Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 4162–4174
Research Article
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Some applications with new admissibility contractions in b-metric spaces

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Communicated by W. Shatanawi

Abstract

The work presented in this paper extends the idea of α-β-contractive mappings in the framework of b-metric spaces. Fixed points are investigated for such kind of mappings. An example is given to show the superiority of our results. As applications we discuss Ulam-Hyres stability, well-posedness and limit shadowing of fixed point problem. ©2017 All rights reserved.

Keywords: α-β(b)-admissible mappings, fixed point, b-metric space, stability.
2010 MSC: 47H10, 54H25.

1. Introduction

Metric spaces have been generalized according to requirement and their applicability to solve a particular problem. The problem of convergence of measurable functions with respect to measure leaded to generalize the metric space in such a way that set considered in metric space is replaced with the space and consequently the function 'd' of metric space is replaced with the functional 'd'. The metric space defined in the above is called b-metric space. In 1993, Czerwik introduced and proved contraction mapping principle in b-metric space. Following this pioneer paper, several authors have devoted their attention to research the properties of a b-metric space and have reported the existence and uniqueness of fixed points of various operators in the setting of b-metric spaces. One can go through references [10, 18, 22–24] to have a better understanding of advances in b-metric space.

A classical question in the theory of functional equations is the following:

"When is it true that a function which approximately satisfies a functional equation E must be close to an exact solution of E?"

If the problem accepts a solution, we say that the equation E is stable.

The stability problem of functional equations, originated from a question of Ulam [33], in 1940, concerns the stability of group homomorphisms. In 1941, Hyers [14] made an attempt to answer to the question of Ulam for Banach spaces. Thereafter, this type of stability is called the Ulam-Hyers stability and has attracted attention of several authors. Later on, many fixed point results were checked for the stability of certain classes of functional equations. Bota et al. in [2, 3] Brzdek et al. in [4, 5], Cadariu [6],

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doi:10.22436/jnsa.010.08.12

Received 2017-04-02

Lazar [18] and many other authors presented a wide range of stability problems in the context of various metric spaces to generalize Ulam-Hyers stability.

The concept of well-posedness of a fixed point problems and limit shadowing property of fixed points are generalized by various authors for single as well as multi-valued mappings. One can go through the results of Blassi and Myjak [9] Reich and Zaslavski [25], Lahiri and Das [17], and Popa [20, 21].

In 2012, Samet et al. [28] introduced the notions of α-ψ-contractive and α-admissible self-mappings and proved some fixed-point results for these mappings in complete metric spaces. In 2013, Bota et al. [2] established the existence of fixed point theorems for α-ψ-contractive mapping of type-(b) in the framework of b-metric spaces. They applied these results to prove the Ulam-Hyers stability for fixed points. Recently, Felhi et al. [12] established the stability of Ulam-Hyers and well-posedness for fixed point problems for α-λ-contractions on quasi b-metric spaces.

With this work we have two intentions. Firstly to introduce the notion of α-β-contraction in the settings of b-metric spaces and proves fixed point results for these new classes of mappings. Secondly, to verify Ulam-Hyres stability, well-posedness and limit shadowing of fixed points. Our efforts generalize and extend the results of Sintunavarat for α-β-contraction mappings and Ulam-Hyres stability in metric spaces.

2. Preliminaries

In this section, we will go through the prelims which would lead to understand the main results clearly.

Definition 2.1 ([1, 8])**.** Let X be a (non-empty) space, and let $s \geqslant 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$,

for all x, y, and $z \in X$.

If d satisfies all the above b-metric axioms then the pair (X, d) is called a b-metric or metric type space.

Definition 2.2 ([8])**.** Let (X, d) be a b-metric space.

(a) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.
(c) The b-metric space (X, d) is called b-complete if every b-Cauchy sequence in X is b-convergent.

The following lemma is very useful to prove our main result.

Lemma 2.3 ([26])**.** *Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences

$$d(x_{m(k)}, x_{n(k)}), \quad d(x_{m(k)}, x_{n(k)+1}), \quad d(x_{m(k)+1}, x_{n(k)}), \quad d(x_{m(k)+1}, x_{n(k)+1}),$$

the following hold:

$$\varepsilon \leqslant \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leqslant \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leqslant \varepsilon s, \tag{2.1}$$

$$\frac{\varepsilon}{s} \leqslant \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leqslant \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leqslant \varepsilon s^2,$$
$$\frac{\varepsilon}{s} \leqslant \liminf_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leqslant \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leqslant \varepsilon s^2,$$
$$\frac{\varepsilon}{s^2} \leqslant \liminf_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leqslant \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leqslant \varepsilon s^3.$$

A new class of β functions is firstly introduced by Geraghty [13] in 1973. The definition is as follows.

Definition 2.4. Let B denote the class of real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0.$$

An example of a function in B may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$.

In order to generalize the Banach contraction principle, Geraghty proved in 1973 the very important theorem stated as follows.

