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Essential norm of a product-type operator from Bergman space to weighted-type space

Ya-Song Chen[a], Zhong-Shan Fang[a], Yeong-Cheng Liou[b,c,*]

[a] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, P. R. China.*

[b] *Department of Healthcare Administration and Medical Informatics, Center for Big Data Analytics & Intelligent Healthcare, and Research Center of Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 807, Taiwan.*

[c] *Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 807, Taiwan.*

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Abstract

In this paper, we discuss the boundedness of a product-type operator introduced by Stević, which acting from Bergman space to the weighted-type spaces or the little weighted-type spaces in the unit ball, and characterize the the essential norm of the product-type operator. From which the sufficient and necessary condition of compactness of this type operator follows immediately.

Keywords: Product-type operator, Bergman space, weighted-type space, unit ball.
2010 MSC: 47B38, 47B33, 45P0.

1. Introduction

Let $\mathbb{B}_n$ be the unit ball of C^n with boundary $\partial\mathbb{B}_n$. The class of all holomorphic functions on domain $\mathbb{B}_n$ will be denoted by $H(\mathbb{B}_n)$. For any $m \in \mathbb{N}$, the operator $\mathcal{R}^m : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ is defined as follows:

$$\mathcal{R}^m(f)(z) = \sum_{k=1}^{\infty} k^m f_k(z), \quad f(z) = \sum_{k=1}^{\infty} f_k(z)$$

for any $f \in H(\mathbb{B}_n)$ with the homogeneous expansions and $z \in \mathbb{B}_n$. It is easy to see that

$$\mathcal{R}f = \langle \nabla f(z), \bar{z} \rangle,$$

where

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right).$$

*Corresponding author
Email addresses: yasongchen@126.com (Ya-Song Chen), fangzhongshan@aliyun.com (Zhong-Shan Fang), simplex_liou@hotmail.com (Yeong-Cheng Liou)

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The weighted-type space $H_\mu^\infty = H_\mu^\infty(\mathbb{B}_n)$ consists of all $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}_n} \mu(z)|f(z)| < \infty,$$

where μ is a positive continuous function on $\mathbb{B}_n$ (weight).

We denote the little weighted-type space by $H_{\mu,0}^\infty$ the subspace of the H_μ^∞ consisting of the those $f \in H_\mu^\infty$ such that $\lim_{|z|\to 1} \mu(z)|f(z)| = 0$. When $\mu(r) = (1-r^2)^\alpha$, it is the space H_α^∞.

The Bergman space A_α^2 consists of those holomorphic functions such that

$$\|f\|_\alpha := \left[\int_{\mathbb{B}_n} |f(z)|^2 \, dv_\alpha(z)\right]^{1/2} < \infty,$$

where dv_α is a probability measure on $\mathbb{B}_n$.

Let φ be a holomorphic self-map of $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, and $m \in \mathbb{N}$. For $f \in H(\mathbb{B}_n)$, the product-type operator is defined by

$$\mathcal{R}_{u,\varphi}^m(f)(z) = u(z)\mathcal{R}^m f(\varphi(z)), \quad z \in \mathbb{B}_n,$$

that is, it is the product of the multiplication, composition and the iterated radial operators $M_u \circ C_\varphi \circ \mathcal{R}^m$.

In 2011, Stević [4] characterized the boundedness and compactness of this type operator from Bloch-type to weighted-type spaces. Motivated by it, here we study the essential norm of this operator from the Bergman space to weighted-type space.

Throughout the remainder of this paper, constants C or C_m depending only on m, are positive and may differ from one occurrence to the other. We write $a \asymp b$, if there exist two positive constants c and C such that $cb \leqslant a \leqslant Cb$ holds.

The essential norm of a continuous linear operator T is defined by $\|T\|_e := \inf\{\|T-K\|; K \text{ is compact}\}$. Since $\|T\|_e = 0$ if and only if T is compact, the estimates on $\|T\|_e$ will lead to conditions for T to be compact.

2. Some lemmas

To begin the discussion, we will state several useful lemmas, which are used in the proofs of the main results.

The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 of [1].

Lemma 2.1. *Assume that $m \in \mathbb{N}$, φ is a holomorphic self-map of $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, and μ is a weight. Then the operator $\mathcal{R}_{u,\varphi}^m : A_\alpha^2 \to H_\mu^\infty$ is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset A_\alpha^2$ converging to zero uniformly on compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|\mathcal{R}_{u,\varphi}^m f_k\|_{H_\mu^\infty} = 0.$$

Lemma 2.2 ([6, Theorem 2.1])**.** *Suppose $\alpha > -1$. Then*

$$|f(z)| \leqslant \frac{\|f\|_\alpha}{(1-|z|^2)^{(n+1+\alpha)/2}}$$

for all $f \in A_\alpha^2$ and $z \in \mathbb{B}_n$.

