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A general iterative algorithm for vector equilibrium problem

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Abstract

In this paper, iterative algorithm for strong vector equilibrium problem (SVEP) is studied. Firstly, an auxiliary problem for SVEP is introduced and the relationships between the auxiliary problem and SVEP are discussed. Then, based on the auxiliary problem, a general iterative algorithm for SVEP is proposed. Moreover, analysis of convergence of this general iterative algorithm is investigated under suitable conditions of cone-continuity and cone-convexity. The main results obtained in this paper extend and develop the corresponding ones of [A. N. Iusem, W. Sosa, Optimization, **52** (2003), 301–316], [S.-H. Wang, Q.-Y. Li, Optimization, **64** (2015), 2049–2063], and [B. Cheng, S.-Y. Liu, J. Lanzhou Univ. Nat. Sci., **45** (2009), 105–109].

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1. Introduction

Let E be a real Hausdorff topological vector space, and X be a nonempty subset of E. Let $g : X \times X \to \mathbb{R}$ be a real-valued bifunction. The scalar equilibrium problem (for short, EP) is to find $\bar{x} \in X$ such that

$$g(\bar{x}, y) \geqslant 0, \ \forall y \in X. \tag{EP}$$

It provides a unifying framework for many important problems, such as, optimization problems, variational inequality problems, complementary problems, minimax inequality problems and fixed point problems, and has been widely applied to study the problems arising in economics, mechanics and engineering science (see [1]).

In recent years, EP has been extended to vector case in different ways.

Let Z be another real Hausdorff topological vector space, and C be a convex cone of Z. Let $f : X \times X \to Z$ be a vector-valued bifunction. The strong vector equilibrium problem (for short, SVEP) is to find $\bar{x} \in X$ such that

$$f(\bar{x}, y) \in C, \quad \forall y \in X, \tag{SVEP}$$

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and the weak vector equilibrium problem (for short, WVEP) is to find $\bar{x} \in X$ such that

$$f(\bar{x}, y) \notin -\text{int}(C), \quad \forall y \in X, \tag{WVEP}$$

where $\text{int}(C)$ denotes the topological interior of C and $\text{int}(C) \neq \varnothing$.

Obviously, if the cone C is proper (i.e., $C \neq Z$), then the solution set of SVEP is contained in the solution set of WVEP.

Existence of solutions is a fundamental problem for vector equilibrium problems. In the past years, it has been intensively studied by many authors, and a great quantity of existence results of solutions have been obtained in the literatures. Usually, monotonicity or coerciveness condition on f and compactness condition on X are necessary. For details, we refer the reader to the monographs [2, 4, 8] and the references therein.

How to find a solution, i.e., algorithm method, is another important problem for vector equilibrium problems. Just in the recent past years, methods for finding a solution of vector equilibrium problems have been explored by some authors. In 2009, by using a scalarization method, Cheng and Liu [3] suggested a projection iterative algorithm for finding solutions of a weak vector equilibrium problem by solving a corresponding convex feasibility problem in an n-dimensional Euclidean space. In 2012, by applying a regularization technique, Li and Wang [15] proposed a viscosity approximation method, which is a develop of the one of Takahashi and Takahashi [18], for finding a common element of the set of fixed points of a nonexpansive mapping and of the set of solutions of a strong vector equilibrium problem in a Hilbert sapce. Shortly afterwards, Shan and Huang [17] extended this viscosity approximation method for finding a common element of the set of fixed points of an infinite family nonexpansive mappings, of the set of solutions of a generalized mixed vector equilibrium problem and of the set of solutions of a variational inequality problem. In 2015, by applying the Gerstewitz nonlinear scalarization function, Wang and Li [20] presented a projection iterative algorithm for finding solutions of a strong vector equilibrium problem by solving a corresponding scalar optimization problem.

Summarizing these methods, we can find that scalarization and regularization are the only two ideas on which algorithms are proposed for solving vector equilibrium problems. The main idea of scalarization method is to transform vector equilibrium problems to all kinds of scalar problems by using Gerstewitz nonlinear scalarization function. It is well-known that the Gerstewitz nonlinear scalarization function is defined through interior elements of a cone C. This means that, in this scalarization method, the cone C must have a nonempty interior. However, in many cases, the ordering cone has an empty interior. For example, in the classical Banach spaces ℓ^p and $L^p(\Omega)$, where $1 < p < \infty$, the standard ordering cone has an empty interior (see [13]). The main idea of regularization method is to transform vector equilibrium problems to fixed points of a uniformly nonexpansive mapping by adding an appropriate disturbing term to the original vector equilibrium problem. The key of this method is the construction of a proper disturbing term, which is indeed a technical job.

Inspired and motivated by the works mentioned above, in this paper, we shall study algorithms for SVEP. For this purpose, we firstly introduce an auxiliary problem for SVEP and discuss the relationships between the auxiliary problem and SVEP. Then, based on the auxiliary problem, we propose a general iterative algorithm for SVEP, which does not require that the cone C must have a nonempty interior and does not need one to construct any disturbing term. We further analyze the convergence of this algorithm method under some suitable conditions of cone-continuity and cone-convexity. The main results obtained in this paper extend and develop the corresponding ones of Iusem and Sosa [12], Wang and Li [20], and Cheng and Liu [3].

2. Preliminaries

In this section, we shall give some definitions and lemmas used in the sequel.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let K be a nonempty closed and convex subset of H. For any $x \in H$, there exists a unique nearest point in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \leqslant \|x - y\|, \quad \forall y \in K.$$

Such a P_K is called the metric projection of H onto K. It is known that $y = P_K(x)$ is equivalent to

$$\langle y - z, y - x\rangle \leqslant 0, \quad \forall z \in K.$$

Definition 2.1 ([16])**.** Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset, $C \subseteq Z$ a closed convex cone. A mapping $g : X \to Z$ is called

(i) C-lower semicontinuous (for short, C-l.s.c.) (resp. C-upper semicontinuous (for short, C-u.s.c.)) at $x_0 \in X$ if, for any neighborhood V of 0 in Z, there exists a neighborhood U of x_0 in E such that

$$g(x) \in g(x_0) + V + C, \quad \forall x \in U \cap X \quad (\text{resp. } g(x) \in g(x_0) + V - C, \quad \forall x \in U \cap X);$$

(ii) C-l.s.c. (resp. C-u.s.c.) on X if it is C-l.s.c. (resp. C-u.s.c.) at every point $x \in X$;

(iii) C-continuous on X if it is both C-u.s.c. and C-l.s.c. on X.