Theorem 2.5 ([13])**.** *Let (X, d) be a complete metric space, and let $F : X \to X$ be a self-map. Suppose that there exists $\beta \in B$ such that*

$$d(Fx, Fy) \leqslant \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then F has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{F^n x\}$ converges to z when $n \to \infty$.

In 2011, Dukic et al. introduced the Geraghty-type β functions in b-metric space as follows.

Definition 2.6 ([10])**.** Let (X, d) be a b-metric space with the given $s > 1$ we will consider the class of functions B_s, where $\beta \in B_s$ if $\beta : [0, +\infty) \to [0, \frac{1}{s})$ and has the property

$$\beta(t_n) \to \frac{1}{s} \quad \text{implies} \quad t_n \to 0.$$

An example of a function in B_s is given by $\beta(t) = \frac{1}{s}e^{-t}$ for $t > 0$ and $\beta(0) \in [0, \frac{1}{s})$.

In 2012, Samet et al. introduced the α-admissible self-mappings as follows.

Definition 2.7 ([28])**.** A self-mapping $F : X \to X$ defined on a nonempty set X is α-admissible if for all $x, y \in X$, one has

$$\alpha(x, y) \geqslant 1 \quad \Rightarrow \quad \alpha(Fx, Fy) \geqslant 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function under consideration.

Example 2.8. Let $X = [0, \infty)$ and assume $F : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Fx = \sqrt{x}$, for all $x \in X$,

$$\alpha(x, y) = \begin{cases} e^{x-y}, & x \geqslant y, \\ 0, & x < y. \end{cases}$$

Then F is an α-admissible mapping.

Definition 2.9 ([30])**.** Let X be a nonempty set. Then the map $\alpha : X \times X \to [0, \infty)$ is called transitive if for $u, v, w \in X$ we have

$$\alpha(u, v) \geqslant 1, \ \alpha(v, w) \geqslant 1 \quad \Rightarrow \quad \alpha(u, w) \geqslant 1.$$

In 2014, Sintunvarat defined the generalized α-β-contraction mapping in metric spaces as follows.

Definition 2.10 ([30]). Let F be a self-mapping on a nonempty set X and there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\beta \in B$. We say that F is α-β-contraction mapping if the following condition holds:

$$[\alpha(x, y) - 1 + \delta_*]^{d(Fx,Fy)} \leqslant \delta^{\beta(d(x,y))d(x,y)}$$

for all $x, y \in X$, where $1 < \delta \leqslant \delta_*$.

Now, we will introduce our notions in the context of b-metric space.

Definition 2.11. Let F be a self-mapping defined on b-metric space (X, d) with the given $s > 1$ and there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\beta \in B_s$. We say that F is α-β(b)-contraction mapping if the following condition holds:

$$[\alpha(x, y) - 1 + \delta]^{d(Fx,Fy)} \leqslant \delta^{\beta(d(x,y))d(x,y)} \tag{2.2}$$

for all $x, y \in X$, where $1 < \delta$.

3. Fixed point theorems

In this section, we give some theorems linking the above concepts. Our results generalize the results of Sintunvarat [30] and many other related results.

In 2014, Sintunvarat proved the following fixed point theorem for generalized α-β-contraction mappings:

Theorem 3.1 ([30]). *Let (X, d) be a complete metric space and $F : X \to X$ be an α-β-contraction mapping satisfying the following conditions:*

(i) *F is α-admissible;*
(ii) *α is transitive;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, F(x_0)) \geqslant 1$;*
(iv) *F is continuous.*

Then the fixed point problem of F has a solution, that is, there exists $x^* \in X$ such that $x^* = F(x^*)$.

Now, we will extend the above said theorem in the framework of b-metric spaces. Our theorem is as follows.

Theorem 3.2. *Let (X, d) be a complete b-metric space and $F : X \to X$ be an α-$\beta(b)$-contraction mapping satisfying the following conditions:*

(i) *F is α-admissible;*
(ii) *α is transitive;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, F(x_0)) \geqslant 1$;*
(iv) *F is continuous.*

Then F has a fixed point that is, there exists $u \in X$ such that $u = F(u)$.

Proof. From condition (iii), we can consider a point $x_0 \in X$ such that $\alpha(x_0, F(x_0)) \geqslant 1$. Let us construct a sequence $\{x_n\}$ in X such that $x_n = Fx_{n+1}$ for all $n \in \mathbb{N}$.

Now if $x_n = x_{n+1}$ for any $n \in \mathbb{N}$ then x_n is a fixed point of F from definition of $\{x_n\}$.

Without loss of generality, we can suppose that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$.

