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Boundedness of high order commutators of Marcinkiewicz integrals associated with Schrödinger operators

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Abstract

Suppose $L = -\Delta + V$ is a Schrödinger operator on $\mathbb{R}^n$, where $n \geqslant 3$ and the nonnegative potential V belongs to reverse Hölder class RH_n. Let b belong to a new Campanato space $\Lambda_\beta^\theta(\rho)$, and let μ_j^L be the Marcinkiewicz integrals associated with L. In this paper, we establish the boundedness of the m-order commutators $[b^m, \mu_j^L]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q = 1/p - m\beta/n$ and $1 < p < n/(m\beta)$. As an application, we obtain the boundedness of $[b^m, \mu_j^L]$ on the generalized Morrey spaces related to certain nonnegative potentials.

Keywords: Schrödinger operator, Marcinkiewicz integral, commutator, Campanato space, Morrey space.
2010 MSC: 42B25, 35J10.

1. Introduction and results

In this paper we consider the Schrödinger operator

$$L = -\Delta + V \ \text{ on } \ \mathbb{R}^n, \ \ n \geqslant 3,$$

where V is a nonnegative potential. We will assume that V belongs to a reverse Hölder class RH_q for some $q \geqslant n/2$, that is to say, V satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)dy$$

for any balls $B \subset \mathbb{R}^n$.

As in [10], for a given potential $V \in RH_q$ with $q \geqslant n/2$, we define the auxiliary function

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leqslant 1\right\}, \ \ x \in \mathbb{R}^n.$$

It is well-known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

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Let $\theta > 0$ and $0 < \beta < 1$, in view of [6], the new Campanato class $\Lambda_\beta^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}}\int_{B(x,r)}|b(y)-b_B|dy \leqslant C\left(1+\frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\beta^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda_\beta^\theta(\rho)$ is the classical Campanato space; if $\beta = 0$, $\Lambda_\beta^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [1].

We define the Marcinkiewicz integral associated with the Schrödinger operator L by

$$\mu_j^L f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} K_j^L(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j^L = \frac{\partial}{\partial x_j}L^{-1/2}, j = 1,\cdots,n$.

Let b be a locally integrable function and m be a positive integer. The m-order commutator generated by μ_j^L and b is defined by

$$[b^m,\mu_j^L]f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} K_j^L(x,y)(b(x)-b(y))^m f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Let $\widetilde{K_j^\Delta}(x,y)$ denote the kernel of the classical Riesz transform $R_j = \frac{\partial}{\partial x_j}\Delta^{-1/2}$. Then $K_j^\Delta(x,y) = \widetilde{K_j^\Delta}(x,y)|x-y| = \frac{(x_j-y_j)/|x-y|}{|x-y|^{n-1}}$. Obviously, $\mu_j^\Delta f(x)$ is the classical Marcinkiewicz integral. Therefore, it will be an interesting thing to study the property of μ_j^L.

The area of Marcinkiewicz integral associated with the Schrödinger operator has been under intensive research recently. Gao and Tang in [5] showed that μ_j^L is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. When b belongs to $BMO_\theta(\rho)$, Chen and Zou in [3] proved that the commutator $[b,\mu_j^L]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chen and Jin in [2] investigated the boundedness of $[b,\mu_j^L]$ on some Morrey space related to nonnegative potential V. In this paper, we consider the boundedness of m-order commutator $[b^m,\mu_j^L]$ on $L^p(\mathbb{R}^n)$ when b belongs to the new Campanato class $\Lambda_\beta^\theta(\rho)$, and get the following result.

Theorem 1.1. *Let* $V \in RH_n$. *Then for any* $b \in \Lambda_\beta^\theta(\rho), 0 < \beta < 1$, *the commutator* $[b^m,\mu_j^L]$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$, *and*

$$\|[b^m,\mu_j^L]f\|_{L^q(\mathbb{R}^n)} \leqslant C([b]_\beta^\theta)^m\|f\|_{L^p(\mathbb{R}^n)},$$

where $\frac{1}{q} = \frac{1}{p}-\frac{m\beta}{n}, 1 < p < \frac{n}{m\beta}$.

The classical Morrey space was introduced by Morrey in [8], since then a large number of investigations have been given to them by mathematicians. It is well-known that the classical Morrey space plays an important role in the theory of partial differential equations. In [2], Chen and Jin showed the boundedness of μ_j^L and $[b,\mu_j^L]$ on the Morrey spaces related to certain nonnegative potentials. In [9], we introduced the generalized Morrey space related to nonnegative potential V, which covers the general Morrey space; see [2, 7, 8, 11].