Remark 2.2. If $g : X \to Z$ is continuous on X, then it is C-continuous on X; conversely, if g is C-continuous on X, then it is continuous only when the cone C has a closed convex bounded base (see [16, Theorem 5.3, pp. 22-23]).

Definition 2.3 ([16])**.** A mapping $g : X \to Z$ is called lower semicontinuous (for short, l.s.c.) (resp. upper semicontinuous (for short, u.s.c.)) on X if, for any $z \in Z$, the set

$$L(z) = \{x \in X : \ g(x) \in z - C\} \quad (\text{resp. } L(z) = \{x \in X : \ g(x) \in z + C\})$$

is closed in X.

Lemma 2.4 ([9])**.** *If g is C-l.s.c. (resp. C-u.s.c.) on X, then it is l.s.c. (resp. u.s.c.) on X.*

Definition 2.5 ([6, 16])**.** Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset, and $C \subseteq Z$ a convex cone. A mapping $h : X \to Z$ is said to be

(i) C-convex if, for any $u_1, u_2 \in X$ and for any $t \in [0, 1]$, one has

$$h(tu_1 + (1 - t)u_2) \in th(u_1) + (1 - t)h(u_2) - C;$$

(ii) C-quasiconvex if, for any $z \in Z$, the set $\{u \in X : h(u) \in z - C\}$ is convex;

(iii) properly C-quasiconvex if, for any $u_1, u_2 \in X$ and for any $t \in [0, 1]$, one has

$$\text{either} \quad h(tu_1 + (1 - t)u_2) \in h(u_1) - C; \quad \text{or} \quad h(tu_1 + (1 - t)u_2) \in h(u_2) - C;$$

(iv) properly C-quasiconcave if $-h$ is properly C-quasiconvex.

Remark 2.6. (1) It is obvious that if h is C-convex or properly C-quasiconvex, then h is C-quasiconvex. (2) The concept of properly C-quasiconvex vector-valued function is important in the study of minimax theorem for vector-vlalued functions, generalized vector quasiequilibrium problems and vector quasivariational inclusion problems (see [6, 7, 10, 11, 19]).

Lemma 2.7 ([11])**.** *Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset, and $C \subseteq Z$ a closed convex cone. Assume that $h : X \to Z$ is C-u.s.c. and properly C-quasiconcave on X. Then, for any $u_1, u_2 \in X$, there exists some $t_0 \in [0, 1]$ such that*

$$h(u_1) \in h(t_0u_1 + (1 - t_0)u_2) - C \quad \text{and} \quad h(u_2) \in h(t_0u_1 + (1 - t_0)u_2) - C.$$

Definition 2.8 ([14])**.** Let Z be a real Hausdorff topological vector space and $C \subseteq Z$ a convex cone. A nonempty set $M \subseteq Z$ is called upward directed if, for every $u_1, u_2 \in M$, there exists $u \in M$ such that $u_1 \in u - C$ and $u_2 \in u - C$.

The following theorem is important in the convergence analysis of our algorithm, which says the existence of maximal points.

Theorem 2.9 ([10])**.** *Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty compact subset, and $C \subseteq Z$ a closed convex cone. Assume that $f : X \to Z$ is u.s.c. and $f(X)$ is upward directed. Then, there exists $\bar{x} \in X$ such that*

$$g(x) \in g(\bar{x}) - C, \quad \forall x \in X.$$

Definition 2.10. Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset, and $C \subseteq Z$ a closed convex pointed cone. A vector-valued mapping $f : X \times X \to Z$ is said to be

(i) C-monotone if, for any $x, y \in X$, one has

$$f(x, y) + f(y, x) \in -C;$$

(ii) C-pseudomonotone if, for any $x, y \in X$, one has

$$f(x, y) \in C \quad \Longrightarrow \quad f(y, x) \in -C;$$

(iii) strongly C-pseudomonotone if, for any $x, y \in X$, one has

$$f(x, y) \notin -C \quad \Longrightarrow \quad f(y, x) \in -C \setminus \{0\}.$$

Remark 2.11. Clearly, if f is C-monotone or strongly C-pseudomonotone, then it is C-pseudomonotone.

The following theorem gives some sufficient conditions for existence of solutions for SVEP, which is a special case of Theorem 3.1 of Hou et al. [11].

Theorem 2.12. *Let E, Z be two real locally convex Hausdorff topological vector spaces, $X \subseteq E$ a nonempty compact convex subset, and $C \subseteq Z$ a closed convex cone. Let $f : X \times X \to Z$ be a vector-valued bifunction. Suppose that:*

(i) *f is C-continuous;*

(ii) *for any $x \in X$, $f(x, x) \in C$;*

(iii) *for each $x \in X$, $f(x, y)$ is properly C-quasiconvex in y.*

Then, there exists $\bar{x} \in X$ such that

$$f(\bar{x}, y) \in C, \quad \forall y \in X.$$

Next, we present an important local property of SVEP, which says that local solutions of SVEP are indeed global ones.

Theorem 2.13. *Let E, Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset, and $C \subseteq Z$ a closed convex cone. Let $f : X \times X \to Z$ be a vector-valued bifunction satisfying: for each $x \in X$, $f(x, x) = 0$ and $f(x, y)$ is C-convex in y. If there exist an open set $U \subseteq E$ and $\bar{x} \in X \cap U$ such that $f(\bar{x}, y) \in C$, for all $y \in X \cap U$, then $\bar{x}$ solves SVEP.*

Proof. Suppose that there exist an open set $U \subseteq E$ and $\bar{x} \in X \cap U$ such that

$$f(\bar{x}, y) \in C, \quad \forall y \in X \cap U.$$

For each $y \in X$, since $\bar{x} \in U$ and U is open, there exists some $t \in (0, 1)$ small enough such that $x_t = ty + (1-t)\bar{x} \in U$. As X is convex, we have $x_t \in X$. Thus $x_t \in X \cap U$, and so $f(\bar{x}, x_t) \in C$. On the other hand, since $f(\bar{x}, y)$ is C-convex in y, we get

$$tf(\bar{x}, y) + (1-t)f(\bar{x}, \bar{x}) \in f(\bar{x}, x_t) + C \subseteq C + C \subseteq C.$$

This, together with the fact that $f(\bar{x}, \bar{x}) = 0$, implies that $tf(\bar{x}, y) \in C$. As C is a cone, we have $f(\bar{x}, y) \in C$. This indicates that $\bar{x}$ solves SVEP. □

Lemma 2.14 ([5, FKKM Theorem]). *Let X be a nonempty subset of a Hausdorff topological vector space E. For each $x \in X$, consider a closed subset $G(x)$ of E. If the following two conditions are satisfied:*

(i) *the convex hull of any finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\cup_{i=1}^n G(x_i)$,*

(ii) *$G(x)$ is compact for at least one $x \in X$,*

then $\cap_{x \in X} G(x) \neq \emptyset$.

