General L_p-mixed width-integral of convex bodies and related inequalities

Yanping Zhou

Department of Mathematics, Shanghai University, Shanghai 200444, China.

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Abstract

The conception of general L_p-mixed width-integral of convex bodies is introduced and related isoperimetric type inequality, Aleksandrov-Fenchel type inequality and a cyclic inequality are established. Further, the extremum values for the general L_p-mixed width-integral are obtained. ©2017 All rights reserved.

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1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [9, 30])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The study of width integral has a long history. The notion of the classical width-integral was first considered by Blaschke (see [4]) and was further studied by Hardy et al. (see [16]). It was generalized to the mixed width-integral by Lutwak [23] in 1977. For the more results associated with the mixed width-integral, we refer the interested reader to [17, 21, 22, 25]. Particularly, Feng [6] generalized the definition of mixed width integral to general mixed width integral: for $\tau \in (-1, 1)$, the general mixed width-integral, $B^{(\tau)}(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n \in \mathcal{K}_o^n$ is defined by

$$B^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_n, u) dS(u),$$

where $b^{(\tau)}(K, u) = f_1(\tau)h(K, u) + f_2(\tau)h(K, -u)$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are chosen as follows

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}.$$

The main aim of this article is to define a corresponding notion of general L_p-mixed width-integral, and to extend Feng's inequalities to the inequalities related to general L_p-mixed width-integral. Moreover, the extremum values and two Brunn-Minkowski type inequalities for the general L_p-mixed width-integral are established.

Now, we define the general L_p-mixed width-integral as follows: for $\tau \in [-1, 1]$ and $p > 0$, the general L_p-mixed width-integral, $B_p^{(\tau)}(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n \in \mathcal{K}_o^n$ is defined by

$$B_p^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K_1, u) \cdots b_p^{(\tau)}(K_n, u) dS(u), \tag{1.1}$$

where $b_p^{(\tau)}(K, u) = (f_1(\tau)h^p(K, u) + f_2(\tau)h^p(K, -u))^{\frac{1}{p}}$ for any $u \in S^{n-1}$, and $f_1(\tau)$, $f_2(\tau)$ are chosen as follows

$$f_1(\tau) = \frac{(1+\tau)^{2p}}{(1+\tau)^{2p} + (1-\tau)^{2p}}, \quad f_2(\tau) = \frac{(1-\tau)^{2p}}{(1+\tau)^{2p} + (1-\tau)^{2p}}.$$

Obviously, $f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau) + f_2(\tau) = 1,$$
$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau).$$

Note that the case $p = 1$ in definition (1.1) is just Feng's general mixed width-integral. Two convex bodies K and L are said to have similar general L_p-width if there exists a constant $\lambda > 0$ such that $b_p^{(\tau)}(K, u) = \lambda b_p^{(\tau)}(L, u)$ for all $u \in S^{n-1}$. If $b_p^{(\tau)}(K, u) = b_p^{(\tau)}(L, u)$ for all $u \in S^{n-1}$, then we call K and L have the same general L_p-width.

Taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$ in (1.1), the general L_p-mixed width-integral, $B_{p,i}^{(\tau)}(K, L)$, of $K, L \in \mathcal{K}_o^n$ is given by

$$B_{p,i}^{(\tau)}(K, L) = \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^{n-i} b_p^{(\tau)}(L, u)^i dS(u). \tag{1.2}$$

Further, let $L = B$ in (1.2). Then we write $B_{p,i}^{(\tau)}(K)$ for $B_{p,i}^{(\tau)}(K, B)$. Since $b_p^{(\tau)}(B, u) = 1$, it follows that

$$B_{p,i}^{(\tau)}(K) = \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^{n-i} dS(u), \tag{1.3}$$

where $B_{p,i}^{(\tau)}(K)$ is called the ith general L_p-mixed width-integral of K. If $i = 0$, then we write $B_{p,0}^{(\tau)}(K) = B_p^{(\tau)}(K)$, which we call the general L_p-width-integral of K, namely,

$$B_p^{(\tau)}(K) = \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^n dS(u).$$

The general operator belongs to the asymmetric Brunn-Minkowski theory which has its starting point in the theory of valuations in connection with isoperimetric and analytic inequalities (see [1, 2, 7, 8, 10–15, 19, 20, 27–29, 31–35]).

The main results are the following: we first establish the isoperimetric type, Aleksandrov-Fenchel type and cyclic inequalities for the general L_p-mixed width-integral.