Definition 1.2 ([9])**.** Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty), 1 \leqslant p < \infty, \alpha \geqslant 0$, and $V \in RH_q(q \geqslant n/2)$. We denote by $M_{p,\varphi}^{\alpha,V} = M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ the generalized Morrey space related to nonnegative potential V, the space of all functions $f \in L_{loc}^p(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1}r^{-n/p}\|f\|_{L^p(B(x,r))}.$$

As an application of Theorem 1.1, we consider the boundedness of $[b^m, \mu_j^L]$ on $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$, get the following result.

Theorem 1.3. *Let $V \in RH_n, b \in \Lambda^\theta_\beta(\rho), 0 < \beta < 1$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leqslant c_0 \varphi_2(x,r), \tag{1.1}$$

where c_0 does not depend on x and r. Then the operator $[b^m, \mu_j^L]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ to $M^{\alpha,V}_{q,\varphi_2}(\mathbb{R}^n)$, and

$$\left\|[b^m, \mu_j^L]f\right\|_{M^{\alpha,V}_{q,\varphi_2}} \leqslant C([b]^\theta_\beta)^m \|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where $\frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}, 1 < p < \frac{n}{m\beta}$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leqslant CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

Proposition 2.1 ([10])**.** *Let $V \in RH_{n/2}$. For the function ρ there exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

Assume that $Q = B(x_0, \rho(x_0))$, for $x \in Q$, Proposition 2.1 tells us that $\rho(x) \approx \rho(y)$, if $|x - y| < C\rho(x)$.

Lemma 2.2 ([9])**.** *Let $k \in \mathbb{N}$ and $x \in 2^{k+1}B(x_0, r) \setminus 2^k B(x_0, r)$. Then we have*

$$\frac{1}{\left(1 + \frac{2^k r}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}.$$

Proposition 2.3 ([4])**.** *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k)), k \geqslant 1$, satisfies*

(i) $\bigcup_k Q_k = \mathbb{R}^n$;

(ii) *there exists $N = N(\rho)$ such that for every $k \in N$,* $\text{card}\{j : 4Q_j \cap 4Q_k\} \leqslant N$.

For $\alpha > 0$, $g \in L^1_{loc}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, we introduce the following maximal functions

$$M_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g(y)|dy, \quad M^\sharp_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g(y) - g_B|dy,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(z,r) : z \in \mathbb{R}^n \text{ and } r \leqslant \alpha\rho(y)\}$.

We have the following Fefferman-Stein type inequality.

Proposition 2.4 ([1])**.** *For $1 < p < \infty$, there exist δ and β such that if $\{Q_k\}_{k=1}^\infty$ is a sequence of balls as in Proposition 2.3 then*

$$\int_{\mathbb{R}^n} |M_{\rho,\delta}g(x)|^p dx \lesssim \int_{\mathbb{R}^n} |M^\sharp_{\rho,\beta}g(x)|^p dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|}\int_{2Q_k} |g|\right)^p$$

for all $g \in L^1_{loc}(\mathbb{R}^n)$.

We give an inequality for the function $b \in \Lambda_\beta^\theta(\rho)$.

Lemma 2.5 ([6])**.** *Let* $1 \leqslant s < \infty, b \in \Lambda_\beta^\theta(\rho)$, *and* $B = B(x,r)$. *Then*

$$\left(\frac{1}{|2^kB|}\int_{2^kB}|b(y)-b_B|^s dy\right)^{1/s} \leqslant C[b]_\beta^\theta(2^kr)^\beta\left(1+\frac{2^kr}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$, *where* $\theta' = (k_0+1)\theta$ *and* k_0 *is the constant appearing in Proposition 2.1.*

The following proposition gives some estimates on the kernel of μ_j^L.

Proposition 2.6 ([10])**.** *Suppose* $V \in RH_q$.

(i) *If* $q \geqslant n$, *then for every* N, *there exists a constant* C *such that*

$$\left|K_j^L(x,z)\right| \leqslant \frac{C\left(1+|x-z|/\rho(x)\right)^{-N}}{|x-z|^{n-1}}.$$

(ii) *If* $q \geqslant n$, *then for every* N *and* $0 < \delta < 1-n/q$, *there exists a constant* C *such that*

$$|K_j^L(x,z)-K_j^L(y,z)| \leqslant \frac{C|x-y|^\delta\left(1+|x-z|/\rho(x)\right)^{-N}}{|x-z|^{n-1+\delta}},$$

where $|x-y| < \frac{2}{3}|x-z|$.

3. Proof of Theorem 1.1

We first prove the following lemmas.

