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Fixed point results for generalized $(\alpha\text{-}\eta)$-Θ contractions with applications

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Abstract

The aim of this paper is to define generalized $(\alpha\text{-}\eta)$-Θ contraction and to extend the results of Jleli and Samet [M. Jleli, B. Samet, J. Inequal. Appl., **2014** (2014), 8 pages] by applying a simple condition on the function Θ. We also deduce certain fixed and periodic point results for orbitally continuous generalized Θ-contractions and certain fixed point results for integral inequalities are derived. Finally, we provide an example to show the significance of the investigation of this paper.

Keywords: Fixed point, complete metric space, α-admissible mapping.
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1. Introduction and preliminaries

Banach's contraction principle [8] is one of the pivotal results of analysis. It establishes that, given a mapping F on a complete metric space (X, d) into itself and a constant $k \in (0, 1)$ such that

$$d(Fx, Fy) \leqslant kd(x, y),$$

holds for all $x, y \in X$. Then F has a unique fixed point in X.

Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1–13, 17] and references therein). In 2012, Samet et al. [21] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces.

Definition 1.1 ([21]). Let F be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that F is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Fx, Fy) \geqslant 1.$$

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Afterwards Salimi et al. [20] and Hussain et al. [15, 16] modified the notions of α-admissible mappings and established certain fixed point theorems.

Definition 1.2 ([20])**.** Let F be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that F is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geqslant \eta(x, y) \quad \Longrightarrow \quad \alpha(Fx, Fy) \geqslant \eta(Fx, Fy).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.1. Also, if we take $\alpha(x, y) = 1$, then we say that F is an η-subadmissible mapping.

Definition 1.3 ([16])**.** Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $F : X \to X$ be functions. We say F is an α-η-continuous mapping on (X, d), if for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty, \ \alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1}) \, \forall \, n \in \mathbb{N} \Longrightarrow \ Fx_n \to Fx.$$

A mapping $F : X \to X$ is called orbitally continuous at $p \in X$ if $\lim_{n\to\infty} F^n x = p$ implies that $\lim_{n\to\infty} FF^n x = Fp$. The mapping F is orbitally continuous on X if F is orbitally continuous for all $p \in X$.

Remark 1.4 ([16]). Let $F : X \to X$ be a self-mapping on an orbitally F-complete metric space X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \eta(x, y) = 1,$$

where $O(w)$ is an orbit of a point $w \in X$. If $F : X \to X$ is an orbitally continuous map on (X, d), then F is α-η-continuous on (X, d).

Very recently, Jleli and Samet [19] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.5. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exist $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $F : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \Longrightarrow \Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^k.$$

Theorem 1.6 ([19])**.** *Let (X, d) be a complete metric space and $F : X \to X$ be a Θ-contraction, then F has a unique fixed point.*

They showed that any Banach contraction is a particular case of Θ-contraction while there are Θ-contractions which are not Banach contractions. To be consistent with Jleli et al. [19], we denote by the Ψ set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions (Θ_1)-(Θ_3).

Hussain et al. [17] modified and extended the above result and proved the following fixed point theorem for generalized Θ-contractive condition in the setting of complete metric spaces.

Theorem 1.7 ([17])**.** *Let (X, d) be a complete metric space and $F : X \to X$ be a self-mapping. If there exist a function $\Theta \in \Psi$ and positive real numbers $\alpha, \beta, \gamma, \delta$ with $0 \leqslant \alpha + \beta + \gamma + 2\delta < 1$ such that*

$$\Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^\alpha \cdot [\Theta(d(x, Fx))]^\beta \cdot [\Theta(d(y, Fy))]^\gamma \cdot [\Theta((d(x, Fy) + d(y, Fx))]^\delta$$

for all $x, y \in X$, then F has a unique fixed point.

Very recently, Ahmad et al. [2, 7] used the following weaker condition instead of the condition (Θ_3) in Definition 1.5.

(Θ_3') Θ is continuous on $(0, \infty)$.