Theorem 1.1. *If $\tau \in [-1, 1]$ and $K_1, \cdots, K_n \in \mathcal{K}_c^n$, then for $p > 0$*

$$V(K_1) \cdots V(K_n) \leqslant B_p^{(\tau)}(K_1, \cdots, K_n)^n, \tag{1.4}$$

with equality if and only if $K_1, \cdots, K_n$ are n-balls.

Theorem 1.2. *If $\tau \in [-1, 1]$, $K_1, \cdots, K_n \in \mathcal{K}_o^n$, and $1 < m \leqslant n$, then for $p > 0$*

$$B_p^{(\tau)}(K_1, \cdots, K_n)^m \leqslant \prod_{i=1}^m B_p^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-i+1}, \cdots, K_{n-i+1}), \quad (1.5)$$

with equality if and only if $K_{n-m+1}, \cdots, K_n$ are all of similar general L_p-width.

Theorem 1.3. *If $\tau \in [-1, 1]$, $K, L \in \mathcal{K}_o^n$, and $p > 0$, then for $i < j < k$,*

$$B_{p,i}^{(\tau)}(K, L)^{k-j} B_{p,k}^{(\tau)}(K, L)^{j-i} \geqslant B_{p,j}^{(\tau)}(K, L)^{k-i}, \quad (1.6)$$

with equality if and only if K and L have similar general L_p-width.

Moreover, the extremum values for the general L_p-mixed width-integral are also established.

Theorem 1.4. *If $\tau \in [-1, 1]$, $K \in \mathcal{K}_o^n$, and $p > 0$, then for $i < n - p$,*

$$B_{p,i}^{(0)}(K) \leqslant B_{p,i}^{(\tau)}(K) \leqslant B_{p,i}^{(\pm 1)}(K); \quad (1.7)$$

for $n - p < i < n$ or $i > n$

$$B_{p,i}^{(0)}(K) \geqslant B_{p,i}^{(\tau)}(K) \geqslant B_{p,i}^{(\pm 1)}(K). \quad (1.8)$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$. When $i = n$, (1.7) and (1.8) are identical.

The proofs of theorems above will be given in Section 3 of this paper. In Section 4, two Brunn-Minkowski type inequalities for the ith general L_p-mixed width-integral will be established.

2. Preliminaries

The radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact star-shaped (about the origin) set K in $\mathbb{R}^n$ is defined, for $u \in S^{n-1}$, by (see [9, 30])

$$\rho(K, u) = \max\{\lambda \geqslant 0 : \lambda \cdot u \in K\}.$$

The polar body, K^*, of $K \in \mathcal{K}^n$ is defined by (see [9, 30])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}.$$

It is easy to check that for $K \in \mathcal{K}_o^n$,

$$(K^*)^* = K \quad \text{and} \quad h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.$$

An extension of the well-known Blaschke-Santaló inequality is as follows (see [24]).

Theorem 2.1. *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leqslant \omega_n^2,$$

with equality if and only if K is an ellipsoid.

For $K, L \in \mathcal{K}_o^n$, $p \geqslant 1$ and $\lambda, \mu \geqslant 0$ (not both zero), the Firey L_p-combination (also called L_p-Minkowski combination), $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [26])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p,$$

where the operation "$+_p$" is called Firey addition and $\lambda \cdot K$ denotes the Firey scalar multiplication.

The polar coordinate formula for volume of a body K in $\mathbb{R}^n$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u).$$

3. Proof of main results

Proof of Theorem 1.1. From Jensen's inequality (see [16]), we obtain

$$
\begin{aligned}
B_p^{(\tau)}(K_1, \cdots, K_n) &= \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K_1, u) \cdots b_p^{(\tau)}(K_n, u) dS(u) \\
&\geqslant n\omega_n^2 \left[\int_{S^{n-1}} b_p^{(\tau)}(K_1, u)^{-1} \cdots b_p^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-1}
\end{aligned}
\tag{3.1}
$$

with equality if and only if $K_1, \cdots, K_n$ have joint constant general L_p-width. Together with Hölder's inequality (see [16]), we have

$$
\left[\int_{S^{n-1}} b_p^{(\tau)}(K_1, u)^{-1} \cdots b_p^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-n} \geqslant \prod_{i=1}^{n} \left[\int_{S^{n-1}} b_p^{(\tau)}(K_i, u)^{-n} dS(u) \right]^{-1},
\tag{3.2}
$$

