suppose that there exists $k\in[0,1)$ with $0\leqslant sk<1$ such that the condition*

$$r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u))\leqslant k[r_{b2}(gu,F(u,v))+r_{b2}(gv,F(v,u))]$$

holds for all $(x,y),(u,v)\in X\times X$. If $F(X\times X)\subset g(X)$ and $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y)\in X\times X$, which satisfies that $gx=F(x,y)=gy=F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u=gu=F(u,u)$.

Let $g=I_X$ (the identity mapping) in Theorem 2.1 and Corollaries 2.2–2.4, 2.6, 2.7. Then we have the following results.

Corollary 2.9. *Let r_{b1} and r_{b2} be two rectangular* b*-metrics on X such that $r_{b2}(x,y)\leqslant r_{b1}(x,y)$ for all $x,y\in X$, (X,r_{b1}) with coefficient $s\geqslant 1$ and $F:X\times X\to X$ be a mapping. Suppose that there exist k_1,k_2, and k_3 in $[0,1)$ with $0\leqslant k_1+k_2+k_3<1$ and $0\leqslant sk_3<1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v))+r_{b1}(F(y,x),F(v,u))\leqslant{}& k_1[r_{b2}(x,u)+r_{b2}(y,v)]+k_2[r_{b2}(x,F(x,y)) \\ &+r_{b2}(y,F(y,x))]+k_3[r_{b2}(u,F(u,v))+r_{b2}(v,F(v,u))] \end{aligned}$$

holds for all $(x,y),(u,v)\in X\times X$. If X is r_{b1}-complete, then F has a unique coupled fixed point of the form (u,u), which satisfies that $u=F(u,u)$.

Corollary 2.10. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, (X, r_{b1}) with coefficient $s \geqslant 1$ and $F : X \times X \to X$ be a mapping. Suppose that there exist k_1, k_2, and k_3 in $[0,1)$ with $0 \leqslant 2(k_1+k_2+k_3) < 1$ and $0 \leqslant 2sk_3 < 1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant k_1[r_{b2}(x,u)+r_{b2}(y,v)] + k_2[r_{b2}(x,F(x,y))+r_{b2}(y,F(y,x))] \\ + k_3[r_{b2}(u,F(u,v))+r_{b2}(v,F(v,u))] \end{aligned}$$

holds for all $(x,y),(u,v) \in X \times X$. If $g(X)$ is r_{b1}-complete, then F and g have a coupled coincidence point $(x,y) \in X \times X$, which satisfies that $gx = F(x,y) = gy = F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Corollary 2.11. *Let (X, r_b) be a r_b-complete rectangular b-metric space with coefficient $s \geqslant 1$, and $F : X \times X \to X$ be a mapping. Suppose that there exist k_1, k_2, and k_3 in $[0,1)$ with $0 \leqslant k_1+k_2+k_3 < 1$ and $0 \leqslant sk_3 < 1$ such that the condition*

$$\begin{aligned} r_b(F(x,y),F(u,v)) + r_b(F(y,x),F(v,u)) \leqslant k_1[r_b(x,u)+r_b(y,v)] + k_2[r_b(x,F(x,y))+r_b(y,F(y,x))] \\ + k_3[r_b(u,F(u,v))+r_b(v,F(v,u))] \end{aligned}$$

holds for all $(x,y),(u,v) \in X \times X$. Then F has a unique coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Corollary 2.12. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, (X, r_{b1}) with coefficient $s \geqslant 1$ and $F : X \times X \to X$ be a mapping. Suppose that there exist $a_i \in [0,1)$ $(i = 1,2,3,\cdots,6)$ with $0 \leqslant a_1+a_2+a_3+\cdots+a_6 < 1$ and $0 \leqslant s(a_5+a_6) < 1$ such that the condition*