Consistent with Ahmad et al. [2], we denote by Ω the set of all functions satisfying the conditions $(\Theta_1), (\Theta_2)$ and (Θ_3').

Example 1.8 ([2])**.** Let $\Theta_1(t) = e^{\sqrt{t}}$, $\Theta_2(t) = e^{\sqrt{te^t}}$, $\Theta_3(t) = e^t$, $\Theta_4(t) = \cosh t$, $\Theta_5(t) = 1 + \ln(1+t)$ and $\Theta_6(t) = e^{te^t}$ for all $t > 0$. Then $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Omega$.

Example 1.9 ([2])**.** Note that the conditions Θ_3 and Θ_3' are independent of each other. Indeed, for $p \geqslant 1$, $\Theta(t) = e^{t^p}$ satisfies the conditions (Θ_1) and (Θ_2) but it does not satisfy (Θ_3), while it satisfies the condition (Θ_3'). Therefore $\Omega \not\subseteq \Psi$. Again for $p > 1$, $m \in (0, \frac{1}{p})$ $\Theta(t) = 1 + t^m(1 + [t])$ where $[t]$ denotes the integral part of t, satisfies the conditions (Θ_1) and (Θ_2) but it does not satisfy (Θ_3'), while it satisfies the condition (Θ_3) for any $k \in (\frac{1}{p}, 1)$. Therefore $\Psi \not\subseteq \Omega$. Also, if we take $\Theta(t) = e^{\sqrt{t}}$, then $\Theta \in \Psi$ and $\Theta \in \Omega$. Therefore $\Psi \cap \Omega \neq \emptyset$.

In this paper, we apply the same weaker condition (Θ_3') to obtain some new fixed point theorems in the context of complete metric spaces.

2. Main results

In this section, we define α-η-Θ-contraction for a new family of functions Ω and establish certain fixed point theorems in the context of complete metric spaces.

Definition 2.1. Let (X, d) be a metric space and F be a self-mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that F is α-η-Θ-contraction if for $x, y \in X$ with $\eta(x, Fx) \leqslant \alpha(x, y)$ and $d(Fx, Fy) > 0$, we have

$$\Theta(d(Fx, Fy)) \leqslant \left[\Theta(d(x, y))\right]^k,$$

where $\Theta \in \Omega$ and $k \in (0, 1)$.

Theorem 2.2. *Let (X, d) be a complete metric space. Let $F : X \to X$ be a self-mapping satisfying the following assertions:*

(i) F *is α-admissible mapping with respect to η;*

(ii) F *is α-η-Θ-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant \eta(x_0, Fx_0)$;*

(iv) F *is α-η-continuous.*

Then F has a fixed point. Moreover, F has a unique fixed point when $\alpha(x, y) \geqslant \eta(x, x)$ for all $x, y \in \mathrm{Fix}(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant \eta(x_0, Fx_0)$. For such x_0, we define the sequence $\{x_n\}$ by $x_n = F^n x_0 = Fx_{n-1}$. Now, since F is α-admissible mapping with respect to η, then $\alpha(x_0, x_1) = \alpha(x_0, Fx_0) \geqslant \eta(x_0, Fx_0) = \eta(x_0, x_1)$. By continuing this process we have

$$\eta(x_{n-1}, Fx_{n-1}) = \eta(x_{n-1}, x_n) \leqslant \alpha(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Fx_{n-1}, Fx_n) > 0$ for all $n \in \mathbb{N}$. Since, F is α-η-Θ-contraction, so we have

$$1 < \Theta(d(x_n, x_{n+1})) = \Theta(d(Fx_{n-1}, Fx_n)) \leqslant [\Theta(d(x_{n-1}, x_n))]^k$$

$$
\begin{aligned}
&= [\Theta(d(Fx_{n-2}, Fx_{n-1}))]^k \leqslant [\Theta(d(x_{n-2}, x_{n-1}))]^{k^2} \\
&\vdots \\
&\leqslant [\Theta(d(x_0, x_1))]^{k^n}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, so by taking limit as $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} \Theta(d(x_n, x_{n+1})) = 1.$$