We need in the sequel the following lemma, which is an infinite-dimensional version of Lemma 2.3 of Iusem and Sosa [12].

Lemma 2.15. *Let X be a nonempty subset of a Hausdorff topological vector space E. For each $x \in X$, consider a closed subset $G(x)$ of E. If the following two conditions are satisfied:*

(i) *the convex hull of any finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\cup_{i=1}^n G(x_i)$,*

(ii) *the closure $\overline{co}X$ of the convex hull of X is compact,*

then $\cap_{x \in X} G(x) \neq \emptyset$.

Proof. For each $x \in X$, let $M(x) = \overline{co}X \cap G(x)$. Then $M(x)$ is closed in E. Consider any finite subset $\{x_1, \cdots, x_n\}$ of X. Obviously, $co\{x_1, \cdots, x_n\} \subseteq \overline{co}X$. Furthermore, by condition (i), we can derive that $co\{x_1, \cdots, x_n\} \subseteq \cup_{i=1}^n G(x_i)$. Thus, we have

$$co\{x_1, \cdots, x_n\} \subseteq \overline{co}X \cap [\cup_{i=1}^n G(x_i)],$$

implying

$$co\{x_1, \cdots, x_n\} \subseteq \cup_{i=1}^n [\overline{co}X \cap G(x_i)] = \cup_{i=1}^n M(x_i).$$

On the other hand, by condition (ii), we know that $M(x)$ is compact for each $x \in X$. Hence, by Lemma 2.14, we have $\cap_{x \in X} M(x) \neq \emptyset$, implying $\cap_{x \in X} G(x) \neq \emptyset$. □

3. Auxiliary problem

In this section, we shall introduce an auxiliary problem for SVEP and discuss the relationships between the auxiliary problem and SVEP. Furthermore, we shall give some sufficient conditions for existence of solutions for auxiliary problem. All of these will help us to propose algorithms for SVEP.

From now on, unless otherwise specified, we shall use the following notations and assumptions.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let X be a nonempty compact convex subset of H. Let Z be a real locally convex Hausdorff topological vector space, C be a closed convex pointed cone of Z and $e \in C \backslash \{0\}$ be any given point. Let $f : X \times X \to Z$ be a vector-valued bifunction satisfying the following assumptions:

(P1) f is C-continuous;

(P2) $\forall x \in X$, $f(x, x) = 0$;

(P3) $\forall x \in X, f(x, y)$ is C-convex in y;

(P4) for any given $y \in X$, for every $x_1, x_2 \in X$, there exists $x \in X : \|x\| \leqslant \max\{\|x_1\|, \|x_2\|\}$, such that

$$f(x_1, y) \in f(x, y) - C \quad \text{and} \quad f(x_2, y) \in f(x, y) - C.$$

The next proposition shows that, under some suitable conditions of cone-continuity and cone-convexity, the assumption (P4) can be satisfied.

Proposition 3.1. *If, for each $y \in X$, $f(\cdot, y)$ is C-u.s.c. and properly C-quasiconcave on X, then the assumption* (P4) *is satisfied.*

Proof. For any fixed $y \in X$, since $f(\cdot, y)$ is C-u.s.c. and properly C-quasiconcave on X, then, we can derive from Lemma 2.7 that, for every $x_1, x_2 \in X$, there exists $t_0 \in [0, 1]$ such that

$$f(x_1, y) \in f(t_0x_1 + (1 - t_0)x_2, y) - C \quad \text{and} \quad f(x_2, y) \in f(t_0x_1 + (1 - t_0)x_2, y) - C.$$

Let $x = t_0x_1 + (1 - t_0)x_2$. Then $x \in X$ as X is convex and

$$\|x\| = \|t_0x_1 + (1 - t_0)x_2\| \leqslant t_0\|x_1\| + (1 - t_0)\|x_2\| \leqslant \max\{\|x_1\|, \|x_2\|\}$$

and

$$f(x_1, y) \in f(x, y) - C \quad \text{and} \quad f(x_2, y) \in f(x, y) - C.$$

□

Also, we have the following result.

Proposition 3.2. *Let H be a normed vector space with norm $\| \cdot \|$, and X be a nonempty compact subset of H. Let $Z = \mathbb{R}, C = \mathbb{R}_+ = [0, +\infty), e = 1 \in C \setminus \{0\}$. If, for each $y \in X$, the real-valued function $f(\cdot, y)$ is continuous on X, then the assumption* (P4) *is also satisfied.*

Proof. For any given $x_1, x_2 \in X$, let $\rho = \max\{\|x_1\|, \|x_2\|\}$ and let $\overline{B(\rho)}$ be the closed ball centered at 0 with radius ρ in H. Since X is compact, the subset $X \cap \overline{B(\rho)}$ is also compact. Then, for any fixed $y \in X$, the real-valued continuous function $f(\cdot, y)$ can attains its maximum on the compact set $X \cap \overline{B(\rho)}$, i.e., there exists some $\bar{x} \in X \cap \overline{B(\rho)}$ such that

$$f(\bar{x}, y) \geqslant f(x, y), \quad \forall x \in X \cap \overline{B(\rho)}.$$

Noting that $x_1, x_2 \in X \cap \overline{B(\rho)}$, we have

$$f(\bar{x}, y) \geqslant f(x_1, y) \ \text{ and } \ f(\bar{x}, y) \geqslant f(x_2, y).$$

That is,

$$f(x_1, y) \in f(\bar{x}, y) - C \quad \text{and} \quad f(x_2, y) \in f(\bar{x}, y) - C.$$

Moreover,

$$\|\bar{x}\| \leqslant \rho = \max\{\|x_1\|, \|x_2\|\}.$$

□

Remark 3.3. In Proposition 3.2, if H is in particular a finite-dimensional Euclidean space, then the compactness condition of X can be replaced by a closedness one. In fact, if H is a finite-dimensional Euclidean space and X is a nonempty closed subset, then the bounded closed ball $\overline{B(\rho)}$ defined in the proof of Proposition 3.2 is clearly compact. And so the subset $X \cap \overline{B(\rho)}$ is also compact. Then, by proceeding the rest arguments as that in the proof of Proposition 3.2, we know that the assumption (P4) is satisfied.