Lemma 2.3. *Suppose that $f \in A_\alpha^2$, and for fixed $0 < \delta < 1$, let $G = \{z \in \mathbb{B}_n : |z| \leqslant \delta\}$. Then*

$$\lim_{r\to 1} \sup_{\|f\|_\alpha \leqslant 1} \sup_{z\in G} |\mathcal{R}^m(z) - \mathcal{R}^m(rz)| = 0.$$

Proof.

$$\begin{aligned}\sup_{z\in G}|\mathcal{R}^m(z)-\mathcal{R}^m(rz)| &\leqslant \sup_{z\in G}\left|\int_r^1 \frac{\mathcal{R}^{m+1}f(tz)}{t}\,dt\right| \\ &\leqslant \sup_{z\in G}\int_r^1\left|\frac{\mathcal{R}^{m+1}f(tz)(1-|z|^2)^{m+1+(n+1+\alpha)/2}}{t(1-|z|^2)^{m+1+(n+1+\alpha)/2}}\right|dt \\ &\leqslant C_{m+1}(1-r)\|f\|_\alpha \sup_{z\in G}\frac{1}{(1-|z|^2)^{m+1+(n+1+\alpha)/2}} \\ &\leqslant \frac{C_{m+1}(1-r)\|f\|_\alpha}{(1-\delta^2)^{m+1+(n+1+\alpha)/2}}.\end{aligned}$$

The lemma follows as $r\to 1$. □

By Theorem 2.19 in [7], we obtain the following lemma immediately.

Lemma 2.4. *Assume that $m\in\mathbb{N}$ and holomorphic functions $f\in A^2_\alpha$. Then the following asymptotic relations hold*

$$\|f\|^2_\alpha \asymp \int_{\mathbb{B}_n}\left|(1-|z|^2)^m\mathcal{R}^m f(z)\right|^2 dv_\alpha(z)\asymp \sum_{|k|\leqslant m}\left|\frac{\partial^k f}{\partial z^k}(0)\right| + \sum_{|k|=m}\int_{\mathbb{B}_n}\left|(1-|z|^2)^m\frac{\partial^k f}{\partial z^k}(z)\right|^2 dv_\alpha(z).$$

Lemma 2.5. *For any $z\in\mathbb{B}_n$, let*

$$f_w(z)=\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}.$$

Then

$$\mathcal{R}^m f_w(z)=\frac{P_m(\langle z,w\rangle)}{(1-\langle z,w\rangle)^{n+1+\alpha+m}},$$

where

$$P_m(t)=p^{(m)}_m t^m+p^{(m)}_{m-1}t^{m-1}+\cdots+p^{(m)}_2 t^2+(n+1+\alpha)t,$$

with nonnegative numbers $p^{(m)}_j, j=1,2,\ldots,m$.

Proof. A direct calculation with mathematical induction leads to the lemma. □

Lemma 2.6. *For any fixed $0<r<1$, assume that the operator $\mathcal{R}^m_{u,\varphi}: A^2_\alpha\to H^\infty_\mu$ is bounded, then $\mathcal{R}^m_{u,r\varphi}$ is compact.*

Proof. Assume that $\mathcal{R}^m_{u,\varphi}$ is bounded. By taking the test functions $f_i(z_1,\ldots,z_n)=z_i\in A^2_\alpha, i=1,2,\ldots,n$, we get

$$\sup_{z\in\mathbb{B}_n}\mu(z)|u(z)||\varphi(z)|=\sum_{i=1}^n\sup_{z\in\mathbb{B}_n}\mu(z)|u(z)||\varphi_i(z)|=\left\|\mathcal{R}^m_{u,\varphi}f_i\right\|_{H^\infty_\mu}<\infty.$$

For every bounded sequence $(f_k)_{k\in\mathbb{N}}\subset A^2_\alpha$ converging to 0 uniformly on compact subsets of $\mathbb{B}_n$ as $k\to\infty$, we have

$$\begin{aligned}\lim_{k\to\infty}\|\mathcal{R}^m_{u,r\varphi}f_k\|_{H^\infty_\mu}=\sup_{z\in\mathbb{B}_n}\mu(z)|u(z)||\mathcal{R}^m f_k(r\varphi(z))| &\leqslant \sup_{z\in\mathbb{B}_n}\mu(z)|u(z)||r\varphi(z)||\nabla\mathcal{R}^{m-1}f_k(r\varphi(z))| \\ &\leqslant \sup_{|\zeta|\leqslant r}|\nabla\mathcal{R}^{m-1}f_k(\zeta)|\sup_{z\in\mathbb{B}_n}\mu(z)|u(z)||\varphi(z)|.\end{aligned}$$

Since $(f_k)_{k\in\mathbb{N}}$ converges to zero uniformly on compact subsets, the sequence $(|\nabla\mathcal{R}^{m-1}f_k|)_{k\in\mathbb{N}}$ also converges to zero on compacts for any $m\in\mathbb{N}$. So by Lemma 2.1, $\mathcal{R}^m_{u,r\varphi}$ is compact. □

3. Main theorems

In this section we characterize the boundedness of the operator $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_\mu$. We also give the essential norm estimates of the operators $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_\mu \ (H^\infty_{\mu,0})$.