By (Θ_2), we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.1}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence, then we assume that there exist $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d(x_{p(n)}, x_{q(n)}) \geqslant \varepsilon.$$

Then

$$d(x_{p(n)-1}, x_{q(n)}) < \varepsilon \tag{2.2}$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (2.2), we have

$$\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \leqslant d(x_{p(n)-1}, x_{p(n)}) + \varepsilon.$$

By taking the limit and using inequality (2.2), we get

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{2.3}$$

From (2.1), we can choose a natural number $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \text{ and } d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4} \tag{2.4}$$

for all $n \geqslant n_0$. Next, we claim that $Fx_{p(n)} \neq Fx_{q(n)}$ for all $n \geqslant n_0$, that is

$$d(x_{p(n)+1}, x_{q(n)+1}) = d(Fx_{p(n)}, Fx_{q(n)}) > 0. \tag{2.5}$$

Arguing by contradiction, there exists $N_0 \geqslant n_0$ such that $d(x_{p(n)+1}, x_{q(n)+1}) = 0$. It follows from (2.1), (2.4), and (2.5) that

$$
\begin{aligned}
\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) &\leqslant d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)}) \\
&\leqslant \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}
$$

a contradiction. Thus the relation (2.4) holds. Then by the assumption, we get

$$\Theta(d(Fx_{p(n)}, Fx_{q(n)})) \leqslant [\Theta(d(x_{p(n)}, x_{q(n)}))]^k. \tag{2.6}$$

By taking limit as $n \to +\infty$ and using (Θ_3'), (2.3) and (2.6), we get

$$\Theta(\varepsilon) \leqslant [\Theta(\varepsilon)]^k,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now, since F is α-η-continuous and $\eta(x_{n-1}, x_n) \leqslant \alpha(x_{n-1}, x_n)$, so

$$d(z, Fz) = \lim_{n\to\infty} d(x_n, Fx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = d(z, z) = 0.$$

Hence, z is a fixed point of F. Now we show the uniqueness of fixed point. We suppose on the contrary that there exists another fixed point u of F distinct from z, that is

$$Fz = z \neq u = Fu \quad \text{that is} \quad Fz \neq Fu.$$

Then from assumption of theorem, we obtain

$$\Theta(d(z,u)) = \Theta(d(Fz,Fu)) \leqslant [\Theta(d(z,u))]^k,$$

which is contradiction because $k \in (0,1)$. Thus z is the unique fixed point of F. □

Theorem 2.3. *Let (X,d) be a complete metric space. Let $F : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *F is an α-admissible mapping with respect to η;*

(ii) *F is α-η-Θ-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant \eta(x_0, Fx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta(Fx_n, F^2x_n) \leqslant \alpha(Fx_n, x), \quad \textit{or} \quad \eta(F^2x_n, F^3x_n) \leqslant \alpha(F^2x_n, x),$$

holds for all $n \in \mathbb{N}$.

Then F has a fixed point. Moreover, F has a unique fixed point whenever $\alpha(x,y) \geqslant \eta(x,x)$ for all $x, y \in \mathrm{Fix}(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant \eta(x_0, Fx_0)$. As in proof of Theorem 2.2 we can conclude that

$$\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1}) \quad \text{and} \quad x_n \to x^* \text{ as } n \to \infty,$$

where, $x_{n+1} = Tx_n$. So, from (iv), either

$$\eta(Fx_n, F^2x_n) \leqslant \alpha(Fx_n, x^*) \quad \text{or} \quad \eta(F^2x_n, F^3x_n) \leqslant \alpha(F^2x_n, x^*),$$

holds for all $n \in \mathbb{N}$. This implies

$$\eta(x_{n+1}, x_{n+2}) \leqslant \alpha(x_{n+1}, x) \quad \text{or} \quad \eta(x_{n+2}, x_{n+3}) \leqslant \alpha(x_{n+2}, x),$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Fx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leqslant \alpha(x_{n_k}, x^*), \tag{2.7}$$

and so from (2.7) we deduce that

$$\Theta(d(Fx_{n_k}, Fx^*)) \leqslant [\Theta(d(x_{n_k}, x^*))]^\lambda < \Theta(d(x_{n_k}, x^*)).$$