Now we consider the following Auxiliary Problem (for short, AP).

$$\text{Find } \bar{x} \in X \ \text{ such that } \ \bar{x} \in \cap_{y \in X} L_f(y), \tag{AP}$$

where $L_f(y)$ is defined by

$$L_f(y) = \{x \in X : f(y, x) \in -C\}, \quad \forall y \in X.$$

For each $y \in X$, by (P2), $y \in L_f(y)$, and so $L_f(y)$ is nonempty. Also, by (P1) and Lemma 2.4, $L_f(y)$ is closed. Moreover, by (P3) and Remark 2.6, $L_f(y)$ is convex.

The AP is also called Convex Feasibility Problem (CFP) (see [12]). Its relevance lies in the fact that the solution set of AP contains in the solution set of SVEP.

Theorem 3.4. *The solution set of AP is a subset of the solution set of SVEP.*

Proof. Let $x \in X$ be a solution of AP. Then

$$f(y, x) \in -C, \quad \forall y \in X. \tag{3.1}$$

For each $y \in X$ and $t \in (0, 1)$, let $x_t = ty + (1-t)x$. Then $x_t \in X$ as X is convex. And so, by (3.1), we have $f(x_t, x) \in -C$. On the other hand, since $f(x_t, \cdot)$ is C-convex, we can get

$$tf(x_t, y) + (1-t)f(x_t, x) \in f(x_t, x_t) + C = 0 + C = C.$$

Noting that C is a convex cone, we have

$$tf(x_t, y) \in -(1-t)f(x_t, x) + C \subseteq C + C \subseteq C.$$

Thus $f(x_t, y) \in C$ as C is a cone. Furthermore, since f is C-continuous, it is C-u.s.c.. Then, by Lemma 2.4, we know that f is u.s.c.. And so, we can get $f(x, y) \in C$, which implies that x is a solution of SVEP. □

The following theorem shows that the converse of Theorem 3.4 is also true when f is equipped with suitable condition of monotonicity.

Theorem 3.5. *If the mapping f is C-pseudomonotone, then the solution set of SVEP is a subset of the solution set of AP.*

Proof. Suppose that $x \in X$ is a solution of SVEP. Then, for each $y \in X$, one has $f(x, y) \in C$. As f is C-pseudomonotone, we have $f(y, x) \in -C$. This indicates that x is solution of AP. □

As a consequence of Theorems 2.12, 3.4, and 3.5, we can obtain the following existence result of solutions for AP.

Theorem 3.6. *Suppose that the mapping f satisfies* (P1)-(P2) *and the following conditions:*

(i) *f is C-pseudomonotone;*

(ii) *for each $x \in X$, $f(x, y)$ is properly C-quasiconvex in y.*

Then, AP has at least one solution in X.

Proof. By Theorem 2.12, we know that SVEP has at least one solution in X. On the other hand, by Theorems 3.4 and 3.5, we know that the solution set of SVEP coincides with the solution set of AP. Thus, AP has at least one solution in X. □

We also have the following result.

Theorem 3.7. *If the mapping f is strongly C-pseudomonotone, then, for any sequence $\{y_m\} \subseteq X$, one has $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$.*

Proof. For any given sequence $\{y_m\} \subseteq X$, let $D = \{y_m\}$ and $P(y) = L_f(y)$ for all $y \in D$. Then $D \subseteq X$ and for each $y \in D$, $P(y)$ is nonempty closed and convex. Noting that X is a compact convex subset of H, we know that $\overline{co}D \subseteq X$ is compact. On the other hand, for any finite subset $\{y_{m_1}, \cdots, y_{m_p}\} \subseteq D$, we claim that

$$co\{y_{m_1}, \cdots, y_{m_p}\} \subseteq \cup_{i=1}^{p} P(y_{m_i}).$$

In fact, suppose to the contrary that there exists some $\bar{y} = \sum_{i=1}^{p} \lambda_i y_{m_i}$ with $\sum_{i=1}^{p} \lambda_i = 1$ and $\lambda_i \geqslant 0$ ($i = 1, \cdots, p$) such that $\bar{y} \notin \cup_{i=1}^{p} P(y_{m_i})$. It follows that $f(y_{m_i}, \bar{y}) \notin -C$, $i = 1, \cdots, p$. Then, by strongly C-pseudomonotonicity of f, we have

$$f(\bar{y}, y_{m_i}) \in -C \setminus \{0\}, \quad i = 1, \cdots, p.$$

As $f(\bar{y},\cdot)$ is C-convex, we can get

$$0 = f(\bar{y},\bar{y}) \in \sum_{i=1}^{p} \lambda_i f(\bar{y}, y_{m_i}) - C \subseteq -C\setminus\{0\} - C \subseteq -C\setminus\{0\}.$$

This is impossible, and so the assertion holds. Then, by applying Lemma 2.15, we can obtain that $\cap_{m=1}^{\infty} P(y_m) \neq \emptyset$, i.e., $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$. □

4. Algorithm and its convergence

In this section, we shall propose an algorithm for solving SVEP and analyze its convergence.
Based on AP, we can propose a general iterative algorithm for SVEP.

Algorithm 4.1.

Step 0 (initial step). Choose initial $x_0 \in X$ and let $\rho_0 = \|x_0\|$. Set $m = 0$.

Step 1. Define

$$X_m = \{x \in X : \|x\| \leqslant \rho_m + 1\}. \tag{4.1}$$

Step 2. Find $y_m \in X_m$ satisfying

$$f(y, x_m) \in f(y_m, x_m) + \varepsilon_m e - C, \quad \forall y \in X_m. \tag{4.2}$$

Step 3. Compute x_{m+1} as

$$x_{m+1} = x_m + \lambda_m [P_{L_f(y_m)}(x_m) - x_m]. \tag{4.3}$$

Step 4. Calculate ρ_{m+1} as

$$\rho_{m+1} = \max\{\rho_m, \|x_{m+1}\|\}.$$

Step 5. Set $m = m+1$ and return to Step 1.
where $P_{L_f(y_m)}(\cdot)$ denotes the metric projection onto $L_f(y_m) = \{x \in X : f(y_m, x) \in -C\}$, $\{\varepsilon_m\}$ and $\{\lambda_m\}$ are two sequence of $\mathbb{R}$, satisfying $\varepsilon_m \geqslant 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$ and $\{\lambda_m\} \subseteq [\alpha, 1]$ for some $\alpha \in (0,1)$.

We start now the convergence analysis of Algorithm 4.1.

