



Second-order accurate numerical approximations for the fractional percolation equations

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Abstract

First, we examine a practical numerical method which based on the classical Crank-Nicholson (CN) method combined with Richardson extrapolation is used to solve a class of one-dimensional initial-boundary value fractional percolation equation (FPE) with variable coefficients on a finite domain. Secondly, we present ADI-CN method for the two-dimensional fractional percolation equation. Stability and convergence of these methods are proved. Using these methods, we can achieve second-order convergence in time and space. Finally, numerical examples are presented to verify the order of convergence. ©2017 All rights reserved.

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1. Introduction

The fractional percolation equations, which are used to model problems in seepage hydraulics, groundwater hydraulics, groundwater dynamics, and fluid dynamics in porous media [5, 10, 17, 25], are generated from classical integer order percolation equations. Under the hypotheses of continuity and Darcy's law, the traditional integer percolation equation for incompressible, single phase percolation flow can be written as

$$\frac{1}{v} \frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(k(x) \frac{\partial p(x, t)}{\partial x} \right) + h(x, t), x \in \Omega,$$

where $k(x)$ is the percolation coefficient along the x direction, $p = p(x, t)$ is the pressure, v is the velocity, $h(x, t)$ is the source term, and Ω denotes the percolation domain. In fact, some people studied the seepage flow and found that the process is neither continuous nor rigid, therefore a more general equation for seepage flow was considered. He [9] first considered a modification of Darcy's law to treat the seepage flow movement in a non-homogeneous porous medium. It can be written as

$$\frac{1}{v} \frac{\partial p}{\partial t} = \frac{\partial^\beta}{\partial x^\beta} \left(k(x) \frac{\partial^\alpha p(x, t)}{\partial x^\alpha} \right) + h(x, t), x \in \Omega, \quad (1.1)$$

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where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. In this FPE, we denote the rate of the fluid mass flux $q = k(x) \frac{\partial^\alpha p(x,t)}{\partial x^\alpha}$. And the above defined fractional derivative is the Riemann-Liouville fractional derivative [16, 18].

The Riemann-Liouville fractional derivative $\frac{\partial^\gamma p(x,t)}{\partial x^\gamma}$ of the order γ is defined by

$$\frac{\partial^\gamma p(x,t)}{\partial x^\gamma} = \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial x^n} \int_0^x p(x,t)(x-s)^{-\gamma+n-1} ds,$$

where, for any $\gamma > 0$, n is an integer, and $n-1 < \gamma \leq n$.

If the seepage flow is rigid, then $\beta = 0$, and the FPE (1.1) can be written as

$$\frac{1}{v} \frac{\partial p}{\partial t} = \frac{\partial^0}{\partial x^0} \left(k(x) \frac{\partial^\alpha p(x,t)}{\partial x^\alpha} \right) + h(x,t), x \in \Omega.$$

The equation (1.1) is called the fractional percolation equation. This paper will focus on a class of initial-boundary value fractional percolation equation (FPE) with variable coefficients on a finite domain.

As we all know, most fractional partial differential equations are not easy to obtain analytic solutions, so some researchers resort to numerical solution for fractional partial differential equations [8, 11–13, 15, 20]. In recent years, many authors presented some efficient numerical methods. Tchier et al. [24] introduced the residual power series method for solving nonlinear time fractional reaction-diffusion equations. Zhang et al. [26] used the series expansion method within local fractional derivative to obtain the solutions of both homogeneous and non-homogeneous transport equations. Bhrawy et al. [1] reported a new space-time spectral algorithm for obtaining an approximate solution for the space-time fractional Burger's equation (FBE) based on spectral shifted Legendre collocation (SLC) method in combination with the shifted Legendre operational matrix of fractional derivatives. Singh and Kumar [19] presented the homotopy analysis transform method (HATM) to solve fractional Lotka-Volterra equation, which describes the long term survival of species. Srivastava et al. [21] presented an efficient analytical approach based on the q -homotopy analysis transform technique in order to analyze a fractional model of the vibration equation for large membranes.

In the numerical aspect of the FPE, Chen et al. [2] developed a novel implicit finite difference method for the one-dimensional fractional percolation equation. Guo et al. [6] proposed an implicit finite difference method for the one-dimensional fractional percolation equation with Dirichlet and fractional boundary conditions. Chen et al. [4] considered an alternating direction implicit difference method for the two-dimensional case. Chen et al. [3] discussed the two-dimensional variable-order fractional percolation equation. Guo et al. [7] proposed a second order finite difference method for the two-dimensional fractional percolation equation. Liu et al. [14] proposed two finite difference methods for the three-dimensional non-continued seepage flow problem with constant percolation coefficients and continued seepage flow problem with variable percolation coefficients. However, these estimates of numerical solution for the one-dimensional FPE are only first-order accurate in their papers. To our knowledge, the study on one-dimensional FPE with better than first-order accuracy is still limited. Here, we present numerical method, which combines a fractional Crank-Nicholson method with the extrapolation of the Crank-s solution, and get second-order accurate both in time and space. This method can be extended to two-dimensional fractional percolation equation. This work has appeared in the literature [7]. We proposed ADI-CN method for a two-dimensional percolation equation, compared with the CN method, can reduce the amount of computation, thus reducing the computational cost.

The rest of this paper is organized as follows. In Section 2, the Crank-Nicholson method for the one-dimensional fractional percolation equation is proposed. In Section 3, its stability and convergence are discussed in two cases. In Section 4, we present ADI-CN method for the two-dimensional percolation equation. In Section 5, we study the stability and convergence of the method. In Section 6, we carry out numerical experiments to verify the accuracy of the methods.

2. The Crank-Nicholson method for the one-dimensional FPE

Consider a one-dimensional fractional percolation equation

$$\frac{\partial p}{\partial t} = \frac{\partial^\beta}{\partial x^\beta} (k(x) \frac{\partial^\alpha p(x, t)}{\partial x^\alpha}) + f(x, t), \quad 0 < x < L, \quad 0 < t < T. \tag{2.1}$$

We assume an initial condition $p(x, 0) = s(x), 0 \leq x \leq L$, and Dirichlet boundary conditions of the form $p(0, t) = 0, p(L, t) = u(t), 0 \leq t \leq T$, where $0 < \alpha < 1, 0 < \beta \leq 1$, and we assume the percolation coefficient $k(x) > 0$.

For the Crank-Nicholson numerical approximation scheme, define $t_n = n\Delta t$ to be integration time $0 \leq t_n \leq T$, for $n = 0, 1, 2, \dots, N$ and $\Delta x = h = \frac{L}{m}$ to be the grid size in space with $x_i = i\Delta x$ for $i = 0, 1, \dots, m$. Define p_i^n to be the numerical solution of $p(x_i, t_n)$. The initial conditions are set by $p_i^0 = s(x_i)$. Similarly, we define $k_i = k(x_i)$ and $f_i^n = f(x_i, t_n)$. The Dirichlet boundary conditions are set by $p_0^n = 0, p_m^n = u(t_n)$.

