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A class of fractional order systems with not instantaneous impulses

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Abstract

This paper is concerned with a kind of fractional order systems with Caputo-Hadamard derivative (of order $q \in \mathbb{C}$ and $\Re(q) \in (1,2)$) and not instantaneous impulses. The obtained result uncovers that there exists a general solution for these impulsive systems, which means that the state trajectory of these impulsive systems is non-unique, and it is expounded by a numerical example. ©2017 All rights reserved.

Keywords: Fractional differential equations, impulsive fractional differential equations, not instantaneous impulses, general solution, state trajectory.

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1. Introduction

Fractional calculus has gained much attention since it provides a key tool to describe hereditary properties of some materials and processes in many fields of science and engineering, and the general theory of fractional calculus can be referred in [5, 12, 15], and there are some advances in numerical calculation, controllability, chaos synchronization, etc. for fractional differential equations [4, 11, 20–22].

Impulsive differential equations have been focused since it serves as an important tool to characterize the phenomena in which sudden, discontinuous jumps occur in various fields of science and engineering, and impulsive fractional (partial) differential equations have received much attentions [1, 2, 6, 8, 16–18, 23, 26]. However, impulses are instantaneous impulses in many existing papers about the impulsive models, and it can not describe some processes such as evolution processes in pharmacotherapy. Therefore, the authors in [9] presented a kind of impulsive differential equations with not instantaneous impulses, and the authors in [14] continued studying on differential equations with not instantaneous impulses in a PC_α-normed Banach space. Next, the fractional differential equations with not instantaneous impulses were considered in [13, 19].

Furthermore, the works in [7, 10] developed fractional calculus in frame of Caputo-Hadamard fractional derivative, and Caputo-Hadamard fractional differential equations were studied in [3]. Next, the recent results in [24, 25] discovered that Caputo-Hadamard fractional differential equations with instantaneous impulses have general solution, which uncovered that there may be general solution for Caputo-Hadamard fractional differential equations with not instantaneous impulses. Therefore, we will try to

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seek general solution for the following Caputo-Hadamard fractional differential equations with not instantaneous impulses:

$$\begin{cases} {}_{C-H}D^{q}_{a^+}z(t) = f(t, z(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, N, \\ z(t) = g_k(t, z(t)), & t \in (t_k, s_k], k = 1, 2, \ldots, N, \\ z(a) = z_a,\ z'(a) = \bar{z}_a, & z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \tag{1.1}$$

Here $q \in \mathbb{C}$, $\Re(q) \in (1, 2)$ and $a > 0$, ${}_{C-H}D^{q}_{a^+}$ denotes left-sided Caputo-Hadamard fractional derivative of order q, $f : [a, T] \times \mathbb{C} \to \mathbb{C}$ is an appropriate continuous function, $g_k : (t_k, s_k] \times \mathbb{C} \to \mathbb{C}$ which denote not instantaneous impulses are some appropriate continuous functions, $g'_k(s_k, z(s_k))$ exist (here $k = 1, 2, \ldots, N$), and $a = t_0 = s_0 < t_1 \leqslant s_1 \leqslant t_2 \leqslant \cdots \leqslant t_N \leqslant s_N \leqslant t_{N+1} = T$.

Next, we will introduce some definitions and conclusions in Section 2, and give the equivalent integral equations for a kind of Caputo-Hadamard fractional differential equations with not instantaneous impulses in Section 3. Finally, an example is provided to illustrate the obtained result.

2. Preliminaries

Definition 2.1 ([12, p.110])**.** Let $0 < a < b < \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}$ of function $x(t)$ is defined by

$${}_{H}\mathcal{J}^{\alpha}_{a^+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, \quad (a < t < b),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([10, p.4])**.** Let $\Re(\alpha) \geqslant 0$ and $n = [\Re(\alpha)] + 1$, $x \in \left\{x : [a, b] \to \mathbb{C} : \delta^{(n-1)}x(t) \in AC[a, b]\right\}$, $0 < a < b < \infty$. The left-sided Caputo-Hadamard fractional derivatives ${}_{C-H}D^{\alpha}_{a^+}x(t)$ exist everywhere on $[a, b]$ and if $\alpha \notin \mathbb{N}_0$,

$${}_{C-H}D^{\alpha}_{a^+}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \delta^n x(s)\frac{ds}{s} = {}_{H}\mathcal{J}^{n-\alpha}_{a^+}\delta^n x(t),$$

where differential operator $\delta = t\frac{d}{dt}$ with $\delta^0 x(t) = x(t)$. Moreover, if $\alpha = n \in \mathbb{N}_0$, ${}_{C-H}D^{\alpha}_{a^+}x(t) = \delta^n x(t)$. In particular, ${}_{C-H}D^{0}_{a^+}x(t) = x(t)$.