Lemma 4.2. *Algorithm 4.1 is well-defined.*

Proof. Clearly, $\rho_m = \max\{\|x_0\|, \|x_1\|, \cdots, \|x_m\|\}$, and so the sequence $\{\rho_m\}$ is nondecreasing. This indicates that $X_m \subseteq X_{m+1}$ for all m. As x_0 belongs to X_0, all the sets X_m are nonempty and trivially closed. Notice that $X_m \subseteq X$ and X is compact. All the sets X_m are compact. For any given $x_m \in X_m$, by (P4), we know that the set $f(X_m, x_m) = \{f(y, x_m) : y \in X_m\}$ is upward directed. Moreover, by (P1), we know that the vector-valued mapping $f(\cdot, x_m)$ is C-u.s.c. and so is u.s.c.. Thus, it follows from Theorem 2.9 that there exists some $y_m \in X_m$ such that

$$f(y, x_m) \in f(y_m, x_m) - C, \quad \forall y \in X_m.$$

This, together with $\varepsilon_m \geqslant 0$ and $e \in C\setminus\{0\}$, implies that y_m satisfies (4.2). In addition, since $L_f(y_m)$ is nonempty closed and convex, the metric projection of x_m onto $L_f(y_m)$ is existing and unique. And so x_{m+1} is uniquely defined by (4.3). □

Lemma 4.3. *Let $\{x_m\}$ and $\{y_m\}$ be the sequences generated by Algorithm 4.1.*

(i) *For each $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$, it holds that*

$$\|x_{m+1} - \bar{x}\|^2 \leqslant \|x_m - \bar{x}\|^2 - \lambda_m(2-\lambda_m)\|x_m - P_{L_f(y_m)}(x_m)\|^2 \leqslant \|x_m - \bar{x}\|^2; \tag{4.4}$$

(ii) $\{\|x_m - \bar{x}\|\}$ *is convergent for all* $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$.

Proof.

(i) For each $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$, by (4.3), we have

$$\begin{aligned} \|x_{m+1} - \bar{x}\|^2 &= \|x_m - \bar{x}\|^2 + \lambda_m^2 \|x_m - P_{L_f(y_m)}(x_m)\|^2 + 2\lambda_m \langle x_m - \bar{x}, P_{L_f(y_m)}(x_m) - x_m \rangle \\ &= \|x_m - \bar{x}\|^2 + \lambda_m^2 \|x_m - P_{L_f(y_m)}(x_m)\|^2 - 2\lambda_m \|x_m - P_{L_f(y_m)}(x_m)\|^2 \\ &\quad + 2\lambda_m \langle P_{L_f(y_m)}(x_m) - \bar{x}, P_{L_f(y_m)}(x_m) - x_m \rangle \\ &= \|x_m - \bar{x}\|^2 - \lambda_m(2 - \lambda_m) \|x_m - P_{L_f(y_m)}(x_m)\|^2 \\ &\quad + 2\lambda_m \langle P_{L_f(y_m)}(x_m) - \bar{x}, P_{L_f(y_m)}(x_m) - x_m \rangle. \end{aligned} \tag{4.5}$$

Noting that $\bar{x} \in L_f(y_m)$, by the well-known property of metric projection, we have

$$\langle P_{L_f(y_m)}(x_m) - \bar{x}, P_{L_f(y_m)}(x_m) - x_m \rangle \leqslant 0.$$

In addition, we can get $\lambda_m(2 - \lambda_m) > 0$ as $0 < \alpha \leqslant \lambda_m \leqslant 1$. Thus, by (4.5), we obtain

$$\|x_{m+1} - \bar{x}\|^2 \leqslant \|x_m - \bar{x}\|^2 - \lambda_m(2 - \lambda_m)\|x_m - P_{L_f(y_m)}(x_m)\|^2 \leqslant \|x_m - \bar{x}\|^2.$$

(ii) For each $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$, by (4.4), we can get $\|x_{m+1} - \bar{x}\| \leqslant \|x_m - \bar{x}\|$ for all m. This indicates that the nonnegative sequence $\{\|x_m - \bar{x}\|\}$ is nonincreasing, and hence, it is convergent. □

The following theorem gives convergence properties of Algorithm 4.1.

Theorem 4.4. *Let* $\{x_m\}$ *and* $\{y_m\}$ *be the sequences generated by Algorithm 4.1.*

(i) *If* $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$, *then* $\{x_m\}$ *converges to a solution of SVEP.*

(ii) *If SVEP has no solution, then* $\{x_m\}$ *does not converge.*

Proof.

(i) As the sequence $\{x_m\}$ is contained in the compact set X, it has a convergent subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k} \to x^* \in X$ as $k \to \infty$. Similarly, since the sequence $\{y_{m_k}\}$ is contained in the compact set X, it also has a convergent subsequence. Without loss of generality, we may assume that $y_{m_k} \to y^*$ for some $y^* \in X$ as $k \to \infty$,

Next, we shall show that x^* is a solution of SVEP and $x_m \to x^*$ as $m \to \infty$. The proof is divided into three steps.

(I) $f(y^*, x^*) = 0$.

In fact, we can take any $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$ and let $\bar{x}$ be fixed. Then, by rewriting (4.4) and using the conditions on $\{\lambda_m\}$, we can get

$$\alpha(2-\alpha)\|x_m - P_{L_f(y_m)}(x_m)\|^2 \leqslant \lambda_m(2-\lambda_m)\|x_m - P_{L_f(y_m)}(x_m)\|^2 \leqslant \|x_m - \bar{x}\|^2 - \|x_{m+1} - \bar{x}\|^2. \tag{4.6}$$

Note that $\alpha(2 - \alpha) > 0$. Then, by (4.6) and Lemma 4.3 (ii), we have

$$\lim_{m \to \infty} (x_m - P_{L_f(y_m)}(x_m)) = 0.$$

It follows that

$$\lim_{k \to \infty} P_{L_f(y_{m_k})}(x_{m_k}) = x^*.$$

In addition, by the definition of $L_f(y_{m_k})$ and the fact $P_{L_f(y_{m_k})}(x_{m_k})$ belongs to $L_f(y_{m_k})$, we can get

$$f(y_{m_k}, P_{L_f(y_{m_k})}(x_{m_k})) \in -C.$$

Since f is C-l.s.c., it is l.s.c.. Thus, we have

$$f(y^*, x^*) \in -C. \tag{4.7}$$

On the other hand, it is clear that $\rho_m = \max\{\|x_0\|, \|x_1\|, \cdots, \|x_m\|\}$, and so, by (4.1), we know that $x_m \in X_m$ for all m. Thus, by (P2) and (4.2), we have, for each m,

$$0 = f(x_m, x_m) \in f(y_m, x_m) + \varepsilon_m e - C,$$

implying

$$f(y_m, x_m) + \varepsilon_m e \in C. \tag{4.8}$$

Since $\varepsilon_m \to 0$ $(m \to \infty)$ and f is C-u.s.c., we can prove that