From (Θ_1) we have

$$d(x_{n_k+1}, Fx^*) < d(x_{n_k}, x^*).$$

By taking limit as $k \to \infty$ in the above inequality we get $d(x^*, Fx^*) = 0$, i.e., $x^* = Fx^*$. Uniqueness follows similarly as in Theorem 2.2. □

Taking $\alpha(x,y) = \eta(x,y) = 1$ for all $x, y \in X$, then we deduce the following result as corollary.

Corollary 2.4. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping. If for all* $x, y \in X$ *with* $d(Fx, Fy) > 0$, *we have*

$$\Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^k,$$

where $F \in \Omega$. *Then* F *has a fixed point.*

Recall that a self-mapping T is said to have the property P, if $\mathrm{Fix}(T^n) = F(T)$ for every $n \in \mathbb{N}$.

Theorem 2.5. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be an* α*-continuous self-mapping. Assume that there exists some* $k \in (0, 1)$ *such that*

$$\Theta(d(Fx, F^2x)) \leqslant [\Theta(d(x, Fx))]^k, \tag{2.8}$$

holds for all $x \in X$ *with* $d(Fx, F^2x) > 0$ *where* $\Theta \in \Omega$. *If* F *is an* α*-admissible mapping and there exists* $x_0 \in X$ *such that* $\alpha(x_0, Fx_0) \geqslant 1$, *then* F *has the property* P.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant 1$. For such x_0, we define the sequence $\{x_n\}$ by $x_n = F^n x_0 = Fx_{n-1}$. Now, since F is α-admissible mapping, so $\alpha(x_1, x_2) = \alpha(Fx_0, Fx_1) \geqslant 1$. By continuing this process, we have

$$\alpha(x_{n-1}, x_n) \geqslant 1$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Fx_{n_0}$, then x_{n_0} is fixed point of F and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Fx_{n-1}, F^2x_{n-1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.8) we have

$$1 < \Theta(d(Fx_{n-1}, F^2x_{n-1})) \leqslant [\Theta(d(x_{n-1}, Fx_{n-1}))]^k,$$

which implies

$$1 < \Theta(d(x_n, x_{n+1})) \leqslant [\Theta(d(x_{n-1}, x_n))]^k,$$

and so

$$1 < \Theta(d(x_n, x_{n+1})) \leqslant [\Theta(d(x_{n-1}, x_n))]^k.$$

Therefore,

$$1 < \Theta(d(x_n, x_{n+1})) \leqslant [\Theta(d(x_{n-1}, x_n))]^k \leqslant [\Theta(d(x_{n-2}, x_{n-1}))]^{k^2} \leqslant \cdots \leqslant [\Theta(d(x_0, x_1))]^{k^n}.$$

By taking limit as $n \to \infty$ in above inequality, we have $\lim_{n\to\infty} \Theta(d(x_n, x_{n+1})) = 1$, and since $\Theta \in \Omega$ we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.9}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not Cauchy then we assume there exist $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d(x_{p(n)}, Fx_{q(n)-1}) = d(x_{p(n)}, x_{q(n)}) \geqslant \varepsilon. \tag{2.10}$$

Then

$$d(x_{p(n)-1}, Fx_{q(n)-1}) < \varepsilon$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (2.10), we have

$$\varepsilon \leqslant d(x_{p(n)}, Fx_{q(n)-1}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, Fx_{q(n)-1}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.$$