Lemma 3.1. *Let* $V \in RH_n$, $b \in \Lambda_\beta^\theta(\rho)$, *and* $Q = B(x_0,\rho(x_0))$. *Then for any* $1 < s < \infty$,

$$\frac{1}{|Q|}\int_Q|[b^m,\mu_j^L]f| \lesssim ([b]_\beta^\theta)^m \inf_{x\in Q} M_{m\beta,s}(f)(x) + \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}\inf_{x\in Q} M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x)$$

holds for all $f \in L^s(\mathbb{R}^n)$, *where*

$$M_{m\beta,s}(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-m\beta s/n}}\int_B|f(y)|^s dy\right)^{1/s}.$$

Proof. By Binomial Theorem we have

$$\begin{aligned}(b(y)-b(z))^m &= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(z))^{m-l}+(\lambda-b(z))^m\\ &= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(y)+b(y)-b(z))^{m-l}+(\lambda-b(z))^m\\ &= \sum_{l=1}^m\sum_{h=0}^{m-l} C_{l,m,h}(b(y)-\lambda)^{l+h}(b(y)-b(z))^{m-l-h}+(\lambda-b(z))^m\\ &= \sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y)-\lambda)^{m-\gamma}(b(y)-b(z))^\gamma+(\lambda-b(z))^m,\end{aligned}$$

then

$$[b^m,\mu_j^L]f(y) = \left(\int_0^\infty \left|\int_{|y-z|\leqslant t} K_j^L(y,z)(b(y)-b(z))^m f(z)dz\right|^2 \frac{dt}{t^3}\right)^{1/2}$$
$$\leqslant \sum_{\gamma=0}^{m-1} C_{\gamma,m}|b(y)-\lambda|^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)+\mu_j^L((b-\lambda)^m f)(y).$$

Let $\lambda = b_{2Q}$. Then by Hölder's inequality and Lemma 2.5 we get

$$\frac{1}{|Q|}\int_Q\left|\sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y)-\lambda)^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)\right|dy$$
$$\lesssim \sum_{\gamma=0}^{m-1}\frac{1}{|Q|}\int_Q \left|(b(y)-b_{2Q})^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)\right|dy$$
$$\lesssim \sum_{\gamma=0}^{m-1}\left(\frac{1}{|Q|}\int_Q |b(y)-b_{2Q}|^{(m-\gamma)s'}dy\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |[b^\gamma,\mu_j^L]f(y)|^s dy\right)^{1/s}$$
$$\lesssim \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}(\rho(x_0))^{\beta(m-\gamma)}\left(\frac{1}{|Q|}\int_Q |[b^\gamma,\mu_j^L]f(y)|^s dy\right)^{1/s}$$
$$\lesssim \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}\inf_{x\in Q} M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x),$$

where $1<s<\infty$, and $1/s+1/s'=1$.

For the second term, we split $f=f_1+f_2$ with $f_1=f\chi_{2Q}$. Let $1<\tilde{s}<s<\infty$, and $\nu=s\tilde{s}/(s-\tilde{s})$, by the boundedness of μ_j^L on $L^{\tilde{s}}(\mathbb{R}^n)$, Hölder's inequality, and Lemma 2.5 we obtain

$$\frac{1}{|Q|}\int_Q |\mu_j^L((b-b_{2Q})^m f_1)(y)|dy \leqslant \left(\frac{1}{|Q|}\int_Q |\mu_j^L((b-b_{2Q})^m f_1)(y)|^{\tilde{s}}\right)^{1/\tilde{s}}$$
$$\lesssim \left(\frac{1}{|Q|}\int_{2Q} |(b(y)-b_{2Q})^m f(y)|^{\tilde{s}}dy\right)^{1/\tilde{s}}$$
$$\lesssim \left(\frac{1}{|Q|}\int_{2Q}|f(y)|^s dy\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_{2Q}|^{m\nu}dy\right)^{1/\nu}$$
$$\lesssim ([b]_\beta^\theta)^m \inf_{x\in Q} M_{m\beta,s}f(x).$$

For the remaining term, note that $\rho(y)\approx\rho(x_0)$ for any $y\in Q$, by Proposition 2.6, Minkowski's inequality, and Lemma 2.5 we get