Now, α-admissibility of F implies that $\alpha(x_0, x_1) = \alpha(x_0, F(x_0)) \geqslant 1$. Similarly,

$$\alpha(x_1, x_2) = \alpha(Fx_0, Fx_1) \geqslant 1.$$

By mathematical induction we can easily deduce,

$$\alpha(x_{n-1}, x_n) \geqslant 1, \quad \forall\, n \in \mathbb{N}. \tag{3.1}$$

Now, for each $n \in N$

$$\begin{aligned}\delta^{d(x_n,x_{n+1})} \leqslant \delta^{d(Fx_{n-1},Fx_n)} &\leqslant [\alpha(x_{n-1},x_n)-1+\delta]^{d(Fx_{n-1},Fx_n)}\\ &\leqslant \delta^{\beta(d(x_{n-1},x_n))d(x_{n-1},x_n)}.\end{aligned}$$

This implies that

$$d(x_n,x_{n+1}) \leqslant \beta(d(x_{n-1},x_n))d(x_{n-1},x_n) \leqslant d(x_{n-1},x_n), \quad \forall\, n \in N. \tag{3.2}$$

Consequently, the sequence $\{d(x_{n-1},x_n)\}$ comes out as strictly decreasing sequence and so $d(x_{n-1},x_n) \to r$ as $n \to \infty$ for some $r \geqslant 0$. Next, we claim that $r = 0$. On contrary let us take $r > 0$. Proceeding the limits as $n \to \infty$ in (3.2), we get that

$$r \leqslant \lim_{n\to\infty} \beta(d(x_{n-1},x_n))r,$$

which implies that

$$\frac{1}{s} \leqslant 1 \leqslant \lim_{n\to\infty} \beta(d(x_{n-1},x_n)).$$

Since $\beta \in B_s$, and by definition of beta functions in b-metric spaces we obtain that $\lim_{n\to\infty} d(x_{n-1},x_n) = 0$, which is a contradiction. Therefore, $r = 0$ and thus

$$\lim_{n\to\infty} d(x_{n-1},x_n) = 0. \tag{3.3}$$

Now, we prove that $\{x_n\}$ is a b-Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a b-Cauchy sequence. Then there exist $\varepsilon > 0$ and subsequence of integers n_p and m_p with $n_p > m_p \geqslant 0$ such that

$$d(x_{n_p},x_{m_p}) > \varepsilon, \quad \forall\, p \in N. \tag{3.4}$$

This means that

$$d(x_{m_p},x_{n_{p-1}}) < \varepsilon.$$

From (3.4) and using the triangular inequality, we get

$$\begin{aligned}&\varepsilon \leqslant d(x_{m_p},x_{n_p}) \leqslant s d(x_{m_p},x_{n_{p-1}}) + s d(x_{n_{p-1}},x_{n_p}),\\ &\frac{\varepsilon}{s} \leqslant \frac{1}{s} d(x_{m_p},x_{n_p}) \leqslant d(x_{m_p},x_{n_{p-1}}) + d(x_{n_{p-1}},x_{n_p}).\end{aligned}$$

Proceeding the limit as $p \to \infty$, we get

$$\frac{\varepsilon}{s} \leqslant \frac{1}{s} \lim_{p\to\infty} d(x_{m_p},x_{n_p}) \leqslant \lim_{p\to\infty} d(x_{m_p},x_{n_p}) \leqslant \varepsilon. \tag{3.5}$$

Also, by (2.1) of Lemma 2.3 and (3.5), we have

$$\frac{\varepsilon}{s} \leqslant \lim_{p\to\infty} \inf \frac{1}{s} d(x_{m_p},x_{n_p}) \leqslant \limsup_{p\to\infty} \frac{1}{s} d(x_{m_p},x_{n_p}) \leqslant \varepsilon \frac{s}{s},$$

which in turns implies

$$\lim_{p\to\infty} d(x_{m_p},x_{n_p}) = \varepsilon. \tag{3.6}$$

Since α is transitive and $np > mp$ we can deduce that

$$\alpha(x_{m_p}, x_{n_p}) \geqslant 1.$$

Now, we have

$$\begin{aligned}
\delta^{d(x_{m_p},x_{n_p})} &\leqslant \delta^{sd(x_{m_p},x_{m_{p+1}})+sd(x_{m_{p+1}},x_{n_{p+1}})+sd(x_{n_{p+1}},x_{n_p})} \\
&\leqslant \delta^{sd(x_{m_p},x_{m_{p+1}})+sd(Fx_{m_p},Fx_{n_p})+sd(x_{n_{p+1}},x_{n_p})} \\
&\leqslant \delta^{sd(x_{m_p},x_{m_{p+1}})+sd(x_{n_{p+1}},x_{n_p})}\delta^{sd(Fx_{m_p},Fx_{n_p})} \\
&\leqslant \delta^{sd(x_{m_p},x_{m_{p+1}})+sd(x_{n_{p+1}},x_{n_p})} \times [\alpha(x_{m_p}, x_{n_p}) - 1 + \delta]^{sd(Fx_{m_p},Fx_{n_p})} \\
&\leqslant \delta^{sd(x_{m_p},x_{m_{p+1}})+sd(x_{n_{p+1}},x_{n_p})}\delta^{s\beta(d(x_{m_p},x_{n_p}))d(x_{m_p},x_{n_p})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&d(x_{m_p}, x_{n_p}) \leqslant sd(x_{m_p}, x_{m_{p+1}}) + sd(x_{n_{p+1}}, x_{n_p}) + s\beta(d(x_{m_p}, x_{n_p}))d(x_{m_p}, x_{n_p}), \\
&d(x_{m_p}, x_{n_p}) - sd(x_{m_p}, x_{m_{p+1}}) - sd(x_{n_{p+1}}, x_{n_p}) \leqslant s\beta(d(x_{m_p}, x_{n_p}))d(x_{m_p}, x_{n_p}), \\
&(d(x_{m_p}, x_{n_p}) - sd(x_{m_p}, x_{m_{p+1}}) - sd(x_{n_{p+1}}, x_{n_p}))/(sd(x_{m_p}, x_{n_p})) \leqslant \beta(d(x_{m_p}, x_{n_p})) < \frac{1}{s}.
\end{aligned}$$