By taking the limit and using inequality (2.9), we get

$$\lim_{n\to\infty} d(x_{p(n)}, Fx_{q(n)-1}) = \varepsilon.$$

On the other hand, from (2.9) there exists a natural number $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \text{ and } d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4} \tag{2.11}$$

for all $n \geqslant n_0$. Next, we claim that

$$d(Fx_{p(n)}, F^2x_{q(n)-1}) = d(x_{p(n)+1}, Fx_{q(n)}) > 0 \tag{2.12}$$

for all $n \geqslant n_0$. We suppose on the contrary that there exists $m \geqslant n_0$ such that

$$d(Fx_{p(m)}, F^2x_{q(m)-1}) = d(x_{p(m)+1}, Fx_{q(m)}) = 0. \tag{2.13}$$

Then from (2.11), (2.12) and (2.13), we have

$$\begin{aligned} \varepsilon &\leqslant d(x_{p(m)}, Fx_{q(m)-1}) \leqslant d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, Fx_{q(m)-1}) \\ &\leqslant d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, Fx_{q(m)-1}) \\ &= d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, Fx_{q(m)}) + d(x_{q(m)+1}, x_{q(m)}) \\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

which is a contradiction. Thus

$$\begin{gathered} d(Fx_{p(n)}, F^2x_{q(n)-1}) = d(x_{p(n)+1}, Fx_{q(n)}) > 0, \\ \Theta\big(d(Fx_{p(n)}, F^2x_{q(n)-1})\big) \leqslant [\Theta\big(d(x_{p(n)}, Fx_{q(n)-1})\big)]^k, \end{gathered} \tag{2.14}$$

is established which further implies that

$$\Theta\big(d(x_{p(n)+1}, x_{q(n)+1})\big) \leqslant [\Theta\big(d(x_{p(n)}, x_{q(n)})\big)]^k.$$

From (Θ_3), (2.10) and (2.14), we get

$$\Theta(\varepsilon) \leqslant [\Theta(\varepsilon)]^k,$$

which is a contradiction because $k \in (0, 1)$. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now, since F is α-continuous and $\alpha(x_{n-1}, x_n) \geqslant 1$ then, $x_{n+1} = Fx_n \to Fx^*$ as $n \to \infty$. That is, $x^* = Fx^*$. Thus F has a fixed point and $F(F^n) = F(F)$ for $n = 1$. Let $n > 1$. Assume contrarily that $w \in F(F^n)$ and $w \notin F(F)$. Then, $d(w, Fw) > 0$. Now we have

$$\begin{aligned} 1 < \Theta(d(w, Fw)) &= \Theta(d(F(F^{n-1}w)), F^2(F^{n-1}w))) \\ &\leqslant [\Theta(d(F^{n-1}w, F^n w))]^k \\ &\leqslant [\Theta(d(F^{n-2}w, F^{n-1}w))]^{k^2} \leqslant \cdots \\ &\leqslant [\Theta(d(w, Fw))]^{k^n}. \end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality we have $\Theta(d(w, Fw)) = 1$. Hence, by (Θ_2) we get, $d(w, Fw) = 0$ which is a contradiction. Therefore, $F(F^n) = F(F)$ for all $n \in \mathbb{N}$. □

Let $(X, d, \preceq)$ be a partially ordered metric space. Recall that $F : X \to X$ is nondecreasing if for all $x, y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$ and ordered Θ-contraction if for $x, y \in X$ with $x \preceq y$ and $d(Fx, Fy) > 0$, we have

$$\Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k,$$

where $\Theta \in \Omega$. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 14, 16, 18] and references therein). From Theorems 2.2-2.5, we derive the following new results in partially ordered metric spaces.

Theorem 2.6. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) *F is nondecreasing and ordered Θ-contraction;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$;*

(iii) *either for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \to x \text{ as } n \to \infty \quad \text{and} \quad x_n \preceq x_{n+1}, \ \forall n \in \mathbb{N} \ \text{ we have } \ Fx_n \to Fx,$$

or if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either

$$Fx_n \preceq x, \quad \text{or} \quad F^2x_n \preceq x,$$

holds for all $n \in \mathbb{N}$.