with equality if and only if $K_1, \cdots, K_n$ have similar general L_p-width. Using Minkowski's inequality (see [16]), we have

$$
\begin{aligned}
\left[\frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K_i, u)^{-n} dS(u) \right]^{-\frac{p}{n}} &= \left[\frac{1}{n} \int_{S^{n-1}} (f_1(\tau) h^p(K_i, u) + f_2(\tau) h^p(K_i, -u))^{-\frac{n}{p}} dS(u) \right]^{-\frac{p}{n}} \\
&\geqslant \left[\frac{1}{n} \int_{S^{n-1}} (f_1(\tau) h^p(K_i, u))^{-\frac{n}{p}} dS(u) \right]^{-\frac{p}{n}} \\
&\quad + \left[\frac{1}{n} \int_{S^{n-1}} (f_2(\tau) h^p(K_i, -u))^{-\frac{n}{p}} dS(u) \right]^{-\frac{p}{n}} \\
&\geqslant \left[\frac{1}{n} \int_{S^{n-1}} h(K_i, u)^{-n} dS(u) \right]^{-\frac{p}{n}} = V(K_i^*)^{-\frac{p}{n}},
\end{aligned}
\tag{3.3}
$$

with equality if and only if K_i is origin-symmetric. Since $p > 0$, it follows from Theorem 2.1 and inequality (3.3) that,

$$
\left[\frac{1}{n\omega_n^2} \int_{S^{n-1}} b_p^{(\tau)}(K_i, u)^{-n} dS(u) \right]^{-1} \geqslant V(K_i),
\tag{3.4}
$$

with equality if and only if K_i is an n-dimensional ellipsoid. By inequalities (3.1), (3.2), and (3.4), this gives

$$
V(K_1) \cdots V(K_n) \leqslant B_p^{(\tau)}(K_1, \cdots, K_n)^n.
$$

By the equality conditions of inequalities (3.1), (3.2), and (3.4), equality holds in (1.4) if and only if $K_1, \cdots, K_n$ are n-balls. $\qquad\square$

Lemma 3.1 ([22]). *If $f_0, f_1, \cdots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \cdots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$
\int_{S^{n-1}} f_0(u) f_1(u) \cdots f_m(u) dS(u) \leqslant \prod_{i=1}^{m} \left[\int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u) dS(u) \right]^{\frac{1}{\lambda_i}},
\tag{3.5}
$$

with equality if and only if there exist positive constants $\alpha_1, \cdots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.2. Let in Lemma 3.1

$$
\begin{aligned}
\lambda_i &= m \quad (1 \leqslant i \leqslant m); \\
f_0 &= b_p^{(\tau)}(K_1, u) \cdots b_p^{(\tau)}(K_{n-m}, u) \quad (f_0 = 1 \ \text{if} \ m = n);
\end{aligned}
$$

$$f_i = b_p^{(\tau)}(K_{n-i+1}, u) \quad (1 \leqslant i \leqslant m).$$

Then

$$\int_{S^{n-1}} b_p^{(\tau)}(K_1, u) \cdots b_p^{(\tau)}(K_n, u) dS(u)$$

$$\leqslant \prod_{i=1}^m \left[\int_{S^{n-1}} b_p^{(\tau)}(K_1, u) \cdots b_p^{(\tau)}(K_{n-m}, u) b_p^{(\tau)}(K_{n-i+1}, u)^m dS(u) \right]^{\frac{1}{m}}.$$

Combining with (1.1), we have

$$B_p^{(\tau)}(K_1, \cdots, K_n)^m \leqslant \prod_{i=1}^m B_p^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-i+1}, \cdots, K_{n-i+1}).$$

The equality condition of inequality (3.5) implies that equality holds in (1.5) if and only if K_{n-m+1}, $\cdots$, K_n are all of similar general L_p-width. $\qquad \square$

Proof of Theorem 1.3. It follows from Hölder's inequality (see [16]) that

$$B_{p,i}^{(\tau)}(K, L)^{\frac{k-j}{k-i}} B_{p,k}^{(\tau)}(K, L)^{\frac{j-i}{k-i}} = \left(\frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^{n-i} b_p^{(\tau)}(L, u)^i dS(u) \right)^{\frac{k-j}{k-i}}$$

$$\times \left(\frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^{n-k} b_p^{(\tau)}(L, u)^k dS(u) \right)^{\frac{j-i}{k-i}}$$

$$\geqslant \frac{1}{n} \int_{S^{n-1}} b_p^{(\tau)}(K, u)^{n-j} b_p^{(\tau)}(L, u)^j dS(u) = B_{p,j}^{(\tau)}(K, L).$$