$$f(y^*, x^*) \in C. \tag{4.9}$$

Then, by combining (4.7) with (4.9), we obtain

$$f(y^*, x^*) = 0.$$

Thus, there remains to prove that (4.9) is true. In fact, suppose to the contrary that (4.9) is not true, i.e.,

$$f(y^*, x^*) \notin C.$$

Since C is closed, there exists some neighborhood V_0 of 0 in Z such that

$$[f(y^*, x^*) + V_0] \cap C = \varnothing.$$

Noting that C is a convex cone, we can get

$$[f(y^*, x^*) + V_0 - C] \cap C = \varnothing. \tag{4.10}$$

For the above neighborhood V_0 of 0 in Z, since Z is a topological vector space, there exists a balanced neighborhood V_1 of 0 in Z such that $V_1 + V_1 \subseteq V_0$. Since f is C-u.s.c., there exist neighborhoods $U(x^*)$ and $U(y^*)$ of x^* and y^*, respectively, such that

$$f(y, x) \in f(y^*, x^*) + V_1 - C, \quad \forall\, (y, x) \in (U(y^*) \cap X) \times (U(x^*) \cap X). \tag{4.11}$$

Observe that $\{(y_{m_k}, x_{m_k})\} \subseteq X \times X$ and $(y_{m_k}, x_{m_k}) \to (y^*, x^*) \in X \times X$. There must exist some k_0 such that, for any $k \geqslant k_0$,

$$(y_{m_k}, x_{m_k}) \in (U(y^*) \cap X) \times (U(x^*) \cap X).$$

Then, it follows from (4.11) that

$$f(y_{m_k}, x_{m_k}) \in f(y^*, x^*) + V_1 - C, \quad \forall\, k \geqslant k_0.$$

Also, since $\varepsilon_{m_k} \to 0$ $(k \to \infty)$, there exists some k_1 such that, for any $k \geqslant k_1$, $\varepsilon_{m_k} e \in V_1$. Take $\bar{k} = \max\{k_0, k_1\}$. Then, for any $k \geqslant \bar{k}$,

$$f(y_{m_k}, x_{m_k}) + \varepsilon_{m_k} e \in f(y^*, x^*) + V_1 - C + V_1 \subseteq f(y^*, x^*) + V_0 - C.$$

Thus, by (4.10), we obtain

$$f(y_{m_k}, x_{m_k}) + \varepsilon_{m_k} e \notin C,$$

which contradicts (4.8). Therefore $f(y^*, x^*) \in C$.

(II) x^* solves SVEP.

Indeed, since the set X is compact, it is bounded. And so, the sequence $\{x_m\}$ is bounded as it is contained in X. Then, by noting that $\rho_m = \max\{\|x_0\|, \|x_1\|, \cdots, \|x_m\|\}$, we know the sequence $\{\rho_m\}$ is also bounded. Let $\rho^* = \sup\{\rho_m\}$. Take any $\delta \in (0, 1)$ and let $B(\delta)$ be the open ball centered at 0 with radius $\rho^* + 1 - \delta$. Then, we have

$$\|x_m\| \leqslant \rho_m \leqslant \rho^* < \rho^* + 1 - \delta, \ \ \forall m.$$

It follows that x^* belongs to the interior of $B(\delta)$. We claim that

$$f(y, x^*) \in -C, \ \ \forall y \in X \cap B(\delta). \tag{4.12}$$

This means that x^* solves AP on $X \cap B(\delta)$. Then, by invoking Theorems 3.4 and 2.13, we conclude that x^* solves SVEP on X.

Hence, there remains to show that (4.12) holds. Indeed, we can choose an m_0 satisfying $\rho_{m_0} \geqslant \rho^* - \delta$. Then, by observing that $\{\rho_m\}$ is nondecreasing, we have $\rho_m + 1 \geqslant \rho^* + 1 - \delta$ for all $m \geqslant m_0$. Thus $X \cap B(\delta) \subseteq X_m$ for all $m \geqslant m_0$. Therefore, for each $y \in X \cap B(\delta)$, we have $y \in X_m$ for all $m \geqslant m_0$. And then, by (4.2), we know that, for each $m \geqslant m_0$, there exists some $y_m \in X_m$ such that

$$f(y, x_m) \in f(y_m, x_m) + \varepsilon_m e - C. \tag{4.13}$$

Then, by the C-continuity of f, we can conclude that

$$f(y, x^*) \in f(y^*, x^*) - C. \tag{4.14}$$

In fact, suppose to the contrary that (4.14) is not true, i.e.,

$$f(y, x^*) - f(y^*, x^*) \notin -C.$$

Note that C is closed. There exists some neighborhood V of 0 in Z such that

$$[f(y, x^*) - f(y^*, x^*) + V] \cap (-C) = \varnothing.$$

As C is a convex cone, we can further get

$$[f(y, x^*) - f(y^*, x^*) + V + C] \cap (-C) = \varnothing. \tag{4.15}$$

For the above neighborhood V of 0 in Z, since Z is a topological vector space, there exists a balanced neighborhood V' of 0 in Z such that $V' + V' + V' \subseteq V$. For the balanced neighborhood V', since f is C-u.s.c., there exist neighborhoods $U(y^*)$ and $U_1(x^*)$ of y^* and x^*, respectively, such that

$$f(u, x) \in f(y^*, x^*) + V' - C, \ \ \forall (u, x) \in (U(y^*) \cap X) \times (U_1(x^*) \cap X). \tag{4.16}$$

Observe that $\{(y_{m_k}, x_{m_k})\} \subseteq X \times X$ and $(y_{m_k}, x_{m_k}) \to (y^*, x^*) \in X \times X$. There exists some k_0 such that, for every $k \geqslant k_1$, $(y_{m_k}, x_{m_k}) \in (U(y^*) \cap X) \times (U_1(x^*) \cap X)$. Then, it follows from (4.16) that

$$f(y_{m_k}, x_{m_k}) \in f(y^*, x^*) + V' - C, \ \ \forall k \geqslant k_1. \tag{4.17}$$

Furthermore, since $f(y, \cdot)$ is C-l.s.c., there exists some neighborhood $U_2(x^*)$ of x^*, such that

$$f(y, x) \in f(y, x^*) + V' + C, \ \ \forall x \in (U_2(x^*) \cap X). \tag{4.18}$$

As $\{x_{m_k}\} \subseteq X$ and $x_{m_k} \to x^* \in X$, there exists k_2 such that, for every $k \geqslant k_2$, $x_{m_k} \in (U_2(x^*) \cap X)$. And then, by (4.18), we have

$$f(y, x_{m_k}) \in f(y, x^*) + V' + C, \ \ \forall k \geqslant k_2. \tag{4.19}$$