Theorem 3.1. *Assume that fixed $m \in \mathbb{N}$, φ is a holomorphic self-map of $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, and μ is a weight. Then the operator $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}_n} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} < \infty.$$

Proof. Note that $\mathcal{R}^{m-1}f \in A^2_{\alpha+2m-2}$, whenever $f \in A^2_\alpha$. And by Lemma 2.4, we have $\frac{\partial \mathcal{R}^{m-1}f}{\partial z_k} \in A^2_{\alpha+2m}$, and

$$\begin{aligned}
\mu(z)|\mathcal{R}^m_{u,\varphi}f| &\leqslant \mu(z)|u(z)||\varphi(z)||\nabla\mathcal{R}^{m-1}f(\varphi(z))| \\
&\leqslant \mu(z)|u(z)||\varphi(z)| \left(\sum_{k=1}^n \left|\frac{\partial\mathcal{R}^{m-1}f}{\partial w_k}(\varphi(z))\right|\right) \\
&\leqslant \mu(z)|u(z)||\varphi(z)| \frac{\sum_{k=1}^n \|\frac{\partial\mathcal{R}^{m-1}f}{\partial z_k}\|_{\alpha+2m}}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} \\
&\leqslant n\mu(z)|u(z)||\varphi(z)| \frac{C_m\|f\|_\alpha}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}}.
\end{aligned}$$

The last inequality comes from Lemma 2.2. So the sufficiency follows by the above estimate.

The proof of necessity is analogous to Theorem 1 of [3]. We give the proof here for completeness. Assume that $\mathcal{R}^m_{u,\varphi}$ is bounded. Taking the test functions $f_i(z_1,\dots,z_n) = z_i \in A^2_\alpha, i = 1,2,\dots,n$, we get

$$\sup_{z\in\mathbb{B}_n} \mu(z)|u(z)||\varphi(z)| = \sum_{i=1}^n \sup_{z\in\mathbb{B}_n} \mu(z)|u(z)||\varphi_i(z)| = \sum_{i=1}^n \left\|\mathcal{R}^m_{u,\varphi}f_i\right\|_{H^\infty_\mu} < \infty.$$

The proof can be divided into two cases.

Case 1: if $|\varphi(z)| \leqslant \frac{1}{2}$, then

$$\sup_{|\varphi(z)|<1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} \leqslant C \sup_{|\varphi(z)|<1/2} \mu(z)|u(z)||\varphi(z)| < \infty.$$

Case 2: if $|\varphi(z)| > \frac{1}{2}$, let

$$f_{\varphi(w)}(z) = \frac{(1-|\varphi(w)|^2)^{(n+1+\alpha)/2}}{(1-\langle z, \varphi(w)\rangle)^{n+1+\alpha}}.$$

The boundedness of $\mathcal{R}^m_{u,\varphi}$ implies that for each $w \in \mathbb{B}_n$,

$$\begin{aligned}
C \geqslant \|\mathcal{R}^m_{u,\varphi}f_{\varphi(w)}\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{B}_n} \mu(z)|u(z)||\mathcal{R}^m f_{\varphi(w)}(\varphi(z))| \\
&\geqslant C\frac{\mu(w)|u(w)|P_m(|\varphi(w)|^2)}{(1-|\varphi(w)|^2)^{m+(n+1+\alpha)/2}} \geqslant C\frac{\mu(w)|u(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^{m+(n+1+\alpha)/2}}.
\end{aligned}$$

Therefore,

$$C \geqslant \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u(z)||\varphi(z)|^2}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} \geqslant \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{2(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}}.$$

Combining the two cases, the theorem follows. □

Theorem 3.2. *Assume that* $m \in \mathbb{N}$, φ *is a holomorphic self-map of* $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, *and* μ *is a weight. Suppose that* $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_\mu$ *is bounded, then the essential norm satisfies*

$$\|\mathcal{R}^m_{u,\varphi}\|_e \asymp \lim_{\delta\to 1} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}}.$$