Lemma 2.3 ([10, p.5])**.** *Let $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$, and $x \in C[a, b]$. If $\Re(\alpha) \neq 0$ or $\alpha \in \mathbb{N}$, then*

$${}_{C-H}D^{\alpha}_{a^+}\left({}_{H}\mathcal{J}^{\alpha}_{a^+}x\right)(t) = x(t).$$

Lemma 2.4 ([10, p.6])**.** *Let $x \in AC^n_\delta[a, b]$ or $C^n_\delta[a, b]$ and $\alpha \in \mathbb{C}$, then*

$${}_{H}\mathcal{J}^{\alpha}_{a^+}\left({}_{C-H}D^{\alpha}_{a^+}x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!}\left(\ln \frac{t}{a}\right)^k.$$

Considering a special case in Corollary 17 in [25], we can draw the following conclusion.

Lemma 2.5. *Let $w \in \mathbb{C}$, $\Re(w) \in (1, 2)$, $\sum_{i=1}^{0} y_i = 0$, and ξ, ζ are two constants. The impulsive system*

$$\begin{cases} {}_{C-H}D^{w}_{a^+}x(t) = g(t, x(t)), & t \in (a, T],\ t \neq t_k (k = 1, 2, \ldots, m), \\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k\left(x(t_k^-)\right) \in \mathbb{C},\ k = 1, 2, \ldots, m, \\ \Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-) = \bar{I}_k\left(x(t_k^-)\right) \in \mathbb{C},\ k = 1, 2, \ldots, m, \\ x(a) = x_a \in \mathbb{C},\ x'(a) = \bar{x}_a \in \mathbb{C} \end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= x_a + a\bar{x}_a \ln\frac{t}{a} + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} t_i \bar{I}_i(x(t_i^-)) \ln\frac{t}{t_i} + \frac{1}{\Gamma(w)}\int_a^t \left(\ln\frac{t}{s}\right)^{w-1} g\frac{ds}{s} \\ &+ \sum_{i=1}^{k} \left[\xi I_i(x(t_i^-)) + \zeta t_i \bar{I}_i(x(t_i^-))\right] \left\{ \frac{1}{\Gamma(w)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{w-1} g\frac{ds}{s} + \int_{t_i}^{t} \left(\ln\frac{t}{s}\right)^{w-1} g\frac{ds}{s} \right.\right. \\ &\left.\left. - \int_a^t \left(\ln\frac{t}{s}\right)^{w-1} g\frac{ds}{s} \right] + \frac{\ln\frac{t}{t_i}}{\Gamma(w-1)} \int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{w-2} g\frac{ds}{s} \right\}, \textit{ for } t \in (t_k, t_{k+1}],\ k = 0, \ldots, m, \end{aligned} \tag{2.2}$$

provided that the integral in (2.2) *exists, here* $g = g(s, x(s))$.

3. Main results

For convenience, let $f = f(\tau, z(\tau))$ in this section. Let

$$z(t) = A_k + B_k \ln t + z_a + a\bar{z}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{q-1} f\frac{d\tau}{\tau}, \quad \text{for } \ t \in (s_k, t_{k+1}] \subset (0, T], \tag{3.1}$$

where A_k and B_k $(k = 0, 1, \ldots, N)$ are some constants. In fact, Eq. (3.1) satisfies fractional derivative condition of system (1.1) by Definition 2.2. Substituting (3.1) into system (1.1), we have

$$z(t) = \begin{cases} A_k + B_k \ln t + z_a + a\bar{z}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{q-1} f\frac{d\tau}{\tau}, & \text{for } t \in (s_k, t_{k+1}],\ k = 0, \ldots, N, \\ g_k(t, z(t)), & \text{for } t \in (t_k, s_k],\ k = 1, 2, \ldots, N. \end{cases}$$