In addition, since $\varepsilon_{m_k} \to 0 (k \to \infty)$, there exists k_3 such that, for any $k \geqslant k_3$,

$$\varepsilon_{m_k} e \in V', \ \ \forall k \geqslant k_3. \tag{4.20}$$

Take $\bar{k} = \max\{k_1, k_2, k_3\}$. Then, for each $k \geqslant \bar{k}$, (4.17), (4.19), and (4.20) hold at the same time. Noting that V' is balanced and C is a convex cone, we can get, for every $k \geqslant \bar{k}$,

$$\begin{aligned} f(y, x_{m_k}) - [f(y_{m_k}, x_{m_k}) + \varepsilon_{m_k} e] &\in f(y, x^*) + V' + C - f(y^*, x^*) - V' + C - V' \\ &= f(y, x^*) - f(y^*, x^*) + V' + V' + V' + C + C \\ &\subseteq f(y, x^*) - f(y^*, x^*) + V + C. \end{aligned}$$

Combining with (4.15), we can get

$$f(y, x_{m_k}) - [f(y_{m_k}, x_{m_k}) + \varepsilon_{m_k} e] \notin -C.$$

That is

$$f(y, x_{m_k}) \notin f(y_{m_k}, x_{m_k}) + \varepsilon_{m_k} e - C,$$

which contradicts (4.13). Hence $f(y, x^*) \in f(y^*, x^*) - C$.

(III) $x_m \to x^*$ as $m \to \infty$.

In fact, by (4.12), we can get, for any $\delta \in (0, 1)$,

$$f(y, x^*) \in -C, \quad \forall\, y \in X, \ \|y\| < \rho^* + 1 - \delta.$$

As δ can be chosen as any point in $(0, 1)$, we conclude that

$$f(y, x^*) \in -C, \quad \forall\, y \in X, \ \|y\| < \rho^* + 1. \tag{4.21}$$

Since $f(\cdot, x^*)$ is C-l.s.c., it is l.s.c.. Then, by (4.21), we can further obtain

$$f(y, x^*) \in -C, \quad \forall\, y \in X, \ \|y\| \leqslant \rho^* + 1.$$

Observe that $y_m \in X_m$, and so $\|y_m\| \leqslant \rho_m + 1 \leqslant \rho^* + 1$. Thus,

$$f(y_m, x^*) \in -C, \quad \forall\, m.$$

This means $x^* \in \cap_{m=1}^{\infty} L_f(y_m)$. Then, by Lemma 4.3 (i), we conclude that $\{\|x_m - x^*\|\}$ is nonincreasing. Obviously, the sequence $\{\|x_m - x^*\|\}$ is nonnegative, and so it is convergent. Furthermore, since the sequence $\{\|x_m - x^*\|\}$ has a subsequence $\{\|x_{m_k} - x^*\|\}$, which converges to 0. It follows that the whole sequence $\{\|x_m - x^*\|\}$ converges to 0, i.e., $x_m \to x^*$ as $m \to \infty$.

(ii) Assume that the sequence $\{x_m\}$ converges to some point $x^* \in X$. Note that the hypothesis $\cap_{m=1}^{\infty} L_f(y_m) \neq \varnothing$ (missing in this item) was used in the proof of item (i) only to establish that $\lim\limits_{m\to\infty} (x_m - P_{L_f(y_m)}(x_m)) = 0$. When $\{x_m\}$ converges, we can prove that this fact occurs. And then, by proceeding the arguments as that in the proof of item (i), we can show that x^* is a solution of SVEP, contradicting the hypothesis of this item. Therefore $\{x_m\}$ does not converge.

So, it remains to show that, when $\{x_m\}$ converges, $\lim\limits_{m\to\infty} (x_m - P_{L_f(y_m)}(x_m)) = 0$. In fact, if $\{x_m\}$ converges, then, by (4.3) and the assumptions on $\{\lambda_m\}$, we have

$$\| x_{m+1} - x_m \| = \lambda_m \| P_{L_f(y_m)}(x_m) - x_m \| \geqslant \alpha \| P_{L_f(y_m)}(x_m) - x_m \| .$$

As $\alpha > 0$ and $\{x_m\}$ converges, we know that $\| P_{L_f(y_m)}(x_m) - x_m \| \to 0$ as $m \to \infty$, i.e., $\lim\limits_{m\to\infty} (x_m - P_{L_f(y_m)}(x_m)) = 0$. □

Remark 4.5. From the proof of Theorem 4.4 (ii), we know that if $\{x_m\}$ is convergent, then the limit of $\{x_m\}$ must be a solution of SVEP.

Remark 4.6. Theorem 4.4 extends and develops the corresponding results of Wang and Li [20] and Cheng and Liu [3]. More precisely,

(i) Theorem 4.4 extends the main result (Theorem 4.1) of Wang and Li [20] mainly in the following aspects: (a) the space H is generalized from an n-dimensional Euclidean space to a Hilbert space; (b) the condition that C has a nonempty interior is canceled, and the condition that $e \in \text{int}(C)$ is weakened to $e \in C \setminus \{0\}$; (c) the methods constructing the iterative sequence $\{x_m\}$ and analyzing its convergence are both different. In [20], by using the Gerstewitz nonlinear scalarization function, Wang and Li transformed the vector equilibrium problem to a scalar problem called convex feasibility problem for constructing the iterative sequence $\{x_m\}$ and analyzed its convergence, where the Gerstewitz nonlinear scalarization function, which has many superior properties, played a significant role; while, in Theorem 4.4 of this paper, we constructed the iterative sequence $\{x_m\}$ directly based on a vector auxiliary problem and analyzed its convergence.

(ii) Theorem 4.4 develops the main results (Theorems 3 and 4) of Cheng and Liu [3] in the aspects (a)-(c) listed in item (i) and the following ones: (d) the studied problem is enhanced from weak vector equilibrium problem to strong one; (e) the requirement of continuity on the mapping f is weakened. In fact, in Theorems 3 and 4 of Cheng and Liu [3], the mapping f needs to be continuous; but, in Theorem 4.4 of this paper, the mapping f just needs to be C-continuous.