This gives

$$B_{p,i}^{(\tau)}(K, L)^{k-j} B_{p,k}^{(\tau)}(K, L)^{j-i} \geqslant B_{p,j}^{(\tau)}(K, L)^{k-i}.$$

The equality condition of Hölder's inequality implies that equality holds in (1.6) if and only if K and L have similar general L_p-width. $\qquad \square$

Taking $i = 0$, $j = i$, and $k = n$ in inequality (1.6), we have the following.

Corollary 3.2. *If* $\tau \in [-1, 1]$, $K, L \in \mathcal{K}_o^n$, *and* $p > 0$, *then for* $0 \leqslant i \leqslant n$,

$$B_{p,i}^{(\tau)}(K, L)^n \leqslant B_p^{(\tau)}(K)^{n-i} B_p^{(\tau)}(L)^i; \tag{3.6}$$

for $i < 0$ *or* $i > n$, *inequality* (3.6) *is reversed, with equality in every inequality if and only if* $i = n$ *or, when* $i \neq n$, *K and L have similar general* L_p-*width.*

Let $i = 1$ in Corollary 3.2. The dual Minkowski type inequalities for the general L_p-mixed width-integral are as follows.

Corollary 3.3. *If* $\tau \in [-1, 1]$ *and* $K, L \in \mathcal{K}_o^n$, *then*

$$B_{p,1}^{(\tau)}(K, L)^n \leqslant B_p^{(\tau)}(K)^{n-1} B_p^{(\tau)}(L),$$

with equality if and only if K and L have similar general L_p-*width.*

Corollary 3.4. *If* $\tau \in [-1, 1]$ *and* $K, L \in \mathcal{K}_o^n$, *then*

$$B_{p,-1}^{(\tau)}(K, L)^n \geqslant B_p^{(\tau)}(K)^{n+1} B_p^{(\tau)}(L)^{-1},$$

with equality if and only if K and L have similar general L_p-*width.*

Proof of Theorem 1.4. From (1.3) and using Minkowski's inequality (see [16]), we obtain that for $i < n - p$

$$
\begin{aligned}
B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} &= \left[\frac{1}{n}\int_{S^{n-1}} b_p^{(\tau)}(K,u)^{n-i} dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} (f_1(\tau)h^p(K,u) + f_2(\tau)h^p(K,-u))^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&\leqslant \left[\frac{1}{n}\int_{S^{n-1}} (f_1(\tau)h^p(K,u))^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} (f_2(\tau)h^p(K,-u))^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} (h^p(K,u))^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \text{ or } \left[\frac{1}{n}\int_{S^{n-1}} (h^p(K,-u))^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} (h(K,u))^{n-i} dS(u)\right]^{\frac{p}{n-i}}.
\end{aligned}
\tag{3.7}
$$

Similarly, from Minkowski's inequality we have that for $n - p < i < n$ or $i < n$,

$$
B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} \geqslant \left[\frac{1}{n}\int_{S^{n-1}} (h(K,u))^{n-i} dS(u)\right]^{\frac{p}{n-i}}.
$$

From the equality condition of Minkowski's inequality, we see that equalities hold in (3.6) and (3.7) if and only if K and $-K$ are dilates of one another, i.e., K is centered.

From $f_1(+1) = 1$ and $f_2(+1) = 0$, we have

$$
B_{p,i}^{(+1)}(K) = \frac{1}{n}\int_{S^{n-1}} (h^p(K,u))^{\frac{n-i}{p}} dS(u) = \frac{1}{n}\int_{S^{n-1}} (h(K,u))^{n-i} dS(u).
$$

Together with (3.7), we have

$$
B_{p,i}^{(\tau)}(K) \leqslant B_{p,i}^{(\pm 1)}(K),
$$

which is just the right inequality of (1.3).

According to the equality condition of (3.7), we know that if $\tau \neq \pm 1$, then equality holds in the right inequality of (1.7) if and only if K is centered, i.e., if K is not origin-symmetric, equality holds in the right inequality if and only if $\tau = \pm 1$.