To approximate the mixed fractional spatial derivative by substituting the shifted Grunwald estimates into the differential equation, chen et al. [2] defined

$$\frac{\partial^\beta}{\partial x^\beta} (k_x(x) \frac{\partial^\alpha p}{\partial x^\alpha})|_{(x_i, t_n)} \sim \frac{1}{h^{\alpha+\beta}} \sum_{j=1}^{i+1} \left(\sum_{l=0}^{i-j+1} g_{\beta, l} g_{\alpha, i-j-l+1} k_{i-l} \right) p_j^n + O(\Delta x), \tag{2.2}$$

where $g_{\beta, l} = (-1)^l \binom{\beta}{l}$; $g_{\alpha, j} = (-1)^j \binom{\alpha}{j}$.

Substitute (2.2) into the percolation equation (2.1) to get the Crank-Nicholson type numerical approximation. The resulting finite difference equations are as follows:

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = \frac{1}{2} (\delta_x p_i^{n+1} + \delta_x p_i^n) + f_i^{n+\frac{1}{2}}, \tag{2.3}$$

where the above fractional partial differentiation operator is defined as

$$\delta_x p_i^n = \frac{1}{h^{\alpha+\beta}} \sum_{j=1}^{i+1} \left(\sum_{l=0}^{i-j+1} g_{\beta, l} g_{\alpha, i-j-l+1} k_{i-l} \right) p_j^n.$$

This implies that the implicit finite difference scheme defined by (2.3) is consistent with order $O(\Delta t^2) + O(\Delta x)$. Eq. (2.3) may be rearranged and written as the fractional Crank-Nicholson discretization in the form

$$\left(1 - \frac{\Delta t}{2} \delta_x\right) p_i^{n+1} = \left(1 + \frac{\Delta t}{2} \delta_x\right) p_i^n + f_i^{n+\frac{1}{2}} \Delta t, \tag{2.4}$$

and the operator form (2.4) may be written in matrix form.

$$(I - A)P^{n+1} = (I + A)P^n + F^{n+\frac{1}{2}} \Delta t, \tag{2.5}$$

where

$$P^n = (p_1^n, \dots, p_{m-1}^n)^T, \\ F^{n+\frac{1}{2}} = \left(f_1^{n+\frac{1}{2}} \Delta t, f_2^{n+\frac{1}{2}} \Delta t, \dots, f_{m-1}^{n+\frac{1}{2}} \Delta t + rk_{m-1}(p_m^{n+1} + p_m^n) \right)^T,$$

where $r = \frac{\Delta t}{2h^{\alpha+\beta}}$, I is the $(m-1) \times (m-1)$ identity matrix, and $A = (A_{i,j})_{(m-1) \times (m-1)}$, defined by

$$A_{i,j} = \begin{cases} 0, & \text{when } j > i + 1, \\ rk_i, & \text{when } j = i + 1, \\ r \sum_{l=0}^1 g_{\beta, l} g_{\alpha, -l+1} k_{i-l}, & \text{when } j = i, \\ r \sum_{l=0}^{i-j+1} g_{\beta, l} g_{\alpha, i-j-l+1} k_{i-l}, & \text{when } j \leq i - 1. \end{cases} \tag{2.6}$$

3. Stability and convergence of the fractional CN

Having developed a numerical scheme and showing that it is consistent, we now show that it is also stable and thus convergent. First, we establish stability of the CN method under the assumption of continuity of seepage flow ($\beta = 1$).

Theorem 3.1. *If $\beta = 1$ and $k(x)$ decreases monotonically in the interval $[0, L]$, then the CN method defined by (2.5) is unconditionally stable.*

Proof. For $\beta = 1$, we have $g_{\beta,0} = 1$, $g_{\beta,1} = -1$, and $g_{\beta,n} = 0$ ($n \geq 2$). Hence (2.6) can be rewritten as

$$A_{i,j} = \begin{cases} 0, & \text{when } j > i + 1, \\ rk_i, & \text{when } j = i + 1, \\ r(-\alpha k_i - k_{i-1}), & \text{when } j = i, \\ r(g_{\alpha,i-j+1}k_i - g_{\alpha,i-j}k_{i-1}), & \text{when } j \leq i - 1. \end{cases}$$

Since $0 < \alpha < 1$, so $g_{\alpha,0} = 1$, $g_{\alpha,1} = -\alpha$, and $\sum_{k=0}^{\infty} g_{\alpha,k} = 0$. Now $g_{\alpha,k} = (1 - \frac{\alpha+1}{k}g_{\alpha,k-1})$ so $g_{\alpha,k-1} < g_{\alpha,k} < 0$ ($k \geq 2$). Since $k(x)$ decreases monotonically in the interval $[0, L]$, $0 < k_i < k_{i-1}$ ($i = 1, 2, \dots, m-1$). Therefore $g_{\alpha,i-j+1}k_i - g_{\alpha,i-j}k_{i-1} > 0$ for $j < i$. According to the Gerschgorin theorem (see [16]), the eigenvalues of the matrix A lie in the disks centered at $A_{i,i}$ with radius r_i ($r_i = \sum_{j=1, j \neq i}^{m-1} |A_{i,j}|$).

$$\begin{aligned} r_i &= \sum_{j=1, j \neq i}^{m-1} |A_{i,j}| = r \sum_{j=1}^{i-1} |g_{\alpha,i-j+1}k_i - g_{\alpha,i-j}k_{i-1}| + rk_i \\ &= rk_i \sum_{j=1}^{i-1} g_{\alpha,i-j+1} - rk_{i-1} \sum_{j=1}^{i-1} g_{\alpha,i-j} + rk_i \\ &= rk_i \sum_{s=0, s \neq 1}^i g_{\alpha,s} - rk_{i-1} \sum_{s=1}^{i-1} g_{\alpha,s} \\ &= rk_i(-g_{\alpha,1} - \sum_{s=i+1}^{\infty} g_{\alpha,s}) - rk_{i-1} \sum_{s=1}^{i-1} g_{\alpha,s} \\ &= rk_i\alpha - rk_i \sum_{s=i+1}^{\infty} g_{\alpha,s} - rk_{i-1} \sum_{s=1}^{i-1} g_{\alpha,s} \\ &< rk_i\alpha - rk_{i-1} \sum_{s=i+1}^{\infty} g_{\alpha,s} - rk_{i-1} \sum_{s=1}^{i-1} g_{\alpha,s} \\ &= rk_i\alpha - rk_{i-1} \sum_{s=1, s \neq i}^{\infty} g_{\alpha,s} < rk_i\alpha + rk_{i-1} = -A_{i,i}. \end{aligned}$$

We can get these Gerschgorin disks are within the left half of the complex plane. Therefore, the eigenvalues of the matrix A have negative real-parts.

Next, if λ is an eigenvalue of matrix A , we know $1 - \lambda$ is an eigenvalue of the matrix $I - A$, and $(1 + \lambda)/(1 - \lambda)$ is an eigenvalue of the matrix $(I - A)^{-1}(I + A)$. We found that all the eigenvalues of the matrix $(I - A)$ have a magnitude larger than 1, and thus this matrix is invertible. Furthermore, since the real part of λ is negative, it is not hard to prove that $|(1 + \lambda)/(1 - \lambda)| < 1$. Therefore, the spectral radius of the system matrix $(I - A)^{-1}(I + A)$ is less than one. Thus, the system of finite difference (2.5) is unconditionally stable. \square

Next, we discuss stability of the implicit finite difference method under the assumption of the function $k(x) = C$ for $x \in [0, L]$, where C is a positive constant.