Corollary 4.7. *If AP has solutions, then the sequence $\{x_m\}$ generated by Algorithm 4.1 converges to a solution of SVEP.*

Proof. If AP has solutions, then $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$. And so the conclusion follows immediately from Theorem 4.4 (i). □

Corollary 4.8. *Let $\{x_m\}$ and $\{y_m\}$ be the sequences generated by Algorithm 4.1. If f is strongly C-pseudomonotone, then $\{x_m\}$ converges to a solution of SVEP.*

Proof. If f is strongly C-pseudomonotone, then, by Theorem 3.7, we have $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$. And so the conclusion follows immediately from Theorem 4.4 (i). □

As a consequence of Theorem 3.6 and Corollary 4.7, we have the following result of convergence.

Theorem 4.9. *Suppose that the vector-valued mapping f satisfies (P1)-(P4) and the following conditions:*

(i) *f is C-pseudomonotone;*

(iii) *for each $x \in X$, $f(x, y)$ is properly C-quasiconvex in y.*

Then, the sequence $\{x_m\}$ generated by Algorithm 4.1 converges to a solution of SVEP.

Proof. By Theorem 3.6, AP has at least one solution in X. And so, by Corollary 4.7, the sequence $\{x_m\}$ generated by Algorithm 4.1 converges to a solution of SVEP. □

In Theorem 4.4, if E is a finite-dimensional Euclidean space $\mathbb{R}^n$, then the compactness of X can be replaced by closedness.

Theorem 4.10. *Let E be an n-dimensional Euclidean space $\mathbb{R}^n$, and X be a nonempty closed convex subset of E. Suppose that $\{x_m\}$ and $\{y_m\}$ are the sequences generated by Algorithm 4.1.*

(i) *If $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$, then $\{x_m\}$ converges to a solution of SVEP.*

(ii) *If SVEP has no solution, then $\{x_m\}$ does not converge.*

Proof. First of all, Algorithm 4.1 is well-defined. In fact, the compactness condition of X is only used in the proof of Lemma 4.2 to obtain the compactness of the subsets $X_m, m \in \mathbb{N}_+$. When E is an n-dimensional Euclidean space $\mathbb{R}^n$ and X is a closed subset of E, $X_m, m \in \mathbb{N}_+$ are trivially compact as they are nonempty bounded and closed subsets of X. And then, by proceeding the arguments as that in the proof of Lemma 4.2, we can show that Algorithm 4.1 is well-defined.

Next, we shall prove the conclusions are true.

(i) Suppose that $\cap_{m=1}^{\infty} L_f(y_m) \neq \emptyset$, then, by Lemma 4.3 (ii), we know that, for any $\bar{x} \in \cap_{m=1}^{\infty} L_f(y_m)$, the sequence $\{\|x_m - \bar{x}\|\}$ is convergent. And so $\{x_m\}$ is bounded, i.e., there exists some $r > 0$ such that

$\|x_m\| \leqslant r$ for all $m \in \mathbb{N}_+$. And thus,

$$\rho_m = \max\{\|x_0\|, \|x_1\|, \cdots, \|x_m\|\} \leqslant r, \quad \text{for all } m \in \mathbb{N}_+.$$

This means that the set X_m is contained in the closed ball centered at 0 with radius $r+1$. And so, for each $m \in \mathbb{N}_+$, $\|y_m\| \leqslant r+1$ as $y_m \in X_m$. That is to say $\{y_m\}$ is also bounded. Let x^* be a cluster point of $\{x_m\}$. Then, there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k} \to x^*$ as $k \to \infty$. As X is closed, we have $x^* \in X$. Moreover, since $\{y_{m_k}\} \subseteq X$ is bounded, without loss of generality, we may assume that $y_{m_k} \to y^*$ for some $y^* \in X$ as $k \to \infty$. Then, by proceeding the rest of arguments as that in the proof of Theorem 4.4 (i), we can show that x^* is a solution of SVEP and $x_m \to x^*$ as $m \to \infty$.

(ii) Assume that the sequence $\{x_m\}$ converges to some point $x^* \in X$. Note that the hypothesis $\cap_{m=1}^{\infty} L_f(y_m) \neq \varnothing$ (missing in this item) was used in the proof of item (i) only to establish that $\{x_m\}$ is bounded and, as a consequence, $\lim_{m\to\infty}(x_m - P_{L_f(y_m)}(x_m)) = 0$. When $\{x_m\}$ converges, we can prove that these two facts occur. And then, by proceeding the arguments as that in the proof of (i), we can show that x^* is a solution of SVEP, contradicting the hypothesis of this item. Therefore $\{x_m\}$ does not converge.

So, it remains to show that, when $\{x_m\}$ converges, $\{x_m\}$ is bounded and $\lim_{m\to\infty}(x_m - P_{L_f(y_m)}(x_m)) = 0$. In fact, if $\{x_m\}$ converges, then it is clearly bounded. Furthermore, by (4.3) and the assumptions on $\{\lambda_m\}$, we have

$$\| x_{m+1} - x_m \| = \lambda_m \| P_{L_f(y_m)}(x_m) - x_m \| \geqslant \alpha \| P_{L_f(y_m)}(x_m) - x_m \| .$$

As $\alpha > 0$ and $\{x_m\}$ converges, we know that $\| P_{L_f(y_m)}(x_m) - x_m \| \to 0$ as $m \to \infty$, i.e., $\lim_{m\to\infty}(x_m - P_{L_f(y_m)}(x_m)) = 0$. □

Remark 4.11. In Theorem 4.10, the subset X may be unbounded.

Remark 4.12. In Theorem 4.10, if $Z = \mathbb{R}, C = \mathbb{R}_+ = [0, +\infty), e = 1 \in C \setminus \{0\}$, then, the condition (P4) can be removed, and so Theorem 4.10 reduces to Theorem 3.3 of Iusem and Sosa [12]. In fact, the condition (P4) is only used as one of conditions for guaranteeing the existence of maximal point y_m in (4.2). When $Z = \mathbb{R}, C = \mathbb{R}_+ = [0, +\infty), e = 1 \in C \setminus \{0\}$, we can also prove the existence of maximal point y_m in (4.2) without assumption (P4). Indeed, when $Z = \mathbb{R}, C = \mathbb{R}_+ = [0, +\infty), e = 1 \in C \setminus \{0\}$, the C-continuity of f coincides with the usual continuity of function. Moreover, for each $m \in \mathbb{N}_+$, X_m is clearly compact. Thus, the real-valued function $f(\cdot, x_m)$ can attain its maximum value on X_m, i.e., there exists some point $y_m \in X_m$ such that

$$f(y, x_m) \leqslant f(y_m, x_m) \quad \text{for all } y \in X_m.$$

That is,

$$f(y, x_m) \in f(y_m, x_m) - C \quad \text{for all } y \in X_m.$$

From this, it is easy to see that y_m satisfies (4.2).

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