Taking the limit as $p \to \infty$, and using (3.3) we get

$$\frac{1}{s} \leqslant \lim_{p\to\infty} \beta(d(x_{m_p}, x_{n_p})) < \frac{1}{s}.$$

That is,

$$\lim_{p\to\infty} \beta(d(x_{m_p}, x_{n_p})) = \frac{1}{s}.$$

Now, $\beta \in B_s$ implies that $\lim_{p\to\infty} d(x_{m_p}, x_{n_p}) = 0$ which contradicts with (3.6). Therefore, $\{x_n\}$ is a b-Cauchy sequence and b-completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$. From the continuity of F, it follows that

$$x_{n+1} = Fx_n \to Fu \text{ as } n \to \infty.$$

By the uniqueness of the limit, we get $Fu = u$, that is, u is a fixed point of F. □

One of the most important advantages of α-admissible mappings is that we can prove the existence of fixed points without the continuity of contraction mappings. Now, we will replace the continuity of F by adding some another condition in hypothesis of Theorem 3.2.

Theorem 3.3. *Let (X, d) be a complete b-metric space and $F : X \to X$ an α-$\beta(b)$-contraction mapping satisfying the following conditions:*

(i) *F is α-admissible;*
(ii) *α is transitive;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, F(x_0)) \geqslant 1$;*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n-1}, x_n) \geqslant 1$ for all $n \in N$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in N$.*

Then F has a fixed point, that is, there exists $u \in X$ such that $u = F(u)$.

Proof. We can deduce easily by the methodology used in the proof of Theorem 3.2 that $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (X, d). Therefore, there exists $u \in X$ such that $x_n \to u \in X$ as $n \to \infty$.

Also, from (3.1) and hypothesis (iv) implies that

$$\alpha(x_n, u) \geqslant 1, \quad \forall\, n \in \mathbb{N}. \tag{3.7}$$

Also, we obtain by using (2.2) and (3.7) that

$$\begin{aligned}
\delta^{d(u,Fu)} &\leqslant \delta^{sd(u,x_{n+1})+sd(x_{n+1},Fu)}\\
&= \delta^{sd(u,x_{n+1})+sd(Fx_n,Fu)}\\
&= \delta^{sd(u,x_{n+1})}\delta^{sd(Fx_n,Fu)}\\
&\leqslant \delta^{sd(u,x_{n+1})} \times [\alpha(x_n,u)-1+\delta]^{sd(Fx_n,Fu)}\\
&\leqslant \delta^{sd(u,x_{n+1})}\delta^{s\beta(d(x_n,u))d(x_n,u)}\\
&\leqslant \delta^{sd(u,x_{n+1})+s\beta(d(x_n,u))d(x_n,u)}, \quad \forall\, n \in \mathbb{N},
\end{aligned}$$

which in turns implies that

$$\begin{aligned}
d(u, Fu) &\leqslant sd(u, x_{n+1}) + s\beta(d(x_n, u))d(x_n, u)\\
&< sd(u, x_{n+1}) + sd(x_n, u), \quad \forall\, n \in \mathbb{N}.
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in the above inequality, we get that $d(u, Fu) = 0$, that is $Fu = u$. Therefore, the mapping F has a fixed point. □

We will consider the following hypothesis in order to establish the uniqueness of fixed point:

(H) for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$.

Theorem 3.4. *Adding condition* (H) *to the hypotheses of Theorem 3.2 (resp., Theorem 3.3), we can prove the uniqueness of the fixed point of* F.