Then F has a fixed point.

Theorem 2.7. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) *F is nondecreasing and satisfies (2.8) for all $x \in X$ with $d(Fx, F^2x) > 0$ where $\Theta \in \Omega$ and $\tau > 0$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$;*

(iii) *for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \text{ we have } Fx_n \to Fx.$$

Then F has a property P.

As an application of our results proved above, we deduce certain Suzuki-Samet type fixed point theorems.

Theorem 2.8. *Let (X, d) be a complete metric space and F be a continuous self-mapping on X. If for $x, y \in X$ with $\frac{1}{2}d(x, Fx) \leqslant d(x, y)$ and $d(Fx, Fy) > 0$ we have*

$$\Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k,$$

where $\Theta \in \Omega$. Then F has a unique fixed point.

Proof. Define, $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = d(x, y) \quad \text{and} \quad \eta(x, y) = \frac{1}{2}d(x, y)$$

for all $x, y \in X$. Now, since $\frac{1}{2}d(x, y) \leqslant d(x, y)$ for all $x, y \in X$, so $\eta(x, y) \leqslant \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.2 hold true. Since F is continuous, so F is α-η-continuous. Let $\eta(x, Fx) \leqslant \alpha(x, y)$ with $d(Fx, Fy) > 0$. Equivalently, if $\frac{1}{2}d(x, Fx) \leqslant d(x, y)$ with $d(Fx, Fy) > 0$, then we have

$$\Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k.$$

That is, F is α-η-Θ-contraction mapping. Hence, all conditions of Theorem 2.2 hold and F has a unique fixed point. □

Theorem 2.9. *Let* (X, d) *be a complete metric space and* F *be a self-mapping on* X*. Assume that there exists some* $k \in (0, 1)$ *such that*

$$\frac{1}{2(1+\tau)}d(x, Fx) \leqslant d(x, y) \quad \textit{implies} \quad \Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k \tag{2.15}$$

for $x, y \in X$ *with* $d(Fx, Fy) > 0$ *where* $\Theta \in \Omega$*. Then* F *has a unique fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = d(x, y) \quad \text{and} \quad \eta(x, y) = \frac{1}{2(1+\tau)}d(x, y)$$

for all $x, y \in X$ where $\tau > 0$. Now, since, $\frac{1}{2(1+\tau)}d(x, y) \leqslant d(x, y)$ for all $x, y \in X$, so $\eta(x, y) \leqslant \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.3 hold true. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Assume that $d(Fx_n, F^2x_n) = 0$ for some n. Then $Fx_n = F^2x_n$. That is Fx_n is a fixed point of F and we have nothing to prove. Hence we assume, $Fx_n \neq F^2x_n$ for all $n \in \mathbb{N}$. Since $\frac{1}{2(1+\tau)}d(Fx_n, F^2x_n) \leqslant d(Fx_n, F^2x_n)$ for all $n \in \mathbb{N}$, then from (2.15) we get

$$\Theta\big(d(F^2x_n, F^3x_n)\big) \leqslant [\Theta\big(d(Fx_n, F^2x_n)\big)]^k < \Theta\big(d(Fx_n, F^2x_n)\big),$$

and so from (Θ_1) we get,

$$d(F^2x_n, F^3x_n) < d(Fx_n, F^2x_n). \tag{2.16}$$

Assume there exists $n_0 \in \mathbb{N}$ such that

$$\eta(Fx_{n_0}, F^2x_{n_0}) > \alpha(Fx_{n_0}, x) \quad \text{and} \quad \eta(F^2x_{n_0}, F^3x_{n_0}) > \alpha(F^2x_{n_0}, x),$$

then

$$\frac{1}{2(1+\tau)}d(Fx_{n_0}, F^2x_{n_0}) > d(Fx_{n_0}, x) \quad \text{and} \quad \frac{1}{2(1+\tau)}d(F^2x_{n_0}, F^3x_{n_0}) > d(F^2x_{n_0}, x),$$