Theorem 3.2. *If $1 \leq \alpha + \beta < 2$ and $k(x) = C$ for $x \in [0, L]$, where C is a positive constant, then CN method defined by (2.5) is unconditionally stable.*

Proof. Now $(1 - x)^\alpha = \sum_{k=0}^\infty g_{\alpha,k} x^k$, $(1 - x)^\beta = \sum_{k=0}^\infty g_{\beta,k} x^k$, and $(1 - x)^{\alpha+\beta} = \sum_{k=0}^\infty g_{\alpha+\beta,k} x^k$ for $-1 \leq x \leq 1$. Then we have

$$\left(\sum_{k=0}^\infty g_{\alpha,k} x^k\right)\left(\sum_{k=0}^\infty g_{\beta,k} x^k\right) = \sum_{k=0}^\infty g_{\alpha+\beta,k} x^k.$$

Comparing the coefficients on both sides of the above equation, we have

$$\sum_{k=0}^n g_{\alpha,k} g_{\beta,n-k} = g_{\alpha+\beta,n}, \quad n = 0, 1, \dots$$

With $k(x) = C$ for $x \in [0, L]$, therefore (2.6) can be rewritten as

$$A_{i,j} = \begin{cases} 0, & \text{when } j > i + 1, \\ rC, & \text{when } j = i + 1, \\ rCg_{\alpha+\beta,1}, & \text{when } j = i, \\ rCg_{\alpha+\beta,i-j+1}, & \text{when } j \leq i - 1. \end{cases}$$

Since $1 \leq \alpha + \beta < 2$, $g_{\alpha+\beta,0} = 1$, $g_{\alpha+\beta,1} = -(\alpha + \beta) < 0$, $g_{\alpha+\beta,n} > 0$ ($n \neq 1$), and $\sum_{k=0}^\infty g_{\alpha+\beta,k} = 0$, so $\sum_{l=0, l \neq i}^i g_{\alpha+\beta,l} < -g_{\alpha+\beta,i}$. By the Gerschgorin theorem, the eigenvalues of the matrix A lie in the disks centered at $A_{i,i} = rCg_{\alpha+\beta,i}$ with radius r_i ($r_i = \sum_{j=1, j \neq i}^{m-1} |A_{i,j}|$)

$$\sum_{j=1, j \neq i}^{m-1} |A_{i,j}| = \sum_{j=1, j \neq i}^{i-1} |rCg_{\alpha+\beta,i-j+1}| + rC = rC \sum_{l=0, l \neq 1}^i g_{\alpha+\beta,l} < -rCg_{\alpha+\beta,1} = -A_{i,i}.$$

Similar to the proof of Theorem 3.1, we can get the spectral radius of the system matrix $(I - A)^{-1}(I + A)$ is less than one. Thus, the system of finite difference (2.5) is unconditionally stable. \square

Remark 3.3. The fractional CN method was shown to be stable above. This method is consistent with order $O((\Delta t)^2) + O(\Delta x)$. Therefore, according to Lax’s equivalence Theorem [16], it converges at this rate. In paper [23], the Richardson extrapolated solution is $P_{t_n, x}$, then computed from $P_{t_n, x} = 2P_{t_n, x, (\Delta x)/2} - P_{t_n, x, \Delta x}$, where (x, t_n) is a common grid point, and $P_{t_n, x, (\Delta x)/2}$, $P_{t_n, x, \Delta x}$ denote the CN method solutions at the grid point (x, t_n) on the coarse grid (Δx) and the fine grid $(\Delta x/2)$, we can get second-order accurate both in time and space.

4. The ADI-CN method for the two-dimensional FPE

Consider a two-dimensional fractional percolation equation

$$\frac{\partial p}{\partial t} = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}(k_x(x, y) \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}) + \frac{\partial^{\beta_2}}{\partial x^{\beta_2}}(k_y(x, y) \frac{\partial^{\alpha_2} p}{\partial x^{\alpha_2}}) + f(x, y, t), \quad (x, y) \in \Omega,$$

subject to the initial condition

$$p(x, y, 0) = \phi(x, y), \quad (x, y) \in \Omega,$$

and the Dirichlet boundary conditions

$$p(a_1, y, 0) = p(x, b_1, t) = 0,$$

$$p(a_2, y, 0) = u(y, t), p(x, b_2, t) = v(x, t), 0 \leq t \leq T,$$

where $\Omega = \{(x, y) | a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\}$.

For the ADI-CN scheme, define $t_n = n\Delta t$ to be integration time $0 \leq t_n \leq T$, for $n = 0, 1, 2, \dots, N$ and $\Delta x = \frac{a_2 - a_1}{m_1}$ to be the grid size in x -direction, where m_1 is a positive integer, with $x_i = a_1 + i\Delta x$ for $i = 0, 1, \dots, m_1$. $\Delta y = \frac{b_2 - b_1}{m_2}$ is the grid size in y -direction, where m_2 is a positive integer, with $x_j = b_1 + j\Delta y$ for $i = 0, 1, \dots, m_2$. Define $p_{i,j}^n$ to be the numerical solution of $p(x_i, y_j, t_n)$. The initial conditions are set by $p_{i,j}^0 = \phi_{i,j} = \phi(x_i, y_j)$. Similarly, we define $k_{i,j} = k_x(x_i, y_j)$, $h_{i,j} = k_y(x_i, y_j)$, and $f_{i,j}^n = f(x_i, y_j, t_n)$. The Dirichlet boundary conditions are set by $p_{0,j}^n = p_{i,0}^n = 0$, $p_{m_1,j}^n = u(y_j, t_n)$, $p_{i,m_2}^n = v(x_i, t_n)$, $1 \leq i \leq m_1$, $1 \leq j \leq m_2$.

To approximate the mixed fractional spatial derivative into the differential equation centered at time $t_{n+1/2} = \frac{1}{2}(t_{n+1} + t_n)$, chen et al. [3] defined

$$\begin{aligned} \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}(k_x(x, y) \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}})|_{(x_i, y_j, t_{n+1/2})} &\sim \frac{1}{(\Delta x)^{\alpha_1 + \beta_1}} \sum_{v=1}^{i+1} \left(\sum_{u=0}^{i-v+1} g_{\beta_1, u} g_{\alpha_1, i-u-v+1} k_{i-u, j} \right) p_{v, j}^{n+1/2} + O(\Delta x), \\ \frac{\partial^{\beta_2}}{\partial x^{\beta_2}}(k_y(x, y) \frac{\partial^{\alpha_2} p}{\partial x^{\alpha_2}})|_{(x_i, y_j, t_{n+1/2})} &\sim \frac{1}{(\Delta y)^{\alpha_2 + \beta_2}} \sum_{v=1}^{j+1} \left(\sum_{u=0}^{j-v+1} g_{\beta_2, u} g_{\alpha_2, j-u-v+1} h_{i, j-u} \right) p_{i, v}^{n+1/2} + O(\Delta y). \end{aligned}$$