$$|\mu_j^L((b-b_{2Q})^m f_2)(y)| \lesssim \left(\int_0^\infty \left|\int_{|y-z|\leqslant t}\frac{|f_2(z)||b(z)-b_{2Q}|^m}{|y-z|^{n-1}(1+|y-z|/\rho(y))^N}dz\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$\leqslant \int_{\mathbb{R}^n}\frac{|f_2(z)||b(z)-b_{2Q}|^m}{|y-z|^{n-1}(1+|y-z|/\rho(y))^N}\left(\int_{|y-z|\leqslant t}\frac{dt}{t^3}\right)^{1/2}dz$$
$$\lesssim \rho(x_0)^N\int_{(2Q)^c}\frac{|f(z)||b(z)-b_{2Q}|^m}{|y-z|^{n+N}}dz$$
$$\lesssim \rho(x_0)^N\sum_{k=1}^\infty \frac{(2^k\rho(x_0))^{-N}}{|2^{k+1}Q|}\int_{2^{k+1}Q\setminus 2^kQ}|f(z)||b(z)-b_{2Q}|^m dz$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{\infty} 2^{-kN} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)||b(z)-b_{2Q}|^m dz \\
&\lesssim \sum_{k=1}^{\infty} 2^{-kN} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(z)-b_{2Q}|^{ms'} dz \right)^{1/s'} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^s dz \right)^{1/s} \\
&\lesssim ([b]_\beta^\theta)^m \sum_{k=1}^{\infty} 2^{-kN} \inf_{x\in Q} M_{m\beta,s}(f)(x) \\
&\lesssim ([b]_\beta^\theta)^m \inf_{x\in Q} M_{m\beta,s}(f)(x).
\end{aligned}
$$

This finishes the proof of Lemma 3.1. □

Lemma 3.2. *Let* $V \in RH_n$ *and* $b \in \Lambda_\beta^\theta(\rho)$*, then for any* $s > 1$ *and* $\gamma \geqslant 1$*, there exists a constant* C *such that*

$$
\left|\mu_j^L((b-b_B)^m f_2)(u) - \mu_j^L((b-b_B)^m f_2)(z)\right| \leqslant C([b]_\beta^\theta)^m \inf_{x\in B} M_{m\beta,s} f(x)
$$

holds for all $f \in L^s_{loc}(\mathbb{R}^n)$*,* $u, z \in B = B(x_0, r)$ *with* $r < \gamma\rho(x_0)$ *and* $f_2 = f\chi_{(2B)^c}$*.*

Proof. We write

$$
\begin{aligned}
&\left|\mu_j^L((b-b_B)^m f_2)(u) - \mu_j^L((b-b_B)^m f_2)(z)\right| \\
&\quad \leqslant \left(\int_0^\infty \left| \int_{|u-y|\leqslant t<|z-y|} |K_j^L(u,y) f_2(y)(b(y)-b_{2B})^m| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left| \int_{|z-y|\leqslant t<|u-y|} |K_j^L(u,y) f_2(y)(b(y)-b_{2B})^m| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left| \int_{\{|u-y|\leqslant t, |z-y|\leqslant t\}} |K_j^L(u,y) - K_j^L(z,y)||f_2(y)(b(y)-b_{2B})^m| dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad = J_1 + J_2 + J_3.
\end{aligned}
$$

Due to the estimates for J_1 and J_2 which are similar, then we only consider J_1. Let $Q = B(x_0, \gamma\rho(x_0))$. Since $u, z \in Q$, then $\rho(u) \approx \rho(x_0)$ and $|u-y| \approx |z-y|$. By Minkowski's inequality and Proposition 2.6 we have

$$
\begin{aligned}
J_1 &\leqslant \int_{(2B)^c} |K_j^L(u,y) f(y)(b(y)-b_{2B})^m| \left(\int_{|u-y|\leqslant t<|z-y|} \frac{dt}{t^3} \right)^{1/2} dy \\
&\lesssim r^{1/2} \int_{(2B)^c} \frac{|K_j^L(u,y) f(y)(b(y)-b_{2B})^m|}{|u-y|^{3/2}} dy \\
&\lesssim r^{1/2} \int_{Q\setminus 2B} \frac{|f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+1/2}} dy + r^{1/2}\rho(x_0)^N \int_{Q^c} \frac{|f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+1/2+N}} dy \\
&= J_{11} + J_{12}.
\end{aligned}
$$

Let j_0 be the least integer such that $2^{j_0} \geqslant \gamma\rho(x_0)/r$. Splitting into annuli, we have

$$
J_{11} \leqslant \sum_{j=2}^{j_0} 2^{-j/2} \frac{1}{|2^j B|} \int_{2^j B} |f(y)||b(y)-b_{2B}|^m dy.
$$

By Hölder's inequality, Lemma 2.5, and noting that $2^j r \leqslant \gamma\rho(x_0)$ for $j < j_0$, then we have

$$\begin{aligned}\frac{1}{|2^jB|}\int_{2^jB}|f(y)||b(y)-b_{2B}|^m dy \leqslant&([b]_\beta^\theta)^m(2^jr)^{m\beta}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(y)|^s dy\right)^{1/s}\\ \lesssim&([b]_\beta^\theta)^m\inf_{x\in B}M_{m\beta,s}f(x).\end{aligned}$$