Thus, we obtain $A_0 = 0$, $B_0 = 0$,

$$\begin{aligned} A_k &= g_k(s_k, z(s_k)) - z_a + a\bar{z}_a \ln a - \frac{1}{\Gamma(q)}\int_a^{s_k} \left(\ln\frac{s_k}{\tau}\right)^{q-1} f\frac{d\tau}{\tau} \\ &- \left[s_k g_k'(s_k, z(s_k)) - \frac{1}{\Gamma(q-1)}\int_a^{s_k} \left(\ln\frac{s_k}{\tau}\right)^{q-2} f\frac{d\tau}{\tau} \right] \ln s_k, \end{aligned}$$

and

$$B_k = s_k g_k'(s_k, z(s_k)) - a\bar{z}_a - \frac{1}{\Gamma(q-1)}\int_a^{s_k} \left(\ln\frac{s_k}{\tau}\right)^{q-2} f\frac{d\tau}{\tau}.$$

Therefore,

$$z(t) = \begin{cases} z_a + a\bar{z}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{q-1} f\frac{d\tau}{\tau}, \quad \text{for } t \in (a, t_1], \\ g_k(t, z(t)), \quad \text{for } t \in (t_k, s_k],\ k = 1, 2, \ldots, N, \\ g_k(s_k, z(s_k)) - \frac{1}{\Gamma(q)}\int_a^{s_k} \left(\ln\frac{s_k}{\tau}\right)^{q-1} f\frac{d\tau}{\tau} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{q-1} f\frac{d\tau}{\tau} \\ \quad + \left[s_k g_k'(s_k, z(s_k)) - \frac{1}{\Gamma(q-1)}\int_a^{s_k} \left(\ln\frac{s_k}{\tau}\right)^{q-2} f\frac{d\tau}{\tau} \right] \ln\frac{t}{s_k}, \\ \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, N. \end{cases} \tag{3.2}$$

It is clear that Eq. (3.2) meets all conditions of system (1.1). Moreover, Eq. (3.2) satisfies a hidden condition

$$\lim_{\substack{\left[g_k(t,z(t))-z_a-a\bar{z}_a\ln\frac{t}{a}-\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]=0\\ \text{for } t\in(t_k,s_k] \text{ and all } k\in\{1,2,\ldots,N\}}}\{\text{Eq. (3.2)}\}$$

$$\Leftrightarrow z(t)=z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau},\ \text{for } t\in(a,T], \tag{3.3}$$

$$\Leftrightarrow \lim_{\substack{\left[g_k(t,z(t))-z_a-a\bar{z}_a\ln\frac{t}{a}-\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]=0\\ \text{for } t\in(t_k,s_k] \text{ and all } k\in\{1,2,\ldots,N\}}}\{\text{system (1.1)}\}.$$

On the other hand, substituting $\tilde{z}(t)=z(s_k^+)+\delta z(s_k^+)\ln\frac{t}{s_k}+\frac{1}{\Gamma(q)}\int_{s_k}^t(\ln\frac{t}{\tau})^{q-1}f\frac{d\tau}{\tau}$ for $t\in(s_k,t_{k+1}]$ (here $k=0,1,\ldots,N$) into system (1.1), we have

$$\hat{z}(t)=\begin{cases} z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}, & \text{for } t\in(a,t_1],\\ g_k(t,z(t)), & \text{for } t\in(t_k,s_k],\ k=1,2,\ldots,N,\\ g_k(s_k,z(s_k))+s_kg_k'(s_k,z(s_k))\ln\frac{t}{s_k}+\frac{1}{\Gamma(q)}\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}, & \\ \quad \text{for } t\in(s_k,t_{k+1}],\ k=1,2,\ldots,N. & \end{cases}$$

It is clear that $\hat{z}(t)$ satisfies fractional derivative condition and not instantaneous impulses conditions in system (1.1), but it does not satisfy the corresponding hidden condition (3.3). Hence, $\hat{z}(t)$ is only regarded as an approximate solution of system (1.1), and Eq. (3.2) is only *a particular solution* of system (1.1) since it does not contain $\int_{s_k}^t(\ln\frac{t}{\tau})^{q-1}f\frac{d\tau}{\tau}$, the key part of approximate solution $\hat{z}(t)$ as $t\in(s_k,t_{k+1}]$.