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) \leqslant a_1 r_{b2}(x,u) + a_2 r_{b2}(y,v) + a_3 r_{b2}(x,F(x,y)) + a_4 r_{b2}(y,F(y,x)) \\ + a_5 r_{b2}(u,F(u,v)) + a_6 r_{b2}(v,F(v,u)) \end{aligned}$$

holds for all $(x,y),(u,v) \in X \times X$. If (X, r_{b1}) is complete, then F has a unique coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Corollary 2.13. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exists $k \in [0,1)$ such that the condition*

$$r_{b1}(F(x,y),F(u,v)) + r_{b1}(F(y,x),F(v,u)) \leqslant k[r_{b2}(x,u)+r_{b2}(y,v)]$$

holds for all $(x,y),(u,v) \in X \times X$. If (X, r_{b1}) is complete, then F has a unique coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Corollary 2.14. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exists $k \in [0,1)$ such that the condition*

$$r_{b1}(F(x,y),F(u,v)) + r_{b1}(F(y,x),F(v,u)) \leqslant k[r_{b2}(x,F(x,y))+r_{b2}(y,F(y,x))]$$

holds for all $(x,y),(u,v) \in X \times X$. If (X, r_{b1}) is complete, then F has a unique coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Corollary 2.15. *Let r_{b1} and r_{b2} be two rectangular b-metrics on X such that $r_{b2}(x,y) \leqslant r_{b1}(x,y)$ for all $x,y \in X$, (X, r_{b1}) with coefficient $s \geqslant 1$, and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that there exists $k \in [0,1)$ with $0 \leqslant sk < 1$ such that the condition*

$$r_{b1}(F(x,y),F(u,v)) + r_{b1}(F(y,x),F(v,u)) \leqslant k[r_{b2}(u,F(u,v))+r_{b2}(v,F(v,u))]$$

holds for all $(x,y),(u,v) \in X \times X$. If (X, r_{b1}) is complete, then F has a unique coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

It follows by taking coefficient $s = 1$ (for the space X) in Theorem 2.1, then we have the following corollary.

Corollary 2.16. *Let X be a nonempty set and r_1, r_2 are two rectangular metrics in X such that $r_2(x,y) \leqslant r_1(x,y)$, for all $x, y \in X$. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings, and there exist k_1, k_2, and k_3 in $[0,1)$ with $0 \leqslant k_1 + k_2 + k_3 < 1$ such that the condition*

$$\begin{aligned} &r_1(F(x,y),F(u,v)) + r_1(F(y,x),F(v,u)) \\ &\quad \leqslant k_1[r_2(gx,gu) + r_2(gy,gv)] + k_2[r_2(gx,F(x,y)) + r_2(gy,F(y,x))] + k_3[r_2(gu,F(u,v)) + r_2(gv,F(v,u))] \end{aligned}$$

holds for all $(x,y),(u,v) \in X \times X$. If $F(X \times X) \subset g(X)$ and $g(X)$ is r_1-complete, then F and g have a coupled coincidence point $(x,y) \in X \times X$, satisfying that $gx = F(x,y) = gy = F(y,x)$. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u), which satisfies that $u = gu = F(u,u)$.

Remark 2.17. If we take coefficient $s = 1$ (for the space X) in Corollaries 2.2–2.4, 2.6–2.15, then several new results can be obtained in two rectangular metric spaces.

3. Application to integral equations

Example 3.1. Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ as $r_b(x,y) = |x-y|^k$, $x, y \in X$, where $k \geqslant 1$. Then (X, r_b) is a rectangular b-metric space with coefficient $s = 3^{k-1}$. In fact, the conditions (Rb1) and (Rb2) are clear. Next we show that (Rb3). From the convexity of function $f(x) = x^k$ for $x \geqslant 0$ and by Jensen inequality we have

$$(a+b+c)^k \leqslant 3^{k-1}\left(a^k + b^k + c^k\right)$$

for nonnegative real numbers a, b, c. This means that

$$\begin{aligned} r_b(x,y) = |x-y|^k \leqslant (|x-z| + |z-w| + |w-y|)^k &\leqslant 3^{k-1}\left(|x-z|^k + |z-w|^k + |w-y|^k\right) \\ &= 3^{k-1}[r_b(x,z) + r_b(z,w) + r_b(w,y)], \end{aligned}$$

which implies that (Rb3) holds, hence (X, r_b) is a rectangular b-metric space with coefficient $s = 3^{k-1}$.