Proof. We first consider the upper estimate. For fixed $0 < r < 1$, and for any $0 < \delta < 1$,

$$\begin{aligned}\|\mathcal{R}^m_{u,\varphi}\|_e \leqslant \|\mathcal{R}^m_{u,\varphi} - \mathcal{R}^m_{u,r\varphi}\| &= \sup_{\|f\|_\alpha \leqslant 1} \|(\mathcal{R}^m_{u,\varphi} - \mathcal{R}^m_{u,r\varphi})f\|_{H^\infty_\mu} \\ &\leqslant \sup_{\|f\|_\alpha \leqslant 1} \sup_{|\varphi(z)|\leqslant\delta} \mu(z)|u(z)||\mathcal{R}^m f(\varphi(z)) - \mathcal{R}^m f(r\varphi(z))| \\ &\quad + \sup_{\|f\|_\alpha \leqslant 1} \sup_{|\varphi(z)|>\delta} \mu(z)|u(z)||\mathcal{R}^m f(\varphi(z)) - \mathcal{R}^m f(r\varphi(z))|.\end{aligned}$$

From Lemma 2.3, we can choose r sufficiently close to 1 such that the first term of the right-hand side is less than any given ϵ, and denote the second term by I. Using the estimate in Theorem 3.1, then

$$\begin{aligned} I &\leqslant \sup_{\|f\|_\alpha \leqslant 1} \sup_{|\varphi(z)|>\delta} \mu(z)|u(z)|\left(|\mathcal{R}^m f(\varphi(z))| + |\mathcal{R}^m f(r\varphi(z))|\right) \\ &\leqslant \sup_{|\varphi(z)|>\delta} n\mu(z)|u(z)| \left(\frac{C_m|\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} + \frac{C_m r|\varphi(z)|}{(1-r|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} \right) \\ &\leqslant \sup_{|\varphi(z)|>\delta} \frac{2nC_m\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}}.\end{aligned}$$

First let $r \to 1$ and then $\delta \to 1$, the upper estimate follows.

Now we turn to the lower estimate. Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}_n$ be such that $|\varphi(z_k)| \to 1$ as $k \to \infty$, and

$$f_k(z) = \frac{(1-|\varphi(z_k)|^2)^{(n+1+\alpha)/2}}{(1-\langle z, \varphi(z_k)\rangle)^{n+1+\alpha}}.$$

It is easy to check that $\sup_{k\in\mathbb{N}} \|f_k\|_\alpha < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}_n$. Then for any compact operator K,

$$\begin{aligned} C\|\mathcal{R}^m_{u,\varphi} - K\| &\geqslant \lim_{k\to\infty} \|(\mathcal{R}^m_{u,\varphi} - K)f_k\|_{H^\infty_\mu} \\ &\geqslant \lim_{k\to\infty} \|\mathcal{R}^m_{u,\varphi} f_k\|_{H^\infty_\mu} - \lim_{k\to\infty} \|Kf_k\|_{H^\infty_\mu} \\ &\geqslant \lim_{k\to\infty} \sup_{z\in\mathbb{B}_n} \mu(z)|u(z)||\mathcal{R}^m f_{\varphi(z_k)}(\varphi(z))| - 0 \\ &\geqslant \lim_{k\to\infty} C\frac{\mu(z_k)|u(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{m+(n+1+\alpha)/2}}.\end{aligned}$$

By the definition of essential norm, the lower estimate is obtained. □

Corollary 3.3. *Assume that* $m \in \mathbb{N}$, φ *is a holomorphic self-map of* $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, *and* μ *is a weight. Then operator* $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_\mu$ *is compact if and only if*

$$\lim_{\delta\to 1} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} = 0.$$

Theorem 3.4. *Assume that* $m \in \mathbb{N}$, φ *is a holomorphic self-map of* $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, *and* μ *is a weight. Suppose* $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_{\mu,0}$ *is bounded, then the essential norm satisfies*

$$\|\mathcal{R}^m_{u,\varphi}\|_e \asymp \lim_{\delta\to 1} \sup_{|z|>\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}}.$$

Proof. If $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_{\mu,0}$ is bounded, then as in Lemma 2.6, we have

$$\lim_{|z|\to 1} \mu(z)|u(z)||\varphi(z)| = 0.$$

Notice that the proof of Theorem 3.2 is still valid for this case, and a little modification of Proposition 2.2 of [2] shows that " $\lim\limits_{\delta\to 1} \sup\limits_{|\varphi(z)|>\delta}$ " could be replaced by " $\lim\limits_{\delta\to 1} \sup\limits_{|z|>\delta}$ ". □

Corollary 3.5. *Assume that* $m \in \mathbb{N}, \varphi$ *is a holomorphic self-map of* $\mathbb{B}_n$, $u \in H(\mathbb{B}_n)$, *and* μ *is a weight. Then operator* $\mathcal{R}^m_{u,\varphi} : A^2_\alpha \to H^\infty_{\mu,0}$ *is compact if and only if*

$$\lim_{\delta\to 1} \sup_{|z|>\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{m+(n+1+\alpha)/2}} = 0.$$

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