Theorem 3.1. *Let ξ_k, ζ_k $(k=1,2,\ldots,N)$ are some constants. The system (1.1) is equivalent to an integral equation*

$$z(t)=\begin{cases} z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}, & \textit{for } t\in(a,t_1],\\ g_k(t,z(t)), & \textit{for } t\in(t_k,s_k],\ k=1,2,\ldots,N,\\ \hat{g}_k+s_k\hat{g}_k'\ln\frac{t}{s_k}+z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}+\left[\xi_k\hat{g}_k+\zeta_ks_k\hat{g}_k'\right] & \\ \quad\cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}+\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\right. & \\ \quad\left.+\frac{\ln\frac{t}{s_k}}{\Gamma(q-1)}\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\right\}, & \textit{for } t\in(s_k,t_{k+1}],\ k=1,2,\ldots,N, \end{cases} \tag{3.4}$$

provided that the integral in (3.4) exists, here $\hat{g}_k=g_k(s_k,z(s_k))-z_a-a\bar{z}_a\ln\frac{s_k}{a}-\frac{1}{\Gamma(q)}\int_a^{s_k}(\ln\frac{s_k}{\tau})^{q-1}f\frac{d\tau}{\tau}$ and $\hat{g}_k'=g_k'(s_k,x(s_k))-\frac{a\bar{z}_a}{s_k}-\frac{1}{\Gamma(q-1)}\frac{1}{s_k}\int_a^{s_k}(\ln\frac{s_k}{\tau})^{q-2}f\frac{d\tau}{\tau}$.

Proof.

Sufficiency. By Lemma 2.4, the solution of system (1.1) as $t\in(a,t_1]$ satisfies

$$z(t)=z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau},\ \text{for } t\in(a,t_1],$$

and $z(t)=g_1(t,z(t))$ for $t\in(t_1,s_1]$.

Next, by above discussion, the approximate solution $\hat{z}(t)$ as $t \in (s_1, t_2]$ is given by

$$\hat{z}(t) = g_1(s_1^+, z(s_1^+)) + s_1 g_1'(s_1^+, z(s_1^+)) \ln \frac{t}{s_1} + \frac{1}{\Gamma(q)} \int_{s_1}^{t} \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \quad \text{for } t \in (s_1, t_2].$$

Let error $e_1(t) = z(t) - \hat{z}(t)$ for $t \in (s_1, t_2]$, here $z(t)$ denotes the exact solution of system (1.1). Moreover, by the particular solution (3.2), the solution of system (1.1) satisfies

$$\lim_{\hat{g}_1 \to 0, \hat{g}_1' \to 0} z(t) = z_a + a\bar{z}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \quad \text{for } t \in (s_1, t_2],$$

where $\hat{g}_1 = g_1(s_1, z(s_1)) - z_a - a\bar{z}_a \ln \frac{s_1}{a} - \frac{1}{\Gamma(q)} \int_a^{s_1} (\ln \frac{s_1}{\tau})^{q-1} f \frac{d\tau}{\tau}$. Thus,

$$\begin{aligned} \lim_{\hat{g}_1 \to 0, \hat{g}_1' \to 0} e_1(t) &= \lim_{\hat{g}_1 \to 0, \hat{g}_1' \to 0} \{z(t) - \hat{z}(t)\} \\ &= \frac{1}{\Gamma(q)} \left[\int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_{s_1}^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}\right] \\ &\quad - \frac{\ln \frac{t}{s_1}}{\Gamma(q-1)} \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-2} f \frac{d\tau}{\tau}, \quad \text{for } t \in (s_1, t_2]. \end{aligned} \tag{3.5}$$

By (3.5), we make a hypothesis

$$\begin{aligned} e_1(t) &= \kappa\left(\hat{g}_1, \hat{g}_1'\right) \lim_{\hat{g}_1 \to 0, \hat{g}_1' \to 0} e_1(t) \\ &= \kappa\left(\hat{g}_1, \hat{g}_1'\right) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_{s_1}^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}\right] \right. \\ &\quad \left. - \frac{\ln \frac{t}{s_1}}{\Gamma(q-1)} \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-2} f \frac{d\tau}{\tau} \right\}, \quad \text{for } t \in (s_1, t_2], \end{aligned} \tag{3.6}$$

where $\kappa(\cdot, \cdot)$ denotes an undetermined function with $\kappa(0, 0) = 1$. Therefore,

$$\begin{aligned} z(t) &= \hat{z}(t) + e_1(t) \\ &= g_1(s_1, z(s_1)) - \frac{1}{\Gamma(q)} \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \\ &\quad + \left[s_1 g_1'(s_1, z(s_1)) - \frac{1}{\Gamma(q-1)} \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-2} f \frac{d\tau}{\tau}\right] \ln \frac{t}{s_1} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \\ &\quad + \left[1 - \kappa\left(\hat{g}_1, \hat{g}_1'\right)\right] \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} + \int_{s_1}^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}\right] \right. \\ &\quad \left. + \frac{\ln \frac{t}{s_1}}{\Gamma(q-1)} \int_a^{s_1} \left(\ln \frac{s_1}{\tau}\right)^{q-2} f \frac{d\tau}{\tau} \right\}, \quad \text{for } t \in (s_1, t_2]. \end{aligned} \tag{3.7}$$