Example 3.2. Let $X = \mathbb{R}$ and r_{b1}, r_{b2} are two rectangular b-metrics in X such that

$$r_{b1}(x,y) = (x-y)^2, \quad r_{b2}(x,y) = \frac{(x-y)^2}{2}, \quad \forall\, x, y \in X.$$

Define $F : X \times X \to X$ and $g : X \to X$ respectively by

$$F(x,y) = \frac{x-y}{3}, \quad gx = 2x, \quad \forall\, x, y \in X.$$

It is easy to see that $F(X \times X) \subset g(X)$, $g(X)$ is r_{b1}-complete, and F and g are w-compatible.

On the other hand, we have

$$\begin{aligned} r_{b1}(F(x,y),F(u,v)) &= (F(x,y) - F(u,v))^2 \\ &= \left(\frac{x-y}{3} - \frac{u-v}{3}\right)^2 \\ &= \left(\frac{x-u}{3} + \frac{v-y}{3}\right)^2 \\ &\leqslant 2\left(\left(\frac{x-u}{3}\right)^2 + \left(\frac{y-v}{3}\right)^2\right) \end{aligned}$$

$$= \frac{1}{9}\left(\frac{(2x-2u)^2}{2} + \frac{(2y-2v)^2}{2}\right)$$
$$= \frac{1}{9}\left(\frac{(gx-gu)^2}{2} + \frac{(gy-gv)^2}{2}\right) = \frac{1}{9}[r_{b2}(gx,gu) + r_{b2}(gy,gv)].$$

By similar arguments as above, we can show that

$$r_{b1}(F(y,z),F(v,u)) \leqslant \frac{1}{9}[r_{b2}(gy,gv) + r_{b2}(gx,gu)].$$

Combining the above two inequalities, we obtain

$$r_{b1}(F(x,y),F(u,v)) + r_{b1}(F(y,x),F(v,u)) \leqslant \frac{2}{9}[r_{b2}(gx,gu) + r_{b2}(gy,gv)].$$

Then by Corollary 2.6, F and g have a unique common coupled fixed point, in fact, $(0,0)$ is the unique common coupled fixed point of mappings of F and g.

Next, we assume that $X = C[a,b]$ is the set of all continuous functions. Define two rectangular b-metrics respectively by

$$r_{b1}(x,y) = \max_{t\in[a,b]} |x(t)-y(t)|^k, \quad r_{b2}(x,y) = \frac{\max\limits_{t\in[a,b]} |x(t)-y(t)|^k}{2}, \quad \forall x,y \in X, (k \geqslant 1).$$

Then the coefficient of two rectangular b-metrics is $s = 3^{k-1}$. Consider the nonlinear integral equation set as follows:

$$\begin{cases} x(r) = K(r) + \int_a^b G(r,t)[f(t,x(t)) + g(t,y(t))]dt, \\ y(r) = K(r) + \int_a^b G(r,t)[f(t,y(t)) + g(t,x(t))]dt. \end{cases} \tag{3.1}$$

Next, we will analyze (3.1) under the following conditions:

(i) $f, g : [a,b] \times R \to R$ are three continuous functions.
(ii) $K : [a,b] \to R$ are continuous functions.
(iii) $G : [a,b] \times R \to [0,+\infty)$ are continuous functions.
(iv) There exists $p, q > 0$ such that for all $x, y \in R$,

$$\begin{cases} |f(t,x(t)) - f(t,y(t))| \leqslant p|x-y|, \\ |g(t,x(t)) - g(t,y(t))| \leqslant q|x-y|. \end{cases}$$

(v)

$$\max_{r\in[a,b]} \left(\int_a^b |G(r,t)|dt\right)^k < \frac{1}{2^{k+1}L^k}, \quad \text{with } L = \max\{p,q\}.$$

Theorem 3.3. *Under the conditions* (i)-(v), *the integral equation* (3.1) *has a unique common solution on* $[a,b]$.