Then, we can get the difference equations

$$\begin{aligned} \frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} &= \frac{1}{(\Delta x)^{\alpha_1 + \beta_1}} \sum_{v=1}^{i+1} \left(\sum_{u=0}^{i-v+1} g_{\beta_1, u} g_{\alpha_1, i-u-v+1} k_{i-u, j} \right) p_{v, j}^{n+1/2} \\ &+ \frac{1}{(\Delta y)^{\alpha_2 + \beta_2}} \sum_{v=1}^{j+1} \left(\sum_{u=0}^{j-v+1} g_{\beta_2, u} g_{\alpha_2, j-u-v+1} h_{i, j-u} \right) p_{i, v}^{n+1/2} + f_{i, j}^{n+1/2}, \end{aligned} \tag{4.1}$$

$$p_{i,j}^0 = \phi(x_i, y_j), \tag{4.2}$$

$$p_{0,j}^n = p_{i,0}^n = 0, p_{m_1, j}^n = u(y_j, t_n), p_{i, m_2}^n = v(x_i, t_n). \tag{4.3}$$

This implies that the implicit finite difference scheme defined by (4.1)-(4.3) is consistent with order $O(\Delta t^2) + O(\Delta x) + O(\Delta y)$. Define the following finite difference operator

$$\begin{aligned} \delta_x p_{i,j}^n &= \frac{1}{(\Delta x)^{\alpha_1 + \beta_1}} \sum_{v=1}^{i+1} \left(\sum_{u=0}^{i-v+1} g_{\beta_1, u} g_{\alpha_1, i-u-v+1} k_{i-u, j} \right) p_{v, j}^n, \\ \delta_y p_{i,j}^n &= \frac{1}{(\Delta y)^{\alpha_2 + \beta_2}} \sum_{v=1}^{j+1} \left(\sum_{u=0}^{j-v+1} g_{\beta_2, u} g_{\alpha_2, j-u-v+1} h_{i, j-u} \right) p_{i, v}^n. \end{aligned}$$

Then (4.1) may be rearranged and written in the operator form

$$\left(1 - \frac{\Delta t}{2} \delta_x - \frac{\Delta t}{2} \delta_y\right) p_{i,j}^{n+1} = \left(1 + \frac{\Delta t}{2} \delta_x + \frac{\Delta t}{2} \delta_y\right) p_{i,j}^n + f_{i,j}^{n+\frac{1}{2}} \Delta t.$$

For the AID-CN method, the operator form is written in a directional separation product form

$$\left(1 - \frac{\Delta t}{2} \delta_x\right) \left(1 - \frac{\Delta t}{2} \delta_y\right) p_{i,j}^{n+1} = \left(1 + \frac{\Delta t}{2} \delta_x\right) \left(1 + \frac{\Delta t}{2} \delta_y\right) p_{i,j}^n + f_{i,j}^{n+\frac{1}{2}} \Delta t, \tag{4.4}$$

which introduces an additional perturbation error equal to

$$\frac{1}{4} \{ (\Delta t)^2 (\delta_x \delta_y) (p_{i,j}^{n+1} - p_{i,j}^n) \}.$$

Using Proposition 4.1 in [7], we can conclude that the ADI-CN method is also consistent with order $O(\Delta t^2) + O(\Delta x) + O(\Delta y)$. Eq. (4.4) can be written in the matrix form

$$(I - S)(I - T)P^{n+1} = (I + S)(I + T)P^n + F^{n+1/2}, \tag{4.5}$$

where the matrices A and B represent the operators $1 - \frac{\Delta t}{2}\delta_x$ and $1 - \frac{\Delta t}{2}\delta_y$, which are matrices of size $(m_1 - 1)(m_2 - 1) \times (m_1 - 1)(m_2 - 1)$, $F^{n+1/2}$ absorbs the source terms $f^{n+1/2}$ and the boundary conditions in the discretized equation, and

$$P^n = (p_{1,1}^n, p_{2,1}^n, \dots, p_{m_1-1,1}^n, p_{1,2}^n, p_{2,2}^n, \dots, p_{m_1-1,2}^n, \dots, p_{1,m_2-1}^n, p_{2,m_2-1}^n, \dots, p_{m_1-1,m_2-1}^n).$$

The ADI-CN method defined by (4.4) can now be solved by the following (Peaceman-Rachford type) set of equations:

$$(1 - \frac{\Delta t}{2}\delta_x)p_{i,j}^* = (1 + \frac{\Delta t}{2}\delta_y)p_{i,j}^n + \frac{\Delta t}{2}f_{i,j}^{n+\frac{1}{2}}\Delta t, \tag{4.6}$$

$$(1 - \frac{\Delta t}{2}\delta_y)p_{i,j}^{n+1} = (1 + \frac{\Delta t}{2}\delta_x)p_{i,j}^* + \frac{\Delta t}{2}f_{i,j}^{n+\frac{1}{2}}\Delta t. \tag{4.7}$$

The intermediate solution $p_{i,j}^*$ should be defined carefully on the boundary, prior to solve the system of equations defined by (4.6) and (4.7). Otherwise, the first-order spatial accuracy of the two-step ADI method outlined above will be impacted. This is accomplished by subtracting (4.6) from (4.7) to get the following equation to define $p_{i,j}^*$

$$2p_{i,j}^* = (1 - \frac{\Delta t}{2}\delta_y)p_{i,j}^{n+1} + (1 + \frac{\Delta t}{2}\delta_y)p_{i,j}^n. \tag{4.8}$$

Thus, the boundary conditions for $p_{i,j}^*$ (i.e., $i = 0$ or m_1 for $j = 1, \dots, m_2 - 1$) needed to solve each set of equations in (4.8) are set from

$$p_{0,j}^* = (1 - \frac{\Delta t}{2}\delta_y)p_{0,j}^{n+1} + (1 + \frac{\Delta t}{2}\delta_y)p_{0,j}^n = 0,$$

$$p_{m_1,j}^* = (1 - \frac{\Delta t}{2}\delta_y)p_{m_1,j}^{n+1} + (1 + \frac{\Delta t}{2}\delta_y)p_{m_1,j}^n = (1 - \frac{\Delta t}{2}\delta_y)u_{m_1,j}^{n+1} + (1 + \frac{\Delta t}{2}\delta_y)u_{m_1,j}^n.$$

The corresponding algorithm is implemented as follows:

First, fixed horizontal slice y_l ($l = 1, \dots, m_2 - 1$), solve a set of $m_1 - 1$ equations at the points x_i to obtain an intermediate solution slice $p_{i,l}^*$.

Second, fixed vertical slice x_l ($l = 1, \dots, m_1 - 1$), then solve a set of $m_2 - 1$ equations at the points y_j to obtain the solution slice $p_{l,j}^{n+1}$.