On the other hand, consider a special case in system (1.1)

$$\begin{aligned} &\lim_{t_1 \to s_1} \begin{cases} {}_{C-H}D_{a^+}^q z(t) = f(t, z(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \\ z(t) = g_1(t, z(t)), & t \in (t_1, s_1], \\ z(a) = z_a,\ z'(a) = \bar{z}_a, & z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \\ &= \begin{cases} {}_{C-H}D_{a^+}^q z(t) = f(t, z(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \\ z(t_1) = z(s_1) = g_1(s_1, z(s_1)), \\ z(a) = z_a,\ z'(a) = \bar{z}_a, & z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \end{aligned}$$

$$\Leftrightarrow \begin{cases} {}_{C-H}D^{q}_{a^+}z(t)=f(t,z(t)), \quad t\in(s_k,t_{k+1}],\ k=0,1, \\ z(t_1^+)-z(t_1^-)=g_1(s_1,z(s_1))-z_a-a\bar{z}_a\ln\frac{s_1}{a}-\frac{1}{\Gamma(q)}\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}, \\ z'(t_1^+)-z'(t_1^-)=g_1'(s_1,z(s_1))-\frac{a\bar{z}_a}{s_1}-\frac{1}{\Gamma(q-1)}\frac{1}{s_1}\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}, \\ z(a)=z_a,\ z'(a)=\bar{z}_a,\ \ z_a,\bar{z}_a\in\mathbb{C}. \end{cases} \tag{3.8}$$

Using (3.7) and Lemma 2.5 for system (3.8), respectively, we get

$$\begin{aligned} 1-\kappa\left(\hat{g}_1,\hat{g}_1'\right)&=\xi_1\left[g_1(s_1,z(s_1))-z_a-a\bar{z}_a\ln\frac{s_1}{a}-\frac{1}{\Gamma(q)}\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right] \\ &\quad+\zeta_1\left[s_1g_1'(s_1,z(s_1))-a\bar{z}_a-\frac{1}{\Gamma(q-1)}\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\right]=\xi_1\hat{g}_1+\zeta_1 s_1\hat{g}_1'. \end{aligned}$$

Thus, Eq. (3.7) can be rewritten as

$$\begin{aligned} z(t)&=\hat{g}_1+s_1\hat{g}_1'\ln\frac{t}{s_1}+z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}+\left[\xi_1\hat{g}_1+\zeta_1 s_1\hat{g}_1'\right] \\ &\quad\cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}+\int_{s_1}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\right. \\ &\quad\left.+\frac{\ln\frac{t}{s_1}}{\Gamma(q-1)}\int_a^{s_1}\left(\ln\frac{s_1}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\right\},\quad \text{for } t\in(s_1,t_2], \end{aligned}$$

and $z(t)=g_2(t,z(t))$ for $t\in(t_2,s_2]$.

Next, the approximate solution $\hat{z}(t)$ for $t\in(s_k,t_{k+1}]$ is provided by

$$\hat{z}(t)=g_k(s_k^+,z(s_k^+))+s_kg_k'(s_k^+,z(s_k^+))\ln\frac{t}{s_k}+\frac{1}{\Gamma(q)}\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau},\ \text{for } t\in(s_k,t_{k+1}].$$

Let error $e_k(t)=z(t)-\hat{z}(t)$ for $t\in(s_k,t_{k+1}]$, and $z(t)$ denotes the exact solution of system (1.1). Furthermore, by the particular solution (3.2), the solution of system (1.1) satisfies

$$\lim_{\hat{g}_k\to0,\hat{g}_k'\to0}z(t)=z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau},\ \text{for } t\in(s_k,t_{k+1}],$$

here $\hat{g}_k=g_k(s_k,z(s_k))-z_a-a\bar{z}_a\ln\frac{s_k}{a}-\frac{1}{\Gamma(q)}\int_a^{s_k}(\ln\frac{s_k}{\tau})^{q-1}f\frac{d\tau}{\tau}$. Therefore,

$$\begin{aligned} \lim_{\hat{g}_k\to0,\hat{g}_k'\to0}e_k(t)&=\lim_{\hat{g}_k\to0,\hat{g}_k'\to0}\{z(t)-\hat{z}(t)\} \\ &=\frac{1}{\Gamma(q)}\left[\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right] \\ &\quad-\frac{\ln\frac{t}{s_k}}{\Gamma(q-1)}\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-2}f\frac{d\tau}{\tau},\ \text{for } t\in(s_k,t_{k+1}]. \end{aligned}$$