Proof. Suppose that u and v are two fixed points of F. From (H), there exists $w \in X$ such that

$$\alpha(u, w) \geqslant 1 \quad \text{and} \quad \alpha(v, w) \geqslant 1. \tag{3.8}$$

Since F is α-admissible and u and v are fixed points of F from (3.8), we get

$$\alpha(u, F^n w) \geqslant 1 \quad \text{and} \quad \alpha(v, F^n w) \geqslant 1 \tag{3.9}$$

for all $n \in \mathbb{N}$. Now, (3.9) and (2.2) imply that

$$\begin{aligned}
\delta^{d(u,F^{n+1}w)} &= \delta^{d(Fu,FF^n w)}\\
&\leqslant [\alpha(u, F^n w) - 1 + \delta]^{d(Fu,FF^n w)}\\
&\leqslant \delta^{\beta(d(u,F^n w))d(u,F^n w)}, \quad \forall\, n \in \mathbb{N}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
d(u, F^{n+1}w) &\leqslant \beta(d(u, F^n w))d(u, F^n w)\\
&< \frac{1}{s} d(u, F^n w), \quad \forall\, n \in \mathbb{N}.
\end{aligned} \tag{3.10}$$

Next, we claim that $\lim_{n\to\infty} d(u, F^n w) = 0$. On the contrary, let us assume that

$$0 < \lim_{n\to\infty} d(u, F^n w) < \infty. \tag{3.11}$$

Taking the limit as $n \to \infty$ in (3.10), we get

$$\frac{1}{s} \leqslant \lim_{n\to\infty} \beta(d(u, F^n w)).$$

Now, $\beta \in B_s$, implies that $\lim_{n\to\infty} d(u, F^n w) = 0$, which contradicts (3.11). Thus we get

$$\lim_{n\to\infty} d(u, F^n w) = 0. \tag{3.12}$$

Similarly we can conclude on the same lines

$$\lim_{n\to\infty} d(v, F^n w) = 0. \tag{3.13}$$

Using (3.12) and (3.13), the uniqueness of the limit gives us $u = v$. This finishes the proof. □

Taking $\alpha(x, y) = 1$ and $\delta = 1$ in Theorem 3.4 we get the following variant of Geraghty-theorem.

Corollary 3.5. *Let (X, d) be a complete b-metric space and $F : X \to X$ be self-mapping satisfying the following condition:*

$$d(Fx, Fy) \leqslant \beta(d(x, y))d(x, y)$$

for all $x, y \in X$ and $\beta \in B_s$. Then F has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{F^n x\}$ converges to z when $n \to \infty$.

The following lemma derived from the reference [15] is very useful to prove our next theorem.

Lemma 3.6 ([15]). *Let $\{y_n\}$ be a sequence in b-metric space (X, d) such that*

$$d(y_{n+1}, y_n) \leqslant \lambda d(y_n, y_{n-1})$$

for some λ, $0 < \lambda < \frac{1}{s}$, and each $n = 1, 2, \cdots$. Then $\{y_n\}$ is a Cauchy sequence in X.

Theorem 3.7. *Let (X, d) be a complete b-metric space and $F : X \to X$ is continuous and α-admissible map and α is a transitive mapping. Suppose that there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$[\alpha(x, y) - 1 + \delta]^{d(Fx,Fy)} \leqslant \delta^{\lambda M(x,y)}$$

for all $x, y \in X$ and $1 < \delta$ where

$$M(x, y) = \max\left\{d(x, y), d(x, Fx), d(y, Fy), \frac{d(x, Fy) + d(y, Fx)}{2s}\right\}.$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant 1$, then F has a fixed point.

Proof. Consider $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant 1$. Define a sequence $\{x_n\}$ in X such that $x_n = Fx_{n+1}$ for all $n \in N$.

Now if $x_n = x_{n+1}$ for any $n \in N$ then x_n is a fixed point of F from definition of $\{x_n\}$.

Without loss of generality, we can suppose that $x_n \neq x_{n+1}$ for each $n \in N$. Since, F is α-admissible and $\alpha(x_0, x_1) = \alpha(x_0, F(x_0)) \geqslant 1$, similarly, $\alpha(x_1, x_2) = \alpha(Fx_0, Fx_1) \geqslant 1$. By induction we can easily deduce,

$$\alpha(x_{n-1}, x_n) \geqslant 1, \quad \forall\, n \in N.$$

Now, for each $n \in N$

$$\delta^{d(x_{n+1},x_n)} \leqslant \delta^{d(Fx_n,Fx_{n-1})}$$

$$\leqslant [\alpha(x_n, x_{n-1}) - 1 + \delta]^{d(Fx_n, Fx_{n-1})}$$
$$\leqslant \delta^{\lambda M(x_n, x_{n-1})}.$$

This implies that

$$d(x_{n+1}, x_n) \leqslant \lambda M(x_n, x_{n-1}),$$

or

$$\begin{aligned} d(x_{n+1}, x_n) &\leqslant \lambda \max\left\{ d(x_n, x_{n-1}), d(x_n, Fx_n), d(x_{n-1}, Fx_{n-1}), \frac{d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)}{2s} \right\} \\ &\leqslant \lambda \max\left\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2s} \right\} \\ &\leqslant \lambda \max\left\{ d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\}. \end{aligned} \tag{3.14}$$

Now, if

$$\max\left\{ d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2},$$

then

$$d(x_n, x_{n-1}) < \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} < d(x_{n+1}, x_n).$$

Then (3.14) implies that

$$d(x_{n+1}, x_n) \leqslant \lambda d(x_{n+1}, x_n),$$

which is impossible as $\lambda < 1$. Therefore, we deduce that

$$\max\left\{ d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} = d(x_n, x_{n-1}),$$

which in turn implies that

$$d(x_{n+1}, x_n) \leqslant \lambda d(x_n, x_{n-1}).$$

Using Lemma 3.6, we obtain that $\{x_n\}$ is a b-Cauchy sequence and b-completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$. From the continuity of F, it follows that $x_{n+1} = Fx_n \to Fu$ as $n \to \infty$.