According to the first step gives a set of $m_1 - 1$ linear equations, the system of the equations may be written as

$$(I - A_l)P_l^* = Q_l^n + \frac{\Delta t}{2}F_l^{n+\frac{1}{2}}, \tag{4.9}$$

where

$$P_l^* = [P_{1,l}^*, P_{2,l}^*, \dots, P_{m_1-1,l}^*],$$

$$Q_l^n = [\sum_{v=1}^{l+1} B_{1,v}^n P_{1,v}^n, \sum_{v=1}^{l+1} B_{2,v}^n P_{2,v}^n, \dots, \sum_{v=1}^{l+1} B_{m_1-1,v}^n P_{m_1-1,v}^n],$$

$$F_l^{n+\frac{1}{2}} = [f_{1,l}^{n+\frac{1}{2}}, f_{2,l}^{n+\frac{1}{2}}, \dots, f_{m_1-2,l}^{n+\frac{1}{2}}, f_{m_1-1,l}^{n+\frac{1}{2}} - A_{m_1-1,m_1} P_{m_1,l}^*],$$

and $A_l = [A_{i,v}]$ is the $(m_1 - 1) \times (m_1 - 1)$ matrix of the coefficients resulting from a set of equations at the grid point y_l , therefore the matrix entries $A_{i,v}$ are defined by

$$A_{i,v} = \begin{cases} 0, & \text{when } v > i + 1, \\ r_1 k_{i,l}, & \text{when } v = i + 1, \\ r_1 \sum_{u=0}^1 g_{\beta_1,u} g_{\alpha_1,-u+1} k_{i-u,l}, & \text{when } v = i, \\ r_1 \sum_{u=0}^{i-v+1} g_{\beta_1,u} g_{\alpha_1,i-v-u+1} k_{i-u,l}, & \text{when } v \leq i - 1, \end{cases} \quad (4.10)$$

and the coefficients $B_{i,v}$ for $i = 1, \dots, m_1$ are defined by:

$$B_{i,v} = \begin{cases} r_2 h_{i,l}, & \text{when } v = l + 1, \\ 1 + r_2 \sum_{u=0}^1 g_{\beta_2,u} g_{\alpha_2,-u+1} h_{i,l-u}, & \text{when } v = l, \\ r_2 \sum_{u=0}^{l-v+1} g_{\beta_2,u} g_{\alpha_2,l-v-u+1} h_{i,l-u}, & \text{when } v \leq l - 1, \end{cases}$$

where $r_1 = \frac{\Delta t}{2(\Delta x)^{\alpha_1+\beta_1}}$; $r_2 = \frac{\Delta t}{2(\Delta y)^{\alpha_2+\beta_2}}$.

Similarly, according to the second step gives a set of $m_2 - 1$ linear equations, the system of the equations may be written as

$$(I - \hat{B}_l) P_l^{n+1} = O_l^* + \frac{\Delta t}{2} \hat{F}_l^{n+\frac{1}{2}},$$

where

$$\begin{aligned} P_l^{n+1} &= [P_{l,1}^{n+1}, P_{l,2}^{n+1}, \dots, P_{l,m_2-1}^{n+1}], \\ O_l^* &= [\sum_{v=1}^{l+1} \hat{A}_{1,v}^n P_{v,1}^*, \sum_{v=1}^{l+1} \hat{A}_{2,v}^n P_{v,2}^*, \dots, \sum_{v=1}^{l+1} \hat{A}_{m_2-1,v}^n P_{v,m_2-1}^*], \\ \hat{F}_l^{n+\frac{1}{2}} &= [f_{l,1}^{n+\frac{1}{2}}, f_{l,2}^{n+\frac{1}{2}}, \dots, f_{l,m_2-2}^{n+\frac{1}{2}}, f_{l,m_2-1}^{n+\frac{1}{2}} - \hat{B}_{m_2-1,m_2} P_{l,m_2-1}^*], \end{aligned}$$

and $\hat{B}_l = [\hat{B}_{j,v}]$ is the $(m_2 - 1) \times (m_2 - 1)$ matrix of the coefficients resulting from a set of equations at the grid point x_l , therefore the matrix entries $\hat{B}_{j,v}$ are defined by

$$\hat{B}_{j,v} = \begin{cases} 0, & \text{when } v > j + 1, \\ r_2 h_{l,j}, & \text{when } v = j + 1, \\ r_2 \sum_{u=0}^1 g_{\beta_2,u} g_{\alpha_2,-u+1} h_{l,j-u}, & \text{when } v = j, \\ r_2 \sum_{u=0}^{j-v+1} g_{\beta_2,u} g_{\alpha_2,j-v-u+1} h_{l,j-u}, & \text{when } v \leq j - 1, \end{cases} \quad (4.11)$$

and the coefficients $\hat{A}_{j,v}$ for $j = 1, \dots, m_2$ are defined by:

$$\hat{A}_{j,v} = \begin{cases} r_1 k_{l,j}, & \text{when } v = l + 1, \\ 1 + r_1 \sum_{u=0}^1 g_{\beta_1,u} g_{\alpha_1,-u+1} k_{l-u,j}, & \text{when } v = l, \\ r_1 \sum_{u=0}^{l-v+1} g_{\beta_1,u} g_{\alpha_1,l-v-u+1} k_{l-u,j}, & \text{when } v \leq l - 1. \end{cases}$$

5. Stability and convergence of the fractional ADI-CN method

Having developed a numerical scheme and showing that it is consistent, we now show that it is also stable and thus convergent. First, we establish stability of the ADI-CN method under the assumption of continuity of seepage flow ($\beta_1 = \beta_2 = 1$).

Theorem 5.1. *If $\beta_1 = \beta_2 = 1$, $k_x(x, y)$ decreases monotonically with respect to x , $k_y(x, y)$ decreases monotonically with respect to y in the domain Ω , and the matrices A and B commute, then the ADI-CN method defined by (4.5) is unconditionally stable.*

Proof. By definition, for $\beta_1 = \beta_2 = 1$, we have $g_{\beta_i,0} = 1$, $g_{\beta_i,1} = -1$, and $g_{\beta_i,n} = 0$ ($n \geq 2$), for $i = 1, 2$. Hence (4.10) and (4.11) can be rewritten as

$$A_{i,v} = \begin{cases} 0, & \text{when } v > i + 1, \\ r_1 k_{i,l}, & \text{when } v = i + 1, \\ -r_1[\alpha_1 k_{i,l} + k_{i-1,l}], & \text{when } v = i, \\ r_1[g_{\alpha_1,i-v+1} k_{i,l} + g_{\alpha_1,i-v} k_{i-1,l}], & \text{when } v \leq i - 1, \end{cases} \tag{5.1}$$

$$\hat{B}_{j,v} = \begin{cases} 0 & \text{when } v > j + 1, \\ r_2 h_{l,j}, & \text{when } v = j + 1, \\ -r_2[\alpha_2 h_{l,j} + h_{l,j-1}], & \text{when } v = j, \\ r_2[g_{\alpha_2,j-v+1} h_{l,j} + g_{\alpha_2,j-v} h_{l,j-1}], & \text{when } v \leq j - 1. \end{cases} \tag{5.2}$$