Thus

$$J_{11}\leqslant([b]_\beta^\theta)^m\sum_{j=2}^{j_0}2^{-j/2}\inf_{x\in B}M_{m\beta,s}f(x)\lesssim([b]_\beta^\theta)^m\inf_{x\in B}M_{m\beta,s}f(x).$$

Note that for $j \geqslant j_0$,

$$\begin{aligned}\frac{1}{|2^jB|}\int_{2^jB}|f(y)||b(y)-b_{2B}|^m dy \leqslant&([b]_\beta^\theta)^m(2^jr)^{m\beta}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(y)|^s dy\right)^{1/s}\\ \lesssim&([b]_\beta^\theta)^m\left(\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\inf_{x\in B}M_{m\beta,s}f(x).\end{aligned}$$

Then, by choosing $N \geqslant m\theta'$ we get

$$\begin{aligned}J_{12}&\leqslant r^{1/2}\rho(x_0)^N\int_{Q^c}\frac{|f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+1/2+N}}dy\\ &\leqslant r^{-N}\rho(x_0)^N\sum_{j=j_0}^{\infty}2^{-j(1/2+N)}\frac{1}{|2^jB|}\int_{2^jB}|f(y)||b(y)-b_{2B}|^m dy\\ &\lesssim([b]_\beta^\theta)^m\sum_{j=j_0}^{\infty}2^{-j/2}\left(\frac{\rho(x_0)}{2^jr}\right)^{N-m\theta'}\inf_{x\in B}M_{m\beta,s}f(x)\\ &\lesssim([b]_\beta^\theta)^m\inf_{x\in B}M_{m\beta,s}f(x).\end{aligned}$$

For J_3, note that $\rho(u)\approx\rho(x_0), |u-y|\approx|z-y|$, and $|u-z|<\frac{2}{3}|u-y|$, then by Minkowski's inequality and Proposition 2.6, similar to the estimates for J_{11} and J_{12}, we have

$$\begin{aligned}J_3&\leqslant\int_{(2B)^c}|f(y)||b(y)-b_{2B}|^m|K_j^L(u,y)-K_j^L(z,y)|\left(\int_{\{|u-y|\leqslant t,|z-y|\leqslant t\}}\frac{dt}{t^3}\right)^{1/2}dy\\ &\lesssim\int_{(2B)^c}\frac{|f(y)||b(y)-b_{2B}|^m|K_j^L(u,y)-K_j^L(z,y)|}{|u-y|}dy\\ &\lesssim r^\delta\int_{Q\backslash 2B}\frac{|f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\delta}}dy+r^\delta\rho(x_0)^N\int_{Q^c}\frac{|f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\delta+N}}dy\\ &\lesssim([b]_\beta^\theta)^m\inf_{x\in B}M_{m\beta,s}f(x).\end{aligned}$$

Then the proof of Lemma 3.2 is completed. □

Lemma 3.3. *Let* $s > 1$, $B = B(x_0, r)$ *with* $r \leqslant \gamma\eta(x_0)$, *and* $x \in B$. *Then*

$$M^\sharp_{\rho,\eta}([b^m,\mu_j^L]f)(x)\lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x)+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x).$$

Proof. Since

$$[b^m,\mu_j^L]f(y)=\sum_{\gamma=0}^{m-1}C_{\gamma,m}(b(y)-b_{2B})^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)+\mu_j^L((b-b_{2B})^m f)(y),$$

then

$$\begin{aligned}
&\frac{1}{|B|}\int_B\left|[b^m,\mu_j^L]f(y)-([b^m,\mu_j^L]f)_B\right|dy\\
&\lesssim\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)-\left((b(y)-b_{2B})^{m-\gamma}[b^\gamma,\mu_j^L](f)\right)_B\right|dy\\
&\quad+\frac{1}{|B|}\int_B\left|\mu_j^L((b-b_{2B})^m f)(y)-(\mu_j^L((b-b_{2B})^m f))_B\right|dy\\
&\lesssim\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)\right|dy\\
&\quad+\frac{1}{|B|}\int_B\left|\mu_j^L((b-b_{2B})^m f)(y)-(\mu_j^L((b-b_{2B})^m f))_B\right|dy\\
&=\mathcal{K}_1+\mathcal{K}_2.
\end{aligned}$$