By the uniqueness of the limit, we get $Fu = u$, that is, u is a fixed point of F. □

Next, we consider an example that illustrates the above proved results.

Example 3.8. Let $X = \{0, 1, 3\}$ be a b-metric space with metric d given by $d(x, y) = (x - y)^2$ with $s = 2$. Consider the mapping $F : X \to X$ defined by $F(0) = 1$, $F(1) = 1$, $F(3) = 0$. Let us take $\beta \in B_s$, as $\beta(t) = \frac{1}{2}$ for $t > 0$ and $\beta(0) \in [0, \frac{1}{2})$, and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{when } x, y \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can examine easily that

$$\begin{aligned} d(F0, F1) = d(1, 1) = 0, &\quad \text{and} \quad \beta(d(0, 1))d(0, 1) = \frac{1}{2}, \\ d(F0, F3) = d(1, 0) = 1, &\quad \text{and} \quad \beta(d(0, 3))d(0, 3) = \frac{9}{2}, \end{aligned}$$

$$d(F1, F3) = d(1, 0) = 1, \quad \text{and} \quad \beta(d(1, 3))d(1, 3) = 2.$$

Thus in all cases we get

$$[\alpha(x, y) - 1 + \delta]^{d(Fx, Fy)} \leqslant \delta^{\beta(d(x,y))d(x,y)}$$

for all $x, y \in X$, where $1 < \delta$.

Also we can check that F is α-admissible and α is transitive mapping and satisfies all other axioms of Theorem 3.4. Therefore, we conclude that F has a unique fixed point $x = 1$.

4. Applications

This section deals with applications of fixed point results discussed in previous section. First, we will examine the problem of Ulam-Hyers stability through the fixed point theorems. And then we will discuss the well-posedness and limit shadowing of fixed points.

4.1. Ulam-Hyers stability results through the fixed point problems

Definition 4.1. Let (X, d) be a metric space and $F : X \to X$ an operator defined in X. Then, the fixed point equation

$$x = F(x) \tag{4.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : R^+ \to R^+$ which is increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each $w^* \in X$ an ε-solution of the fixed point equation (4.1), that is, w^* satisfies the inequality

$$d(w^*, F(w^*)) \leqslant \varepsilon. \tag{4.2}$$

There exists a solution $x^* \in X$ of (4.1) such that

$$d(w^*, x^*) \leqslant \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) = c \cdot t$, for each $t \in R^+$, then the fixed point equation (4.1) is said to be Ulam-Hyers stable.

Theorem 4.2. *Let (X, d) be a complete b-metric space with constant $\varepsilon > 1$. We assume that all the conditions of Theorem 3.4 hold and suppose that there exists a strictly increasing and onto operator $\xi : [0, \infty) \to [0, \infty)$ satisfying $\xi(r) := r - s\beta(r)$ with $\beta(0) = 0$. Then if $\alpha(u, v) \geqslant 1$ for all u, v which are ε-solutions of the fixed point equation (4.1), then the fixed point problem of F is generalized Ulam-Hyers stable.*

Proof. According to hypothesis, the operator F has a unique fixed point say, x^*. Let us claim that the fixed point problem of F is generalized Ulam-Hyers stable. Let $\varepsilon > 0$ and $w^* \in X$ be a solution of (4.2), that is,

$$d(w^*, F(w^*)) \leqslant \varepsilon.$$

It is obvious that the fixed point x^* of F satisfies inequality (4.2). From hypothesis, we get $\alpha(x^*, w^*) \geqslant 1$. Also we see that

$$\begin{aligned}
\delta^{d(x^*,w^*)} &= \delta^{d(F(x^*),w^*)} \\
&\leqslant \delta^{sd(F(x^*),F(w^*))+sd(F(w^*),w^*)} \\
&= \delta^{sd(F(x^*),F(w^*))}\delta^{sd(F(w^*),w^*)} \\
&\leqslant \delta^{sd(F(w^*),w^*)} \times [\alpha(x^*, w^*) - 1 + \delta]^{sd(F(x^*),F(w^*))} \\
&\leqslant \delta^{sd(F(w^*),w^*)}\delta^{s\beta(d(x^*,w^*))d(x^*,w^*))}
\end{aligned}$$

$$\leqslant \delta^{s d(F(w^*),w^*)+s\beta(d(x^*,w^*))d(x^*,w^*)}$$
$$\leqslant \delta^{s\varepsilon+s\beta(d(x^*,w^*))d(x^*,w^*)}.$$