Since $0 < \alpha_l < 1$, so $g_{\alpha_l,0} = 1, g_{\alpha_l,1} = -\alpha_l$, and $\sum_{k=0}^{\infty} g_{\alpha_l,k} = 0$. Now $g_{\alpha_l,k} = (1 - \frac{\alpha_l+1}{k} g_{\alpha_l,k-1})$ so $g_{\alpha_l,k-1} < g_{\alpha_l,k} < 0$ for $k \geq 2, l = 1, 2$. Since $k_x(x, y)$ and $k_y(x, y)$ decrease monotonically in the domain Ω , $0 < k_{i,u} < k_{i-1,u}$ ($i = 1, 2, \dots, m_1 - 1$) and $0 < h_{j,u} < h_{j-1,u}$ ($j = 1, 2, \dots, m_2 - 1$). Therefore $g_{\alpha_l,i-j+1} s_{l,u} - g_{\alpha_l,i-j} s_{i-1,u} > 0$ for $j < i, l = 1, 2, s_{l,u} = k_{l,u}$ or $h_{l,u}$. Since

$$\begin{aligned} \sum_{v=1, j \neq i}^{m_1-1} |A_{i,v}| &= r_1 \sum_{v=1}^{i-1} |g_{\alpha_1,i-v+1} k_{i,l} - g_{\alpha_1,i-v} k_{i-1,l}| + r_1 k_{i,l} \\ &= r_1 k_{i,l} \sum_{v=1}^{i-1} g_{\alpha_1,i-v+1} - r_1 k_{i-1,l} \sum_{v=1}^{i-1} g_{\alpha_1,i-v} + r_1 k_{i,l} \\ &= r_1 k_{i,l} \sum_{s=0, s \neq 1}^i g_{\alpha_1,s} - r_1 k_{i-1,l} \sum_{s=1}^{i-1} g_{\alpha_1,s} \\ &= r_1 k_{i,l} (-g_{\alpha_1,1} - \sum_{s=i+1}^{\infty} g_{\alpha_1,s}) - r_1 k_{i-1,l} \sum_{s=1}^{i-1} g_{\alpha_1,s} \\ &= r_1 k_{i,l} \alpha_1 - r_1 k_{i,l} \sum_{s=i+1}^{\infty} g_{\alpha_1,s} - r_1 k_{i-1,l} \sum_{s=1}^{i-1} g_{\alpha_1,s} \\ &< r_1 k_{i,l} \alpha_1 - r_1 k_{i-1,l} \sum_{s=i+1}^{\infty} g_{\alpha_1,s} - r_1 k_{i-1,l} \sum_{s=1}^{i-1} g_{\alpha_1,s} \\ &= r_1 k_{i,l} \alpha_1 - r_1 k_{i-1,l} \sum_{s=1, s \neq i}^{\infty} g_{\alpha_1,s} < r_1 k_{i,l} \alpha_1 + r_1 k_{i-1,l} = -A_{i,i}. \end{aligned}$$

According to the Greschgorin theorem, the eigenvalues of the matrix A_l lie in the union of the disks centered at $A_{i,i}$ with the radius $\sum_{v=1, j \neq i}^{m_1-1} |A_{i,v}|$, therefore, the eigenvalues of the matrix A_l have negative real-parts. Similarly, the eigenvalues of the matrix \hat{B}_l have negative real-parts.

Next, the matrix S is a block diagonal matrix of $(m_2 - 1) \times (m_2 - 1)$ blocks whose blocks are the square $(m_1 - 1) \times (m_1 - 1)$ super-triangular A_l matrices resulting from (5.1). We may write

$$S = \text{diag}(A_1, A_2, \dots, A_{m_2-1}).$$

The matrix T is a block super-triangular matrix of $(m_2 - 1) \times (m_2 - 1)$ blocks whose blocks are the square $(m_1 - 1) \times (m_1 - 1)$ diagonal matrices resulting from (5.2). We may write $T = [T_{j,v}]$, where each $T_{j,v}$ is a $(m_1 - 1) \times (m_1 - 1)$ matrix, such that for $v > j + 1, T_{j,v} = 0$, and for $v \leq j + 1$ each $T_{j,v}$ is a diagonal matrix

$T_{j,v} = \text{diag}((\hat{B}_1)_{j,v}, (\hat{B}_2)_{j,v}, \dots, (\hat{B}_{m_1-1})_{j,v})$, where the notation $(\hat{B}_l)_{j,v}$ refers to the j, v th entry of the matrix (\hat{B}_l) define previously.

Since $S = \text{diag}(A_1, A_2, \dots, A_{m_2-1})$, the eigenvalues of the matrix S are in the union of the Gerschgorin disks for the matrices A_l 's, therefore every eigenvalue of the matrix S has a negative real-part. Similarly, every eigenvalue of the matrix T has a negative real-part.

If λ_1, λ_2 are eigenvalues of matrices S and T , respectively, we can obtain $(1 + \lambda_1)/(1 - \lambda_1)$ and $(1 + \lambda_2)/(1 - \lambda_2)$ are eigenvalues of the matrices $(I - S)^{-1}(I + S)$ and $(I - T)^{-1}(I + T)$, respectively. We observe that the first part of this statement implies that all the eigenvalues of the matrix $(I - A)$ have a magnitude larger than 1, and thus this matrix is invertible. Furthermore, since the real part of λ_1 is negative, it is not hard to check that $|(1 + \lambda_1)/(1 - \lambda_1)| < 1$. Therefore, the spectral radius of matrix $(I - S)^{-1}(I + S)$ is less than one. Similarly the spectral radius of matrix $(I - B)^{-1}(I + B)$ is less than one. Next, the matrices S and T commute, we know the matrices $(I - T)^{-1}, (I + T)^{-1}, I + S$, and $I + T$ commute, so we can obtain $(1 + \lambda_1)/(1 - \lambda_1)(1 + \lambda_2)/(1 - \lambda_2)$ is an eigenvalue of the matrix $(I - A)^{-1}(I + A)(I - B)^{-1}(I + B)$, and the system matrix of $(I - T)^{-1}(I - S)^{-1}(I + S)(I + T) = (I - S)^{-1}(I + S)(I - T)^{-1}(I + T)$, thus the spectral radius of matrix $(I - T)^{-1}(I - S)^{-1}(I + S)(I + T)$ is less than one, then ADI-CN method defined by (4.9) is unconditionally stable. \square

Next, we discuss stability of ADI-CN method under the assumption of the function $k_x(x, y) = C_1$ and $k_y(x, y) = C_2$ for $(x, y) \in \Omega$ where C_1 and C_2 are positive constants.