By Hölder's inequality and Lemma 2.5 we get

$$\begin{aligned}
\mathcal{K}_1&\lesssim\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}[b^\gamma,\mu_j^L](f)(y)\right|dy\\
&\leqslant\sum_{\gamma=0}^{m-1}\left(\frac{1}{|B|}\int_B\left|b(y)-b_{2B}\right|^{(m-\gamma)s'}dy\right)^{1/s'}\left(\frac{1}{|B|}\int_B\left|[b^\gamma,\mu_j^L](f)(y)\right|^s dy\right)^{1/s}\\
&\lesssim\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)}M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L](f))(x).
\end{aligned}$$

For $\mathcal{K}_2$, we split $f=f_1+f_2$ with $f_1=f\chi_{2B}$, we have

$$\mathcal{K}_2\leqslant\frac{1}{|B|}\int_B|\mu_j^L((b-b_{2B})^m f_1)(y)|dy+\frac{1}{|B|}\int_B\left|\mu_j^L((b-b_{2B})^m f)(y)-(\mu_j^L((b-b_{2B})^m f))_B\right|dy=\mathcal{K}_{21}+\mathcal{K}_{22}.$$

As the proof in Lemma 3.1, we obtain

$$\mathcal{K}_{21}\lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).$$

For $\mathcal{K}_{22}$, by Lemma 3.2, we get

$$\mathcal{K}_{22}\lesssim\frac{1}{|B|^2}\int_B\int_B\left|\mu_j^L((b-b_B)^m f_2)(u)-\mu_j^L((b-b_B)^m f_2)(y)\right|du\,dy\lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).$$

□

Now let us prove Theorem 1.1.

Choose numbers t_γ such that $\frac{1}{t_\gamma}=\frac{1}{p}-\frac{\gamma\beta}{n},\gamma=0,1,\cdots,m-1$. Then $\frac{1}{q}=\frac{1}{t_\gamma}-\frac{(m-\gamma)\beta}{n}$. We need to prove the following inequality

$$\left\|[b^m,\mu_j^L]f\right\|_{L^q(\mathbb{R}^n)}^q\lesssim([b]_\beta^\theta)^{mq}\|f\|_{L^p(\mathbb{R}^n)}^q+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)q}\left\|[b^\gamma,\mu_j^L](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)}^q. \tag{3.1}$$

If (3.1) holds, then Theorem 1.1 will be proved by the mathematical induction. In fact, when $m=1$, we have $\gamma=0$ and $p=t_\gamma$. Note that $[b^0,\mu_j^L]=\mu_j^L$, by the boundedness of μ_j^L on $L^p(\mathbb{R}^n)$ for $1<p<\infty$, then $[b,\mu_j^L]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Suppose that the $L^p-L^{t_\gamma}$ boundedness of $[b^\gamma,\mu_j^L]$ holds for $\frac{1}{t_\gamma}=\frac{1}{p}-\frac{\gamma\beta}{n}$, that is

$$\left\|[b^\gamma,\mu_j^L](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)}\lesssim([b]_\beta^\theta)^\gamma\|f\|_{L^p(\mathbb{R}^n)},$$

where $\gamma=2,3,\cdots,m-1$, then by (3.1) we get

$$\left\|[b^m,\mu_j^L](f)\right\|_{L^q(\mathbb{R}^n)}\lesssim([b]_\beta^\theta)^m\|f\|_{L^p(\mathbb{R}^n)}.$$

In the following, we will focus on the proof of (3.1).

Let $1<s<p<\infty, f\in L^p(\mathbb{R}^n)$. By Proposition 2.4 we have

$$\begin{aligned}\left\|[b^m,\mu_j^L]f\right\|_{L^q(\mathbb{R}^n)}^q&\leqslant\int_{\mathbb{R}^n}\left|M_{\rho,\delta}([b^m,\mu_j^L]f)(x)\right|^q dx\\&\leqslant\int_{\mathbb{R}^n}\left|M^\sharp_{\rho,\eta}([b^m,\mu_j^L]f)(x)\right|^q dx+\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}\left|[b^m,\mu_j^L]f(x)\right|dx\right)^q.\end{aligned}$$

By Lemma 3.3,

$$M^\sharp_{\rho,\eta}([b^m,\mu_j^L]f)(x)\lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x)+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x).$$

Since $\frac{1}{q}=\frac{1}{t_\gamma}-\frac{(m-\gamma)\beta}{n}$, and $t_\gamma=p$ when $\gamma=0$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\left|M^\sharp_{\rho,\eta}([b^m,\mu_j^L]f)(x)\right|^q dx&\lesssim([b]_\beta^\theta)^{mq}\int_{\mathbb{R}^n}\left|M_{m\beta,s}(f)(x)\right|^q dx\\&\quad+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)q}\int_{\mathbb{R}^n}\left|M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x)\right|^q dx\\&\lesssim([b]_\beta^\theta)^{mq}\|f\|^q_{L^p(\mathbb{R}^n)}+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)q}\left\|[b^\gamma,\mu_j^L](f)\right\|^q_{L^{t_\gamma}(\mathbb{R}^n)}.\end{aligned}$$