Proof. Define $F : X \times X \to X$ and $g : X \to X$ respectively by

$$F(x,y)(r) = K(r) + \int_a^b G(r,t)[f(t,x(t)) + g(t,y(t))]dt,$$
$$gx = 2x, \forall x \in X, t \in [a,b], x, y \in X,$$
$$r_{b1}(F(x,y),F(u,v)) = \max_{r\in[a,b]} |F(x,y)(r) - F(u,v)(r)|^k, \quad \forall\, x,y,u,v \in X,$$
$$r_{b2}(F(x,y),F(u,v)) = \frac{\max_{r\in[a,b]} |F(x,y)(r) - F(u,v)(r)|^k}{2}, \quad \forall\, x,y,u,v \in X.$$

It is easy to see that $F(X \times X) \subset g(X)$, $g(X)$ is r_{b1}-complete, and F and g are w-compatible.

On the other hand, from (iv) and (v), we have

$$\begin{aligned}
&|F(x,y)(r) - F(u,v)(r)|^k \\
&= \left| \int_a^b G(r,t)[f(t,x(t)) - f(t,u(t))]dt + \int_a^b G(r,t)[g(t,y(t)) - g(t,v(t))]dt \right|^k \\
&\leqslant 2^{k-1} \left| \int_a^b G(r,t)[f(t,x(t)) - f(t,u(t))]dt \right|^k + 2^{k-1} \left| \int_a^b G(r,t)[g(t,y(t)) - g(t,v(t))]dt \right|^k \\
&\leqslant 2^{k-1} \left[\left| \int_a^b G(r,t)[f(t,x(t)) - f(t,u(t))]dt \right|^k + \left| \int_a^b G(r,t)[g(t,y(t)) - g(t,v(t))]dt \right|^k \right] \\
&\leqslant 2^{k-1} \left[p^k \left(\max_{t\in[a,b]} |x(t) - u(t)| \right)^k + q^k \left(\max_{t\in[a,b]} |y(t) - v(t)| \right)^k \right] \cdot \left(\int_a^b G(r,t)dt \right)^k \\
&\leqslant 2^{k-1} L^k \left[\max_{t\in[a,b]} |x(t) - u(t)|^k + \max_{t\in[a,b]} |y(t) - v(t)|^k \right] \cdot \frac{1}{2^{k+1}L^k} \\
&\leqslant \frac{1}{2^{k+2}} \left[\max_{t\in[a,b]} |2x(t) - 2u(t)|^k + \max_{t\in[a,b]} |2y(t) - 2v(t)|^k \right] \\
&= \frac{1}{2^{k+1}} [r_{b2}(gx, gu) + r_{b2}(gy, gv)].
\end{aligned}$$

It follows from the above inequality, we have

$$r_{b1}(F(x,y), F(u,v)) = \max_{r\in[a,b]} |F(x,y)(r) - F(u,v)(r)|^k \leqslant \frac{1}{2^{k+1}} [r_{b2}(gx, gu) + r_{b2}(gy, gv)]. \tag{3.2}$$

By similar arguments as above,

$$r_{b1}(F(y,x), F(v,u)) \leqslant \frac{1}{2^{k+1}} [r_{b2}(gy, gv) + r_{b2}(gx, gu)]. \tag{3.3}$$

It follows from (3.2) and (3.3) that we have

$$r_{b1}(F(x,y), F(u,v)) + r_{b1}(F(y,x), F(v,u)) \leqslant \frac{1}{2^k} [r_{b2}(gx, gu) + r_{b2}(gy, gv)].$$

Consequently, all the conditions of Corollary 2.6 are satisfied. It follows from the result of Corollary 2.6 that F and g have a unique common coupled fixed point (u,u), satisfying $F(u,u) = gu = u$. So (u,u) is the unique solution of integral equation (3.1). □

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