so by (2.16) we have,

$$\begin{aligned}
d(Fx_{n_0}, F^2x_{n_0}) &\leqslant d(Fx_{n_0}, x) + d(F^2x_{n_0}, x) \\
&< \frac{1}{2(1+\tau)}d(Fx_{n_0}, F^2x_{n_0}) + \frac{1}{2(1+\tau)}d(F^2x_{n_0}, F^3x_{n_0}) \\
&< \frac{1}{2(1+\tau)}d(Fx_{n_0}, F^2x_{n_0}) + \frac{1}{2(1+\tau)}d(Fx_{n_0}, F^2x_{n_0}) \\
&= \frac{2}{2(1+\tau)}d(Fx_{n_0}, F^2x_{n_0}) \leqslant d(Fx_{n_0}, F^2x_{n_0}),
\end{aligned}$$

which is a contradiction. Hence, either

$$\eta(Fx_n, F^2x_n) \leqslant \alpha(Fx_n, x) \text{ or } \eta(F^2x_n, F^3x_n) \leqslant \alpha(F^2x_n, x),$$

holds for all $n \in \mathbb{N}$. That is condition (iv) of Theorem 2.3 holds. Let $\eta(x, Fx) \leqslant \alpha(x, y)$. So, $\frac{1}{2(1+\tau)}d(x, Fx) \leqslant d(x, y)$. Then from (2.15) we get $\Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k$. Hence, all conditions of Theorem 2.3 hold and F has a unique fixed point. □

3. Applications

Theorem 3.1. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) *for* $x, y \in O(w)$ *with* $d(Fx, Fy) > 0$ *we have*

$$\Theta\big(d(Fx, Fy)\big) \leqslant [\Theta\big(d(x, y)\big)]^k,$$

where $\Theta \in \Omega$ *and* $k \in (0, 1)$;

(ii) F *is an orbitally continuous function.*

Then F *has a fixed point. Moreover,* F *has a unique fixed point when* $\text{Fix}(F) \subseteq O(w)$.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & \text{if } x,y \in O(w), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \eta(x,y) = 1,$$

where $O(w)$ is an orbit of a point $w \in X$. From Remark 1.4 we know that F is an α-η-continuous mapping. Let $\alpha(x,y) \geqslant \eta(x,y)$, then $x,y \in O(w)$. So $Fx, Fy \in O(w)$. That is, $\alpha(Fx,Fy) \geqslant \eta(Fx,Fy)$. Therefore, F is an α-admissible mapping with respect to η. Since $w, Fw \in O(w)$, then $\alpha(w,Fw) \geqslant \eta(w,Fw)$. Let $\alpha(x,y) \geqslant \eta(x,Fx)$ and $d(Fx,Fy) > 0$. Then, $x,y \in O(w)$ and $d(Fx,Fy) > 0$. Therefore from (i) we have

$$\Theta\big(d(Fx,Fy)\big) \leqslant [\Theta\big(d(x,y)\big)]^k,$$

which implies F is α-η-Θ-contraction mapping. Hence, all conditions of Theorem 2.2 hold true and F has a fixed point. If $\text{Fix}(F) \subseteq O(w)$, then, $\alpha(x,y) \geqslant \eta(x,y)$ for all $x,y \in \text{Fix}(F)$ and so from Theorem 2.2 F has a unique fixed point. □

Theorem 3.2. *Let* (X,d) *be a complete metric space and* $F : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) *for* $x \in X$ *with* $d(Fx, F^2x) > 0$ *we have,*

$$\Theta\big(d(Fx,F^2x)\big) \leqslant [\Theta\big(d(x,Fx)\big)]^k,$$

where $\Theta \in \Omega$ *and* $k \in (0,1)$;