Then we get that

$$d(x^*,w^*) \leqslant s\varepsilon + s\beta(d(x^*,w^*))d(x^*,w^*),$$
$$d(x^*,w^*) - s\beta(d(x^*,w^*))d(x^*,w^*) \leqslant s\varepsilon.$$

Therefore

$$\xi(d(x^*,w^*)) \leqslant s\varepsilon,$$
$$d(x^*,w^*) \leqslant \xi^{-1}(s\varepsilon).$$

Therefore, the fixed point problem of F is generalized Ulam-Hyers stable. □

4.2. Well-posedness and limit shadowing results via the fixed point problems

Before proceeding towards the next applications, we first go through the following definitions.

Definition 4.3 ([9]). Let (X, d) be a metric space and $F : X \to X$ be an operator defined on X. Then the well-posedness of fixed point problem of F exists if the following holds:

(i) F has a unique fixed point x^* in X;
(ii) for any sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$, one has $\lim_{n\to\infty} d(x_n, x^*) = 0$.

Definition 4.4. Let (X, d) be a metric space and $F : X \to X$ be an operator defined on X. We say that the fixed point problem of F has the "limit shadowing property" in X, if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$, it follows that there exists $z \in X$ such that

$$\lim_{n\to\infty} d(F^n(z), x_n) = 0.$$

Theorem 4.5. *Let (X, d) be a complete b-metric space with constant $\varepsilon > 1$. Suppose that all the hypotheses of Theorem 3.4 hold, assume that there exists a strictly increasing and onto operator $\xi : [0,\infty) \to [0,\infty)$ satisfying $\xi(r) := r - s\beta(r)$ with $\beta(0) = 0$. Let us consider*

(i) *if $\alpha(x^*, x_n) \geqslant 1$ for all $n \in N$ such that $\{x_n\}$ is sequence in X in which $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$ and x^* is a fixed point of F, then the fixed point problem of F is well-posed;*
(ii) *if $\alpha(x^*, x_n) \geqslant 1$ for all $n \in N$ such that $\{x_n\}$ is sequence in X in which $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$ and x^* is a fixed point of F then the fixed point problem of F has the limit shadowing property in X.*

Proof. From the proof of Theorem 3.4, we obtain that F has a unique fixed point and so let x^* be a unique fixed point of F. Let $\{x_n\}$ be sequence in X in which $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$.

From assumption, we get $\alpha(x^*, x_n) \geqslant 1$ for all $n \in N$.

Now, we obtain that

$$\begin{aligned}
\delta^d(x^*, x_n) &\leqslant \delta^{s d((x^*),F(x_n))+s d(F(x_n),x_n)}\\
&= \delta^{s d(F(x^*),F(x_n))+s d(F(x_n),x_n)}\\
&= \delta^{s d(F(x^*),F(x_n))}\delta^{s d(F(x_n),x_n)}\\
&\leqslant [\alpha(x^*, x_n) - 1 + \delta]^{s d(F(x^*),F(x_n))}\delta^{s d(F(x_n),x_n)}\\
&\leqslant \delta^{s\beta(d(x^*,x_n))d(x^*,x_n)}\delta^{s d(F(x_n),x_n)}\\
&\leqslant \delta^{s\beta(d(x^*,x_n))d(x^*,x_n)+s d(F(x_n),x_n)}
\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} d(x^*, x_n) &\leqslant s\beta(d(x^*, x_n))d(x^*, x_n) + sd(F(x_n), x_n), \\ \xi(d(x^*, x_n)) &:= d(x^*, x_n) - s\beta(d(x^*, x_n))d(x^*, x_n) \\ &\leqslant sd(F(x_n), x_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which in turns implies that $d(x^*, x_n) \to 0$ as $n \to \infty$ or $x_n \to x^*$ as $n \to \infty$. Therefore, the fixed point problem of F is well-posed.

Secondly, we prove that F has a limit shadowing under assumption (ii). Let $\{x_n\}$ be sequence in X in which $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$ and x^* is a fixed point of F. Similar to case (i), we get

$$d(x^*, x_n) \to 0 \text{ as } n \to \infty.$$

Since x^* is a fixed point of F we have

$$\lim_{n\to\infty} d(x_n, F^n(x^*)) = \lim_{n\to\infty} d(x_n, x^*) = 0.$$

Therefore, F has the limit shadowing property. □

5. Conclusion

Concluding we can say that our results are novel and improved results while concerning the stability and fixed point theorems in the context of b-metric space. Contraction mapping defined here is quiet different one and relevant example supports the results very well. Application part of the article is very interesting and justifies the validity of results.