With similarity to (3.6), suppose

$$\begin{aligned} e_k(t)&=\vartheta\left(\hat{g}_k,\hat{g}_k'\right)\lim_{\hat{g}_k\to0,\hat{g}_k'\to0}e_k(t) \\ &=\vartheta\left(\hat{g}_k,\hat{g}_k'\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\right. \\ &\quad\left.-\frac{\ln\frac{t}{s_k}}{\Gamma(q-1)}\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\right\},\ \text{for } t\in(s_k,t_{k+1}], \end{aligned}$$

where $\vartheta(\cdot,\cdot)$ is an undetermined function. Thus,

$$\begin{aligned} z(t) &= \hat{z}(t) + e_k(t) \\ &= g_k(s_k, z(s_k)) - \frac{1}{\Gamma(q)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \\ &\quad + \left[s_k g_k'(s_k, z(s_k)) - \frac{1}{\Gamma(q-1)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-2} f \frac{d\tau}{\tau} \right] \ln \frac{t}{s_k} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \\ &\quad + \left[1 - \vartheta\left(\hat{g}_k, \hat{g}_k'\right)\right] \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} + \int_{s_k}^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \right] \right. \\ &\quad \left. + \frac{\ln \frac{t}{s_k}}{\Gamma(q-1)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-2} f \frac{d\tau}{\tau} \right\}, \text{ for } t \in (s_k, t_{k+1}]. \end{aligned} \tag{3.9}$$

Considering a special case of system (1.1), we get

$$\begin{aligned} &\lim_{t_k \to s_k} \begin{cases} {}_{C-H}D_{a^+}^q z(t) = f(t, z(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, 1, \ldots, k, \\ z(t) = z_a + a\bar{z}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \text{ for } t \in (t_i, s_i], \ i = 1, 2, \ldots, k-1, \\ z(t) = g_k(t, z(t)), \ t \in (t_k, s_k], \\ z(a) = z_a, \ z'(a) = \bar{z}_a, \quad z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \\ &= \begin{cases} {}_{C-H}D_{a^+}^q z(t) = f(t, z(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, 1, \ldots, k, \\ z(t) = z_a + a\bar{z}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \text{ for } t \in (t_i, s_i], \ i = 1, 2, \ldots, k-1, \\ z(s_k) = g_k(s_k, z(s_k)), \\ z(a) = z_a, \ z'(a) = \bar{z}_a, \quad z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \\ &\Leftrightarrow \begin{cases} {}_{C-H}D_{a^+}^q z(t) = f(t, z(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, 1, \ldots, k, \\ z(t) = z_a + a\bar{z}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \quad t \in (t_i, s_i], \ i = 1, 2, \ldots, k-1, \\ z(s_k^+) - z(s_k^-) = g_k(s_k, z(s_k)) - z_a - a\bar{z}_a \ln \frac{s_k}{a} - \frac{1}{\Gamma(q)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-1} f \frac{d\tau}{\tau}, \\ z'(s_k^+) - z'(s_k^-) = g_k'(s_k, z(s_k)) - \frac{a\bar{z}_a}{s_k} - \frac{1}{\Gamma(q-1)} \frac{1}{s_k} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-2} f \frac{d\tau}{\tau}, \\ z(a) = z_a, \ z'(a) = \bar{z}_a, \quad z_a, \bar{z}_a \in \mathbb{C}. \end{cases} \end{aligned} \tag{3.10}$$

System (3.10) can be regarded as a special case of system (2.1) with single impulsive point. By using (3.9) and Lemma 2.5 for system (3.10), we obtain

$$\begin{aligned} 1 - \kappa\left(\hat{g}_k, \hat{g}_k'\right) &= \xi_k \left[g_k(s_k, z(s_k)) - z_a - a\bar{z}_a \ln \frac{s_k}{a} - \frac{1}{\Gamma(q)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \right] \\ &\quad + \zeta_k \left[s_k g_k'(s_k, z(s_k)) - a\bar{z}_a - \frac{1}{\Gamma(q-1)} \int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-2} f \frac{d\tau}{\tau} \right] = \xi_k \hat{g}_k + \zeta_k s_k \hat{g}_k'. \end{aligned}$$