By Proposition 2.3 and Lemma 3.1 we have

$$\begin{aligned}\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}\left|[b^m,\mu_j^L]f(x)\right|dx\right)^q&\lesssim([b]_\beta^\theta)^{mq}\sum_k\int_{2Q_k}|M_{m\beta,s}(f)|^q dx\\&\quad+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)p}\sum_k\int_{2Q_k}\left|M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)\right|^q dx\\&\lesssim([b]_\beta^\theta)^{mq}\int_{\mathbb{R}^n}\left|M_{m\beta,s}(f)(x)\right|^q dx\\&\quad+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)q}\int_{\mathbb{R}^n}\left|M_{(m-\gamma)\beta,s}([b^\gamma,\mu_j^L]f)(x)\right|^q dx\\&\lesssim([b]_\beta^\theta)^{mq}\|f\|^q_{L^p(\mathbb{R}^n)}+\sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{(m-\gamma)q}\left\|[b^\gamma,\mu_j^L](f)\right\|^q_{L^{t_\gamma}(\mathbb{R}^n)}.\end{aligned}$$

Then the proof of (3.1) is finished.

4. Proof of Theorem 1.3

To prove Theorem 1.3, we first investigate the following local estimate.

Lemma 4.1. *Let* $V \in RH_n, b \in \Lambda_\beta^\theta(\rho)$. *If* $1 < p < \frac{n}{m\beta}, \frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}$, *then the inequality*

$$\|[b^m, \mu_j^L](f)\|_{L^q(B(x_0,r))} \lesssim ([b]_\beta^\theta)^m r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

holds for any $f \in L_{loc}^p(\mathbb{R}^n)$.

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$. Then

$$\left\|[b^m, \mu_j^L](f)\right\|_{L^q(B(x_0,r))} \leqslant \left\|[b^m, \mu_j^L](f_1)\right\|_{L^q(B(x_0,r))} + \left\|[b^m, \mu_j^L](f_2)\right\|_{L^q(B(x_0,r))}.$$

By Theorem 1.1 we know $[b^m, \mu_j^L]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then we get

$$\begin{aligned}
\|[b^m, \mu_j^L](f_1)\|_{L^q(B(x_0,r))} &\lesssim ([b]_\theta^\beta)^m \|f\|_{L^p(B(x_0,2r))} \\
&\lesssim ([b]_\theta^\beta)^m r^{\frac{n}{q}} \|f\|_{L^p(B(x_0,2r))} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim ([b]_\theta^\beta)^m r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{4.1}$$

We now turn to deal with the term $\left\|[b^m, \mu_j^L](f_2)\right\|_{L^q(B(x_0,r))}$. By Binomial Theorem, we have

$$[b^m, \mu_j^L]f_2(x) \leqslant \sum_{\gamma=0}^m C_{\gamma,m}|b(x) - b_{2B}|^\gamma \mu((b - b_{2B})^{m-\gamma}(f_2)(x).$$

By Proposition 2.6 and Lemma 2.2 we have

$$\begin{aligned}
\sup_{x\in B(x_0,r)} \mu_j^L((b - b_{2B})^{m-\gamma} f_2)(x) &\lesssim \int_{(2B)^c} \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{|b - b_{2B}|^{m-\gamma}|f(y)|}{|x_0 - y|^{n-1}} \left(\int_{|x_0-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \\
&\lesssim \sum_{k=1}^\infty \frac{1}{\left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{N/(k_0+1)}} (2^{k+1}r)^{-n} \int_{2^{k+1}B} |b - b_{2B}|^{m-\gamma}|f(y)|dy.
\end{aligned}$$

From Lemma 2.5 we get

$$\begin{aligned}
&(2^{k+1}r)^{-n} \int_{2^{k+1}B} |b(y) - b_{2B}|^{m-\gamma}|f(y)|dy \\
&\lesssim \left((2^{k+1}r)^{-n} \int_{2^{k+1}B} |b(y) - b_{2B}|^{(m-\gamma)p'} dy\right)^{1/p'} (2^{k+1}r)^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,2^{k+1}r))} \\
&\lesssim ([b]_\beta^\theta)^{m-\gamma} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{(m-\gamma)\theta'} (2^k r)^{(m-\gamma)\beta - \frac{n}{p}} \|f\|_{L^p(B(x_0,2^{k+1}r))}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sup_{x\in B(x_0,r)} \mu_j^L((b - b_{2B})^{m-\gamma} f_2)(x) \\
&\lesssim ([b]_\beta^\theta)^{m-\gamma} \sum_{k=1}^\infty \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{(m-\gamma)\theta' - N/(k_0+1)} (2^k r)^{(m-\gamma)\beta - \frac{n}{p}} \|f\|_{L^p(B(x_0,2^{k+1}r))}.
\end{aligned}$$