References

[1] I. A. Bakhtin, *The contraction mapping principle in quasi-metric spaces*, J. Funct. Anal., **30** (1989), 26–37. 2.1

[2] M. F. Bota, E. Karapnar, O. Mlesnite, *Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mapping in (b)-metric space*, Abstr. Appl. Anal., **2013** (2013), 6 pages. 1

[3] M. F. Bota-Boriceanu, A. Petrusel, *Ulam-Hyers stability for operatorial equations*, An. tiin. Univ. Al. I. Cuza Iai. Mat., **57** (2011), 65–74. 1

[4] J. Brzdek, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74** (2011), 6728–6732. 1

[5] J. Brzdek, K. Cieplinski, *A fixed point theorem and the Hyers-Ulam stability in non-Archimedean spaces*, J. Math. Anal. Appl., **400** (2013), 68–75. 1

[6] L. Cadariu, L. Gavruta, P. Gavruta, *Fixed points and generalized Hyers-Ulam stability*, Abstr. Appl. Anal., **2012** (2012), 10 page. 1

[7] S. Chandok, M. Jovanović, S. Radenović, *Ordered b-metric spaces and Geraghty type contractive mappings*, Military Technical Courier, **65** (2017), 331–345.

[8] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 2.1, 2.2

[9] F. S. de Blassi, J. Myjak, *Sur la porosite des contractions sans point fixed*, C. R. Acad. Sci. Paris Sr. I Math., **308** (1989), 51–54. 1, 4.3

[10] D. Dukić, Z. Kadelburg, S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstr. Appl. Anal., **2011** (2011), 13 pages. 1, 2.6

[11] M. Eshaghi Gordji, M. Ramezani, Y. J. Cho, S. Pirbavafa, *A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl., **2012** (2012), 9 pages.

[12] A. Felhi, S. Sahmim, H. Aydi, *Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces*, Fixed Point Theory Appl., **2016** (2016), 20 pages. 1

[13] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608. 2, 2.5

[14] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224. 1

[15] M. Jovanovic, Z. Kadelburg, S. Radenovic, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl., **2010** (2010), 15 pages. 3, 3.6

[16] M. A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl., **2010** (2010), 7 pages.

[17] B. K. Lahiri, P. Das, *Well-posedness and porosity of certain classes of operators*, Demonstratio Mathematica., **38** (2005), 170–176. 1

[18] V. L. Lažar, *Ulam-Hyers stability for partial differential inclusions*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), 19 pages. 1

[19] Z. Mustafa, H. Huang, S. Radenović, *Some remarks on the paper, Some fixed point generalizations are not real generalizations*, J. Adv. Math. Stud., **9** (2016), 110–116.

[20] V. Popa, *Well posedness of fixed point problem in orbitally complete metric spaces*, Stud. Cercet. tiin. Ser. Mat. Univ. Bacu., **16** (2006), 18–20. 1

[21] V. Popa, *Well posedness of fixed point problem in compact metric spaces*, Bul. Univ. Petrol-Gaze, Ploiesti, Sec. Mat. Inform. Fiz., **60** (2008), 1–4. 1

[22] G. S. Rad, S. Radenović, D. Dolićanin-Dekić, *A shorter and simple approach to study fixed point results via* b*-simulation functions*, Iran. J. Math. Sci. Inform., (Accepted). 1

[23] S. Radenović, S. Chandok, W. Shatanawi, *Some cyclic fixed point results for contractive mappings*, University Though, Publication in Nature Sciences, **6** (2016), 38–40.

[24] S. Radenović, T. Došenović, V. Osturk, Ć. Dolićanin, *A note on the paper, Integral equations with new admisibility types in* b*-metric spaces*, J. Fixed Point Theory Appl., **2017** (2017), 9 pages. 1

[25] S. Reich, A. J. Zaslavski, *Well-posedness of fixed point problems*, Far East J. Math. Sci., **3** (2001), 393–401. 1

[26] J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Common fixed point theorems for weakly isotone increasing mappings in ordered* b*-metric spaces*, J. Nonlinear Sci. Appl., **7** (2014), 229–245. 2.3

[27] A. Rus, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10** (2009), 305–320.

[28] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for* α*-*ψ*-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 2.7

[29] R. J. Shahkoohi, A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered* b*-metric spaces*, J. Inequal. Appl., **2014** (2014), 23 pages.

[30] W. Sintunavarat, *Generalized Ulam-Hyers stability, well-posedness, and limit shadowing of fixed point problems for* α*-*β*-contraction mapping in metric spaces*, The Sci. World J., **2014** (2014), 7 pages. 2.9, 2.10, 3, 3.1

[31] W. Sintunavarat, S. Plubtieng, P. Katchang, *Fixed point result and applications on a* b*-metric space endowed with an arbitrary binary relation*, Fixed Point Theory and Appl., **2013** (2013), 13 pages.

[32] F. A. Tise, I. C. Tise, *Ulam-Hyers-Rassias stability for set integral equations*, Fixed Point Theory, **13** (2012), 659–668.

[33] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, (1964). 1