Thus, (3.9) is rewritten as

$$\begin{aligned} z(t) = \hat{g}_k &+ s_k \hat{g}_k' \ln \frac{t}{s_k} + z_a + a\bar{z}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} + \left[\xi_k \hat{g}_k + \zeta_k s_k \hat{g}_k'\right] \\ &\cdot \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{s_k} \left(\ln \frac{s_k}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} + \int_{s_k}^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} - \int_a^t \left(\ln \frac{t}{\tau}\right)^{q-1} f \frac{d\tau}{\tau} \right] \right. \end{aligned}$$

$$+\frac{\ln\frac{t}{s_k}}{\Gamma(q-1)}\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\Bigg\},\ \text{for } t\in(s_k,t_{k+1}],$$

and $z(t)=g_{k+1}(t,z(t))$ for $t\in(t_{k+1},s_{k+2}]$.

Necessity. We will verify that Eq. (3.4) satisfies all conditions of system (1.1). Taking the fractional derivative to Eq. (3.4), we have

(i)

$$\begin{aligned}{}_{C-H}D^q_{a^+}\left[z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]&={}_{C-H}D^q_{a^+}\left[\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\\&=f(t,z(t)),\quad \text{for } t\in(a,t_1],\end{aligned}$$

(ii)

$$\begin{aligned}&{}_{C-H}D^q_{a^+}\Bigg\{\hat{g}_k+s_k\hat{g}_k'\ln\frac{t}{s_k}+z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\\&\quad+\left[\xi_k\hat{g}_k+\zeta_k s_k\hat{g}_k'\right]\Bigg\{\frac{1}{\Gamma(q)}\left[\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}+\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}-\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\\&\quad+\frac{\ln\frac{t}{s_k}}{\Gamma(q-1)}\int_a^{s_k}\left(\ln\frac{s_k}{\tau}\right)^{q-2}f\frac{d\tau}{\tau}\Bigg\}\Bigg\}_{t\in(s_k,t_{k+1}]}\\&=\Bigg\{f(t,z(t))_{t\geqslant a}+\frac{1}{\Gamma(q)}\left[\xi_k\hat{g}_k+\zeta_k s_k\hat{g}_k'\right]\\&\quad\times\left\{{}_{C-H}D^q_{s_k^+}\left[\int_{s_k}^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]-{}_{C-H}D^q_{a^+}\left[\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]\right\}\Bigg\}_{t\in(s_k,t_{k+1}]}\\&=\left\{f(t,z(t))_{t\geqslant a}+\left[\xi_k\hat{g}_k+\zeta_k s_k\hat{g}_k'\right]\left[f(t,z(t))_{t\geqslant s_k}-f(t,z(t))_{t\geqslant a}\right]\right\}_{t\in(s_k,t_{k+1}]}\\&=f(t,z(t))_{t\in(s_k,t_{k+1}]}.\end{aligned}$$

Therefore, Eq. (3.4) satisfies ${}_{C-H}D^q_{a^+}z(t)=f(t,z(t))$ (here $t\in(s_k,t_{k+1}]$ and $k=0,1,\dots,N$) in system (1.1).

Finally, by Eq. (3.4), we have

$$\begin{aligned}&\lim_{\substack{\left[g_k(t,z(t))-z_a-a\bar{z}_a\ln\frac{t}{a}-\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]=0\\ \text{for all } k=1,2,\dots,N}}\{\text{Eq. (3.4)}\}\\&\Leftrightarrow z(t)=z_a+a\bar{z}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau},\ \text{for } t\in(a,T],\\&\Leftrightarrow\begin{cases}{}_{C-H}D^q_{a^+}z(t)=f(t,z(t)),\ t\in(a,T],\\ z(a)=z_a,\ z'(a)=\bar{z}_a,\ \ z_a,\bar{z}_a\in\mathbb{C}.\end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{\substack{\left[g_k(t,z(t))-z_a-a\bar{z}_a\ln\frac{t}{a}-\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]=0\\ \text{for all } k=1,2,\dots,N}}\{\text{Eq. (3.4)}\}\\&\Leftrightarrow\lim_{\substack{\left[g_k(t,z(t))-z_a-a\bar{z}_a\ln\frac{t}{a}-\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{q-1}f\frac{d\tau}{\tau}\right]=0\\ \text{for all } k=1,2,\dots,N}}\{\text{system (1.1)}\}.\end{aligned}$$

Thus, Eq. (3.4) satisfies all conditions of system (1.1).