Notice that

$$\|(b-b_{2B})^{\gamma}\|_{L^q(2B)} \lesssim ([b]_{\beta}^{\theta})^{\gamma} r^{\beta\gamma+\frac{n}{q}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta'\gamma}.$$

Then, taking $N \geqslant (k_0+1)(m)\theta'$ and noticing $m\beta-\frac{n}{p}=-\frac{n}{q}$ we get

$$\begin{aligned}
&\left\|[b^m,\mu_j^L](f_2)\right\|_{L^q(B(x_0,r))}\\
&\lesssim ([b]_{\beta}^{\theta})^m r^{\frac{n}{q}}\sum_{k=1}^{\infty}2^{-\gamma\beta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{m\theta'-N/(k_0+1)}(2^k r)^{m\beta-\frac{n}{p}}\|f\|_{L^p(B(x_0,2^{k+1}r))}\\
&\lesssim ([b]_{\beta}^{\theta})^m r^{\frac{n}{q}}\sum_{k=1}^{\infty}(2^k r)^{-\frac{n}{q}}\|f\|_{L^p(B(x_0,2^{k+1}r))}\\
&\lesssim ([b]_{\beta}^{\theta})^m r^{\frac{n}{q}}\sum_{k=1}^{\infty}\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&= ([b]_{\beta}^{\theta})^m r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.
\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), the proof of Lemma 4.1 is completed. □

Proof of Theorem 1.3. Note the fact that $\|f\|_{L^p(B(x_0,t))}$ is a nondecreasing function of t, and $f\in M^{\alpha,V}_{p,\varphi_1}$, then we have

$$\begin{aligned}
\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}}\\
&\lesssim \sup_{0<s<\infty}\frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,s))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$

Since $\alpha\geqslant 0$, and (φ_1,φ_2) satisfies the condition (1.1), then

$$\begin{aligned}
\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} &= \int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\int_{2r}^{\infty}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).
\end{aligned}$$

Then by Lemma 4.1 we get

$$\begin{aligned}
\|[b^m,\mu_j^L](f)\|_{M^{\alpha,V}_{q,\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/q}\|[b^m,\mu_j^L](f)\|_{L^q(B(x_0,r))}\\
&\lesssim ([b]_{\beta}^{\theta})^m\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim ([b]_{\beta}^{\theta})^m\|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$

□

References

[1] B. Bongioanni, E. Harboure, O. Salinas, *Commutators of Riesz transforms related to Schrödinger operators*, J. Fourier Anal. Appl., **17** (2011), 115–134. 1, 2.4

[2] D. Chen, F. Jin, *The boundedness of Marcinkiewicz integrals associated with Schrödinger operator on Morrey spaces*, J. Funct. Spaces, **2014** (2014), 11 pages. 1, 1

[3] D. Chen, D. Zou, *The boundedness of Marcinkiewicz integral associated with Schrödinger operator and its commutator*, J. Funct. Spaces, **2014** (2014), 10 pages. 1

[4] J. Dziubański, J. Zienkiewicz, *Hardy space* H^1 *associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana, **15** (1999), 279–296. 2.3

[5] W. Gao, L. Tang, *Boundedness for Marcinkiewicz integrals associated with Schrödinger operators*, Proc. Indian Acad. Sci. Math. Sci., **124** (2014), 193–203. 1

[6] Y. Liu, J. Sheng, *Some estimates for commutators of Riesz transforms associated with Schrödinger operators*, J. Math. Anal. Appl., **419** (2014), 298–328. 1, 2.5

[7] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic analysis, ICM-90 Satell. Conf. Proc., Springer, Japan, (1991), 183–189. 1

[8] C. B. Jr. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126–166. 1

[9] B. Ren, H. Wang, *Boundedness of higher order Riesz transforms associated with Schrödinger type operator on generalized Morrey spaces*, J. Nonlinear Sci. Appl., **10** (2017), 2757–2766. 1, 1.2, 2.2

[10] Z. Shen, L^p *estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier, **45** (1995), 513–546. 1, 2.1, 2.6

[11] L. Tang, J. F. Dong, *Boundedness for some Schrödinger type operator on Morrey spaces related to certain nonnegative potentials*, J. Math. Anal. Appl., **355** (2009), 101–109. 1