(ii) F *is an orbitally continuous function.*

Then F *has the property* P.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in O(w), \\ 0, & \text{otherwise}, \end{cases}$$

where $w \in X$. Let $\alpha(x,y) \geqslant 1$, then $x,y \in O(w)$. So $Fx, Fy \in O(w)$. That is, $\alpha(Fx,Fy) \geqslant 1$. Therefore, F is α-admissible mapping. Since $w, Fw \in O(w)$, so $\alpha(w,Fw) \geqslant 1$. By Remark 1.4 we conclude that F is α-continuous mapping. If $x \in X$ with $d(Fx,F^2x) > 0$, then, from (i) we have

$$\Theta\big(d(Fx,F^2x)\big) \leqslant [\Theta\big(d(x,Fx)\big)]^k.$$

Thus all conditions of Theorem 2.5 hold true and F has the property P. □

We can easily deduce following results involving integral inequalities.

Theorem 3.3. *Let* (X,d) *be a complete metric space and* F *be a continuous self-mapping on* X. *If for* $x,y \in X$ *with*

$$\int_0^{d(x,Fx)} \rho(t)dt \leqslant \int_0^{d(x,y)} \rho(t)dt \quad \textit{and} \quad \int_0^{d(Fx,Fy)} \rho(t)dt > 0,$$

we have

$$\Theta\Big(\int_0^{d(Fx,Fy)} \rho(t)dt\Big) \leqslant [\Theta\Big(\int_0^{d(x,y)} \rho(t)dt\Big)]^k,$$

where $\Theta \in \Omega$, $k \in (0,1)$ *and* $\rho : [0,\infty) \to [0,\infty)$ *is a Lebesgue-integrable mapping satisfying* $\int_0^{\varepsilon} \rho(t)dt > 0$ *for* $\varepsilon > 0$. *Then* F *has a unique fixed point.*

Theorem 3.4. *Let* (X, d) *be a complete metric space and* F *be a self-mapping on* X. *Assume that there exists some* $k \in (0, 1)$ *such that*

$$\frac{1}{2(1+\tau)} \int_0^{d(x, Fx)} \rho(t)dt \leqslant \int_0^{d(x,y)} \rho(t)dt \Rightarrow \Theta\Big(\int_0^{d(Fx, Fy)} \rho(t)dt\Big) \leqslant \Big[\Theta\Big(\int_0^{d(x,y)} \rho(t)dt\Big)\Big]^k$$

for $x, y \in X$ *with* $\int_0^{d(Fx, Fy)} \rho(t)dt > 0$ *where* $\Theta \in \Omega$ *and* $\rho : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping satisfying* $\int_0^{\varepsilon} \rho(t)dt > 0$ *for* $\varepsilon > 0$. *Then* F *has a unique fixed point.*

Theorem 3.5. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) *for* $x, y \in O(w)$ *with* $\int_0^{d(Fx, Fy)} \rho(t)dt > 0$ *we have*

$$\Theta\Big(\int_0^{d(Fx, Fy)} \rho(t)dt\Big) \leqslant \Big[\Theta\Big(\int_0^{d(x,y)} \rho(t)dt\Big)\Big]^k,$$

where $\Theta \in \Omega$, $k \in (0, 1)$ *and* $\rho : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping satisfying* $\int_0^{\varepsilon} \rho(t)dt > 0$ *for* $\varepsilon > 0$;

(ii) F *is an orbitally continuous function.*

Then F *has a fixed point. Moreover,* F *has a unique fixed point when* $\mathrm{Fix}(F) \subseteq O(w)$.

Theorem 3.6. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) *for* $x \in X$ *with* $\int_0^{d(Fx, F^2x)} \rho(t)dt > 0$ *we have*

$$\Theta\Big(\int_0^{d(Fx, F^2x)} \rho(t)dt\Big) \leqslant \Big[\Theta\Big(\int_0^{d(x, Fx)} \rho(t)dt\Big)\Big]^k,$$

where $\Theta \in \Omega$, $k \in (0, 1)$ *and* $\rho : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping satisfying* $\int_0^{\varepsilon} \rho(t)dt > 0$ *for* $\varepsilon > 0$;

(ii) F *is an orbitally continuous function.*

Then F *has the property* P.

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