Now, we prove the left inequality of (1.7). Since $f_1(0) = f_2(0) = \frac{1}{2}$, together with Minkowski's inequality, we have

$$
\begin{aligned}
&B_{p,i}^{(0)}(K)^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} b_p^{(0)}(K,u)^{n-i} dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}h^p(K,u) + \frac{1}{2}h^p(K,-u)\right)^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}f_1(\tau)h^p(K,u) + \frac{1}{2}f_2(\tau)h^p(K,-u) + \frac{1}{2}f_1(\tau)h^p(K,-u) + \frac{1}{2}f_2(\tau)h^p(K,u)\right)^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}} \\
&\leqslant \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}f_1(\tau)h^p(K,u) + \frac{1}{2}f_2(\tau)h^p(K,-u)\right)^{\frac{n-i}{p}} dS(u)\right]^{\frac{p}{n-i}}
\end{aligned}
$$

$$+\left[\frac{1}{n}\int_{S^{n-1}}\left(\frac{1}{2}f_1(\tau)h^p(K,-u)+\frac{1}{2}f_2(\tau)h^p(K,u)\right)^{\frac{n-i}{p}}dS(u)\right]^{\frac{p}{n-i}}$$

$$=\left[\frac{1}{n}\int_{S^{n-1}}(f_1(\tau)h^p(K,u)+f_2(\tau)h^p(K,-u))^{\frac{n-i}{p}}dS(u)\right]^{\frac{p}{n-i}}$$

$$=\left[\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{n-i}dS(u)\right]^{\frac{p}{n-i}}$$

$$=B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}.$$

That is,

$$B_{p,i}^{(\tau)}(K)\geqslant B_{p,i}^{(0)}(K).$$

The equality condition of Minkowski's inequality tells us that if $\tau\neq 0$, then equality holds in the left inequality of (1.7) if and only if K is centered. Thus if K is not origin-symmetric, then equality holds in the left inequality of (1.7) if and only if $\tau=0$.

For the case $n-p < i < n$ or $i > n$, the proof is similar. $\square$

4. Brunn-Minkowski type inequalities

In this section, we establish two Brunn-Minkowski type inequalities for the ith general L_p-mixed width-integral.

Theorem 4.1. *If $\tau\in[-1,1]$, $K,L\in\mathcal{K}_o^n$, and $p > 0$, then for $i\leqslant n-p$,*

$$B_{p,i}^{(\tau)}(K+_p L)^{\frac{p}{n-i}}\leqslant B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}},$$

with equality if and only if K and L have similar general L_p-width.

In fact, the more general result than Theorem 4.1 will be obtained as follows.

Theorem 4.2. *If $K,L\in\mathcal{K}_o^n$, $\tau\in[-1,1]$, and $p > 0$, then for $i\leqslant n-p\leqslant j\leqslant n$ and $i\neq j$*

$$\left(\frac{B_{p,i}^{(\tau)}(K+_p L)}{B_{p,j}^{(\tau)}(K+_p L)}\right)^{\frac{p}{j-i}}\leqslant\left(\frac{B_{p,i}^{(\tau)}(K)}{B_{p,j}^{(\tau)}(K)}\right)^{\frac{p}{j-i}}+\left(\frac{B_{p,i}^{(\tau)}(L)}{B_{p,j}^{(\tau)}(L)}\right)^{\frac{p}{j-i}};\tag{4.1}$$

for $j\geqslant n\geqslant i\geqslant n-p$ and $i\neq j$

$$\left(\frac{B_{p,i}^{(\tau)}(K+_p L)}{B_{p,j}^{(\tau)}(K+_p L)}\right)^{\frac{p}{j-i}}\geqslant\left(\frac{B_{p,i}^{(\tau)}(K)}{B_{p,j}^{(\tau)}(K)}\right)^{\frac{p}{j-i}}+\left(\frac{B_{p,i}^{(\tau)}(L)}{B_{p,j}^{(\tau)}(L)}\right)^{\frac{p}{j-i}},\tag{4.2}$$

with equality in every inequality if and only if K and L have similar general L_p-width.

To complete the proof of Theorem 4.2, the following lemmas may be required. An extension of Beckenbach's inequality (see [3]) was obtained by Dresher (see [5]) through the means of moment-space techniques.