Theorem 5.2. *If $1 \leq \alpha_l + \beta_l < 2$ for $l = 1, 2$ and $k_x(x, y) = C_1$ and $k_y(x, y) = C_2$ for $(x, y) \in \Omega$, where C is a positive constant and the matrices A and B commute, then ADI-CN method defined by (4.5) is unconditionally stable.*

Proof. For $l = 1, 2, (1 - x)^{\alpha_l} = \sum_{k=0}^{\infty} g_{\alpha_l, k} x^k, (1 - x)^{\beta_l} = \sum_{k=0}^{\infty} g_{\beta_l, k} x^k$, and $(1 - x)^{\alpha_l + \beta_l} = \sum_{k=0}^{\infty} g_{\alpha_l + \beta_l, k} x^k$ for $-1 \leq x \leq 1$. Then we have

$$\left(\sum_{k=0}^{\infty} g_{\alpha_l, k} x^k\right) \left(\sum_{k=0}^{\infty} g_{\beta_l, k} x^k\right) = \sum_{k=0}^{\infty} g_{\alpha_l + \beta_l, k} x^k.$$

Comparing the coefficients on both sides of the above equation, we have

$$\sum_{k=0}^n g_{\alpha_l, k} g_{\beta_l, n-k} = g_{\alpha_l + \beta_l, n}, n = 0, 1, \dots.$$

With $k_x(x, y) = C_1$ for $(x, y) \in \Omega$, therefore (4.10) and (4.11) can be rewritten as

$$A_{i,v} = \begin{cases} 0, & \text{when } v > i + 1, \\ r_1 C_1, & \text{when } v = i + 1, \\ r_1 C_1 g_{\alpha_1 + \beta_1, 1}, & \text{when } v = i, \\ r_1 C_1 g_{\alpha_1 + \beta_1, i-v+1}, & \text{when } v \leq i - 1, \end{cases}$$

$$\hat{B}_{j,v} = \begin{cases} 0, & \text{when } v > j + 1, \\ r_2 C_2, & \text{when } v = j + 1, \\ r_2 C_2 g_{\alpha_2 + \beta_2, 1}, & \text{when } v = j, \\ r_2 C_2 g_{\alpha_2 + \beta_2, j-v+1}, & \text{when } v \leq j - 1. \end{cases}$$

Since $l = 1, 2, 1 \leq \alpha_l + \beta_l < 2, g_{\alpha_l + \beta_l, 0} = 1, g_{\alpha_l + \beta_l, 1} = -(\alpha_l + \beta_l) < 0, g_{\alpha_l + \beta_l, n} > 0 (n \neq 1)$, and $\sum_{k=0}^{\infty} g_{\alpha_l + \beta_l, k} = 0$, so $\sum_{l=0, l \neq i}^i g_{\alpha_l + \beta_l, l} < -g_{\alpha_l + \beta_l, i}$. Since

$$\sum_{v=1, j \neq m_1-1}^{m-1} |A_{i,j}| = \sum_{v=1, v \neq i}^{i-1} |r_1 C_1 g_{\alpha_1 + \beta_1, i-j+1}| + r_1 C_1 = r_1 C_1 \sum_{l=0, l \neq 1}^i g_{\alpha_1 + \beta_1, l} < -r_1 C_1 g_{\alpha_1 + \beta_1, 1} = A_{i,i}.$$

According to the Gerschgorin theorem, the eigenvalues of the matrix A_l lie in the union of the disks centered at $A_{i,i}$ with the radius $\sum_{v=1, j \neq i}^{m_l-1} |A_{i,v}|$, therefore the eigenvalues of the matrix A_l have negative real-parts. Similarly, the eigenvalues of the matrix \hat{B}_l have negative real-parts. Similar to the proof of Theorem 5.1, the spectral radius of matrix $(I - B)^{-1}(I - A)^{-1}(I + A)(I + B)$ is less than one, thus ADI-CN method defined by (4.9) is unconditionally stable. \square

Remark 5.3. The ADI-CN method is both consistent and unconditionally stable. Therefore, according to Lax's equivalence theorem, it converges at this rate. The Richardson extrapolated [22] solution is $P_{t_n, x}$, then computed from $P_{t_n, x, y} = 2P_{t_n, x, (\Delta x)/2, y, (\Delta y)/2} - P_{t_n, x, \Delta x, y, \Delta y}$, where (x, y) is a common grid point, and $P_{t_n, x, (\Delta x)/2, y, (\Delta y)/2}, P_{t_n, x, \Delta x, y, \Delta y}$ denote the ADI-CN method solutions at the grid point (x, y) on the coarse grid $(\Delta x, \Delta y)$ and the fine grid $(\Delta x/2, \Delta y/2)$, we can get second-order accurate both in time and space.

6. Numerical examples

Example 6.1. The following fractional differential equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(k_x \frac{\partial^\alpha p(x, t)}{\partial x^\alpha} \right) + f(x, t),$$

was considered on a finite domain $0 < x < L$, for $0 < t \leq T_{\text{end}}$, where

$$\alpha = 0.5, L = 1, k(x) = 30 - x^2, f(x, t) = -x^2 e^{-t} - \frac{\Gamma(3)}{\Gamma(2.5)} (45x^{0.5} - 3.5x^{2.5}).$$

Subject to the initial condition

$$p(x, 0) = x^2, 0 < x < L,$$

and the Dirichlet boundary conditions

$$p(0, t) = 0, p(1, t) = e^{-t}, t > 0,$$

the exact solution to this equation is given by

$$p(x, t) = x^2 e^{-t}.$$

In this example, Fig. 1 shows the (unextrapolated) numerical solution obtained by applying the fractional CN method (2.5) discussed above, with $\Delta t = 1/10$ and $\Delta x = 1/10$, at time $T = 1$, which compares well with the exact analytic solution to the FPE in this test case. Table 1 shows the maximum error and error rate of Example 6.1 for CN and CN extrapolated solution of the one-dimensional fractional differential equation ($\beta = 1$). It examines the rate of convergence for these methods.

Table 1: Maximum error and error rate of Example 6.1 for CN and extrapolated CN method when the grid size is reduced at time $T = 1$.

Δt	Δx	Max Error-CN	Error rate	Max Error-Ext CN	Error rate
1/10	1/10	5.2298×10^{-3}	—	2.4950×10^{-3}	-
1/20	1/20	2.4841×10^{-3}	$2.1053 \approx 2$	6.3376×10^{-4}	3.9369
1/40	1/40	1.2357×10^{-3}	$2.0103 \approx 2$	1.5969×10^{-4}	3.9688
1/80	1/80	5.7968×10^{-4}	$2.1317 \approx 2$	4.0199×10^{-5}	3.9724

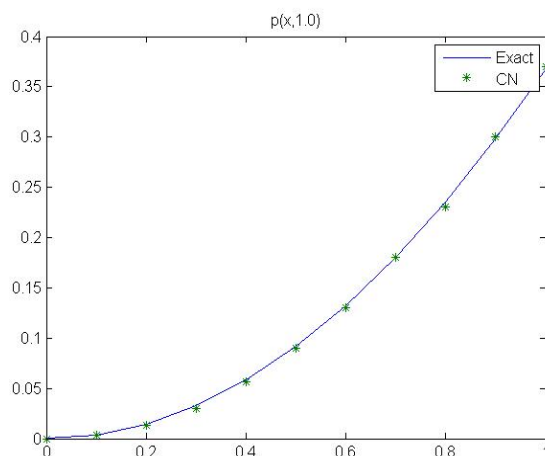


Figure 1: Comparison of exact solution to the CN solution at time $T_{\text{end}} = 1.0$.