According to above proof, the conclusion can be drawn. The proof is now completed. □

4. An example

In this section, we will give an example to illustrate that there exists a general solution for the Caputo-Hadamard fractional differential equations with not instantaneous impulses.

Example 4.1. Let us consider the following impulsive system

$$\begin{cases} {}_{C-H}D_{1^+}^{\frac{3}{2}} z(t) = \ln t, & t \in (1,2] \cup (3,4], \\ z(t) = t^2, & t \in (2,3], \\ z(1) = 1, & z'(1) = 1. \end{cases} \tag{4.1}$$

By Theorem 3.1, the general solution of (4.1) can be obtained as

$$z(t) = \begin{cases} 1 + \ln t + \frac{1}{\Gamma(\frac{7}{2})} (\ln t)^{\frac{5}{2}}, \text{ for } t \in (1,2], \\ t^2, \quad \text{for } t \in (2,3], \\ 9 - \frac{1}{\Gamma(\frac{7}{2})} (\ln 3)^{\frac{5}{2}} + \left[18 - \frac{1}{\Gamma(\frac{5}{2})} (\ln 3)^{\frac{3}{2}} \right] \ln \frac{t}{3} \Big|_{t \geqslant 3} + \frac{1}{\Gamma(\frac{7}{2})} (\ln t)^{\frac{5}{2}} \Big|_{t > 1} \\ \quad + \left\{ \xi \left[8 - \ln 3 - \frac{1}{\Gamma(\frac{7}{2})} (\ln 3)^{\frac{5}{2}} \right] + \zeta \left[17 - \frac{1}{\Gamma(\frac{5}{2})} (\ln 3)^{\frac{3}{2}} \right] \right\} \\ \quad \cdot \left\{ \frac{1}{\Gamma(\frac{7}{2})} \left[(\ln 3)^{\frac{5}{2}} + \left(\ln \frac{t}{3} \right)^{\frac{3}{2}} \left[\ln t + \frac{3 \ln 3}{2} \right]_{t \geqslant 3} - (\ln t)^{\frac{5}{2}} \Big|_{t > 1} \right] + \frac{(\ln 3)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \ln \frac{t}{3} \Big|_{t \geqslant 3} \right\} \\ \quad \text{for } t \in (3,4], \end{cases} \tag{4.2}$$

here ξ and ζ are two constants. To show solution trajectory of system (4.1), we plot (4.2) for several values of ξ and ζ in Fig. 1.

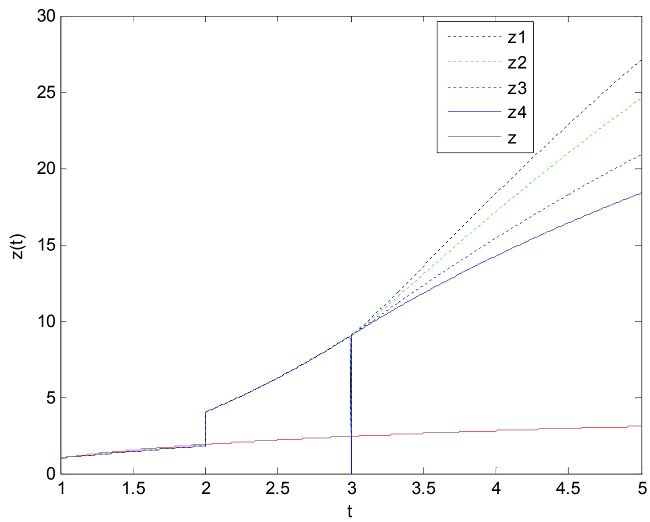

Figure 1: Solution trajectory of system (4.1).

$z1, z2, z3$, and $z4$ denote (4.2) with $\xi = 2$ and $\zeta = 2$, $\xi = 0$ and $\zeta = 2$, $\xi = 2$ and $\zeta = 0$, and $\xi = 0$ and $\zeta = 0$, respectively, and z is the solution trajectory of system (4.1) without impulses in Fig. 1.

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