Lemma 4.3 (Beckenbach-Dresher inequality). *If $p\geqslant 1\geqslant r\geqslant 0$, $p\neq r$, $f,g\geqslant 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}}\leqslant\left(\frac{\int_{\mathbb{E}}f^p d\phi}{\int_{\mathbb{E}}f^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}}g^p d\phi}{\int_{\mathbb{E}}g^r d\phi}\right)^{\frac{1}{p-r}},\tag{4.3}$$

with equality if and only if the functions f and g are positively proportional.

Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

The inverse Beckenbach-Dresher inequality was established in the reference [18].

Lemma 4.4 (Inverse Beckenbach-Dresher inequality). *If $1 \geqslant p \geqslant 0 \geqslant r$, $p \neq r$, $f, g \geqslant 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p \, d\phi}{\int_{\mathbb{E}}(f+g)^r \, d\phi}\right)^{\frac{1}{p-r}} \geqslant \left(\frac{\int_{\mathbb{E}} f^p \, d\phi}{\int_{\mathbb{E}} f^r \, d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p \, d\phi}{\int_{\mathbb{E}} g^r \, d\phi}\right)^{\frac{1}{p-r}}, \tag{4.4}$$

with equality if and only if the functions f and g are positively proportional.

Proof of Theorem 4.2. From (1.3) and (4.3), it follows that for $i \leqslant n - p \leqslant j \leqslant n$

$$\left(\frac{B_{p,i}^{(\tau)}(K +_p L)}{B_{p,j}^{(\tau)}(K +_p L)}\right)^{\frac{p}{j-i}}$$

$$= \left(\frac{\frac{1}{n}\int_{S^{n-1}} b_p^{(\tau)}(K +_p L, u)^{n-i} \, dS(u)}{\frac{1}{n}\int_{S^{n-1}} b_p^{(\tau)}(K +_p L, u)^{n-j} \, dS(u)}\right)^{\frac{p}{j-i}}$$

$$= \left(\frac{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K +_p L, u) + f_2(\tau)h^p(K +_p L, -u)\right)^{\frac{n-i}{p}} \, dS(u)}{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K +_p L, u) + f_2(\tau)h^p(K +_p L, -u)\right)^{\frac{n-j}{p}} \, dS(u)}\right)^{\frac{1}{\frac{n-i}{p} - \frac{n-j}{p}}}$$

$$= \left(\frac{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K, u) + f_2(\tau)h^p(K, -u) + f_1(\tau)h^p(L, u) + f_2(\tau)h^p(L, -u)\right)^{\frac{n-i}{p}} \, dS(u)}{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K, u) + f_2(\tau)h^p(K, -u) + f_1(\tau)h^p(L, u) + f_2(\tau)h^p(L, -u)\right)^{\frac{n-j}{p}} \, dS(u)}\right)^{\frac{1}{\frac{n-i}{p} - \frac{n-j}{p}}}$$

$$\leqslant \left(\frac{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K, u) + f_2(\tau)h^p(K, -u)\right)^{\frac{n-i}{p}} \, dS(u)}{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K, u) + f_2(\tau)h^p(K, -u)\right)^{\frac{n-j}{p}} \, dS(u)}\right)^{\frac{p}{j-i}}$$

$$+ \left(\frac{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(L, u) + f_2(\tau)h^p(L, -u)\right)^{\frac{n-i}{p}} \, dS(u)}{\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(L, u) + f_2(\tau)h^p(L, -u)\right)^{\frac{n-j}{p}} \, dS(u)}\right)^{\frac{p}{j-i}}$$

$$= \left(\frac{B_{p,i}^{(\tau)}(K)}{B_{p,j}^{(\tau)}(K)}\right)^{\frac{p}{j-i}} + \left(\frac{B_{p,i}^{(\tau)}(L)}{B_{p,j}^{(\tau)}(L)}\right)^{\frac{p}{j-i}}.$$

This gets the desired inequality (4.1). Using the same method, inequality (4.2) follows from inequality (4.4).

Together with the equality conditions of inequalities (4.3) and (4.4), we see that equality holds in inequalities (4.1) and (4.2) if and only if $b_p^{(\tau)}(K, u)$ and $b_p^{(\tau)}(L, u)$ are positively proportional, namely, K and L have similar general L_p-width. $\square$

Let $j = n$ in (4.1). Since $B_{p,n}^{(\tau)}(K + L) = B_{p,n}^{(\tau)}(K) = B_{p,n}^{(\tau)}(L) = \omega_n$ is a constant, we obtain Theorem 4.1.

Acknowledgment

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