Example 6.2. The following fractional differential equation

$$\frac{\partial p}{\partial t} = \frac{\partial^\beta}{\partial x^\beta} \left(\frac{\partial^\alpha p(x, t)}{\partial x^\alpha} \right) + f(x, t), \quad 0 < x < L, \quad 0 < t \leq T,$$

was considered on a finite domain $0 < x < L$, for $0 < t \leq T_{\text{end}}$, where $\alpha = 0.9$, $\beta = 0.8$, $L = 1$, $f(x, t) = -x^2 e^{-t} - \frac{\Gamma(3)}{\Gamma(1.3)} x^{0.3} e^{-t}$. Subject to the initial condition

$$p(x, 0) = x^2, \quad 0 < x < L,$$

and the Dirichlet boundary conditions

$$p(0, t) = 0, \quad p(1, t) = e^{-t}, \quad t > 0,$$

the exact solution to this equation is given by

$$p(x, t) = x^2 e^{-t}.$$

In Fig. 2 we compute up at time $T = 2$ by setting $\Delta t = 1/10$, $\Delta x = 1/10$ and using (2.5) with $\alpha = 0.9$ and $\beta = 0.8$. We denote the results of the numerical scheme and the solid curve corresponding to the exact analytical solution for the one-dimensional fractional differential equation ($\beta = 1$). Table 2 shows the maximum error and error rate of Example 6.2 for CN and CN extrapolated solution of the one-dimensional fractional differential equation ($k(x) = 1$). It is proved that the numerical method is effective.

Table 2: Maximum error and error rate of Example 6.2 for C-N and extrapolated C-N method when the grid size is reduced at time $T = 2$.

Δt	Δx	Max Error-CN	Error rate	Max Error-Ext CN	Error rate
1/10	1/10	4.2039×10^{-3}	—	8.5388×10^{-4}	—
1/20	1/20	2.0517×10^{-3}	$2.049 \approx 2$	2.1631×10^{-4}	3.9475
1/40	1/40	9.6148×10^{-4}	$2.1339 \approx 2$	5.4639×10^{-5}	3.9589
1/80	1/80	4.1159×10^{-4}	$2.3360 \approx 2$	1.3719×10^{-5}	3.9827

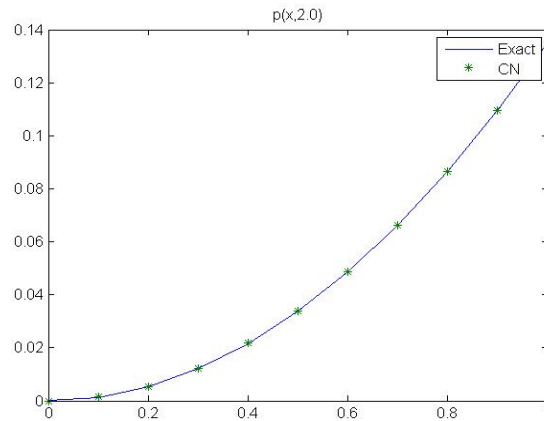


Figure 2: Comparison of exact solution to the CN solution at time $T_{\text{end}} = 2.0$.

Example 6.3. The two-dimensional fractional percolation equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(k_x \frac{\partial^{0.7} p(x, y, t)}{\partial x^{0.7}} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial^{0.8} p(x, y, t)}{\partial y^{0.8}} \right) + f(x, y, t),$$

was considered on a finite rectangular domain $0 < x < 1, 0 < y < 1$, for $0 < t \leq T_{\text{end}}$, where the diffusion coefficients are

$$k_x = 2 - x, \quad k_y = 2 - y,$$

and the source function is

$$f(x, y, t) = -\frac{\Gamma(2)}{\Gamma(1.3)} y e^{-t} \left[\frac{2\Gamma(1.3)}{\Gamma(0.3)} x^{-0.7} - \frac{\Gamma(2.3)}{\Gamma(1.3)} x^{0.3} \right] - \frac{\Gamma(2)}{\Gamma(1.2)} x e^{-t} \left[\frac{2\Gamma(1.2)}{\Gamma(0.2)} y^{-0.8} - \frac{\Gamma(2.2)}{\Gamma(1.2)} x^{0.2} \right],$$

subject to the initial condition

$$p(x, y, 0) = e^{-t} xy,$$

and the Dirichlet boundary conditions

$$p(0, y, t) = p(x, 0, t) = 0, \quad p(1, y, t) = e^{-t} y^2, \quad p(x, 1, t) = e^{-t} x^2.$$

The exact solution to this two-dimensional fractional percolation equation is given by

$$p(x, y, t) = e^{-t} xy.$$

Table 3 shows the maximum error and error rate of Example 6.3 for CN and CN extrapolated solution of the two-dimensional fractional differential equation ($\beta_1 = \beta_2 = 1$). The late column in the Table 3 shows that the maximum error in the extrapolated ADI-CN method approximately decreases by a factor of four as the grid size is halved. It is proved that the numerical method is effective.

Table 3: Maximum error for the Example 6.3 at time $T_{\text{end}} = 1$.

Δt	$\Delta x = \Delta y$	Max Error of ADI-CN	Max Error of Ext ADI-CN
1/5	1/5	7.6714×10^{-3}	2.3863×10^{-3}
1/10	1/10	3.8642×10^{-3}	5.0112×10^{-4}
1/20	1/20	1.9451×10^{-3}	1.2615×10^{-4}
1/40	1/40	9.9824×10^{-4}	3.1513×10^{-5}

Example 6.4. The two-dimensional fractional percolation equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial^{0.7} p(x, y, t)}{\partial x^{0.7}} \right) + \frac{\partial}{\partial y} \left(\frac{\partial^{0.8} p(x, y, t)}{\partial x^{0.8}} \right) + f(x, y, t),$$

was considered on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 < t \leq T_{\text{end}}$, and the source function is

$$f(x, y, t) = -\frac{\Gamma(2)}{\Gamma(1.3)} y e^{-t} \left[\frac{2\Gamma(1.3)}{\Gamma(0.3)} x^{-0.7} - \frac{\Gamma(2.3)}{\Gamma(1.3)} x^{0.3} \right] - \frac{\Gamma(2)}{\Gamma(1.2)} x e^{-t} \left[\frac{2\Gamma(1.2)}{\Gamma(0.2)} y^{-0.8} - \frac{\Gamma(2.2)}{\Gamma(1.2)} x^{0.2} \right],$$

subject to the initial condition

$$p(x, y, 0) = e^{-t} xy,$$

and the Dirichlet boundary conditions

$$p(0, y, t) = p(x, 0, t) = 0, p(1, y, t) = e^{-t} y^2, p(x, 1, t) = e^{-t} x^2.$$

The exact solution to this two-dimensional fractional percolation equation is given by

$$p(x, y, t) = e^{-t} xy.$$

Table 4 shows the maximum error and error rate of Example 6.4 for CN and CN extrapolated solution of the two-dimensional fractional differential equation ($k_x = k_y = 1$). The late column in Table 4 shows that the maximum error in the extrapolated ADI-CN method approximately decreases by a factor of four as the grid size is halved. It is proved that the numerical method is effective.

Table 4: Maximum error for the Example 6.4 at time $T_{\text{end}} = 1$.

Δt	$\Delta x = \Delta y$	Max Error of ADI-CN	Max Error of Ext ADI-CN
1/5	1/5	3.5035×10^{-3}	8.5041×10^{-4}
1/10	1/10	1.7344×10^{-3}	2.1207×10^{-4}
1/20	1/20	8.2471×10^{-4}	5.6089×10^{-5}
1/40	1/40	4.0981×10^{-4}	1.3614×10^{-5}

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