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Smooth solutions for the p-order functional equation $f(\varphi(x)) = \varphi^p(f(x))$

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Abstract

This paper deals with the p-order functional equation

$$\begin{cases} f(\varphi(x)) = \varphi^p(f(x)), \\ \varphi(0) = 1, \quad -1 \leqslant \varphi(x) \leqslant 1, \qquad x \in [-1,1], \end{cases}$$

where $p \geqslant 2$ is an integer, φ^p is the p-fold iteration of φ, and $f(x)$ is smooth odd function on $[-1,1]$ and satisfies $f(0) = 0, -1 < f'(x) < 0, (x \in [-1,1])$. Using constructive method, the existence of unimodal-even-smooth solutions of the above equation on $[-1,1]$ can be proved.

Keywords: Functional equation, constructive method, unimodal-even-smooth solution.
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1. Introduction

The existence of solutions for functional equations has been extensively studied in some literatures. In the last years, many authors considered p-order functional equation

$$\begin{cases} f(\varphi(x)) = \varphi^p(f(x)), \\ \varphi(0) = 1, \quad -1 \leqslant \varphi(x) \leqslant 1, \qquad x \in [-1,1], \end{cases} \tag{1.1}$$

where $p \geqslant 2$ is an integer, φ^p is the p-fold iteration of φ. Eq. (1.1) is known as Cvitanović-Feigenbaum equation when $p = 2$ and $f(x) = -\lambda x, 0 < \lambda < 1$. Feigenbaum [6, 7], and Coullet and Tresser [2] introduced the notion of renormalization for real dynamical systems. Later, Sullivan [10] proved the uniqueness of a fixed point which satisfies the Cvitanović-Feigenbaum equation for period-doubling renormalization operator. For some results about existence of solutions for Cvitanović-Feigenbaum equation, we refer the interested reader to [4, 5, 9, 11].

For $f(x) = -\lambda x, (-1 < \lambda < 1)$, Chen [1] constructed the even C^1 solutions of Eq. (1.1) if $p = 3$. If p is large enough, Eckmann et al. [3] showed that there exists a solution of Eq. (1.1) similar with the function $\varphi(x) = |1 - 2x^2|$. For $p \geqslant 2$, Liao [8] proved that Eq. (1.1) has single-valley continuous solutions on $[0,1]$.

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For $p \geqslant 2$, Zhang and Si [12] proved that Eq. (1.1) has single-valley-extended continuous solutions on $[0, 1]$.

In [11], Yang and Zhang showed that it is interesting to construct smooth solutions and analytic solutions for Eq. (1.1). In this paper, we will discuss the existence of unimodal-even-smooth solutions of Eq. (1.1) on $[-1, 1]$ when $f(x)$ is odd smooth function on $[-1, 1]$ and satisfies $f(0) = 0, -1 < f'(x) < 0, (x \in [-1, 1])$.

2. Basic definitions and lemmas

In this section, we will give some characterizations of unimodal-even continuous solution of Eq. (1.1) where $f(x)$ is a strictly decreasing odd continuous function on $[-1, 1]$ and satisfies $f(0) = 0, |f(x)| < |x|, (x \in [-1, 1], x \neq 0)$. And we will prove them in the Appendix.

Definition 2.1 (unimodal-even solution)**.** We call φ a unimodal-even solution of Eq. (1.1), if (i) φ is an even solution of Eq. (1.1); (ii) φ is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, 1]$.

Lemma 2.2. *Suppose that $\varphi(x)$ is a unimodal-even continuous solution of Eq.* (1.1)*. Then:*

(i) *0 is a recurrent but not periodic point of φ;*

(ii) *$\varphi(x)$ has a unique fixed point $\beta = \varphi(\beta)$ on $[-1, 1]$, and*

$$0 < -\varphi^{p-1}(1) = -f(1) = \lambda < \beta < \alpha < 1, \tag{2.1}$$

where $\varphi(\alpha) = 0$;

(iii) *for $x \in [-\lambda, \lambda]$ and $0 \leqslant i \leqslant p-1$, then $|\varphi^i(x)| = \alpha$ if and only if $|x| = -f(\alpha)$ and $i = p-1$;*

(iv) *for $1 \leqslant i \leqslant p-1$, then $|\varphi^i(x)| > \lambda$, for all $x \in [f(\alpha), -f(\alpha)]$ and $|\varphi^i(x)| > -f(\alpha)$, for all $x \in (-\lambda, f(\alpha)] \cup (-f(\alpha), \lambda]$;*

(v) *for $1 \leqslant i \leqslant p-1$, then φ has no periodic point of period i on $[-\lambda, \lambda]$.*

Lemma 2.3. *Suppose that $\varphi(x)$ is a unimodal-even continuous solution of Eq.* (1.1)*. Let $J = [0, \lambda], J_0 = \varphi(J), J_i = \varphi^i(J_0)$. Then:*

(i) *for all $i = 0, 1, \cdots, p-2$, then $\varphi^i : J_0 \longmapsto J_i$ is a homeomorphism;*

(ii) *for all $i = 0, 1, \cdots, p-2$, then $J_i \subset (\lambda, 1]$ or $J_i \subset [-1, -\lambda)$. And $J_0, J_1, \cdots, J_{p-2}$ are pairwise disjoint.*

Lemma 2.4. *Suppose that $\varphi(x)$ is a unimodal-even continuous solution of Eq.* (1.1)*. Then the equation $\varphi^{p-1}(x) = f(x)$ has only one solution $x = 1$ in $(\varphi(-f(\alpha)), 1]$.*

Lemma 2.5. *Let φ_1, φ_2 be two unimodal-even continuous solutions of Eq.* (1.1)*. If*

$$\varphi_1(x) = \varphi_2(x), \quad x \in [\lambda, 1],$$

then $\varphi_1(x) = \varphi_2(x)$ on $[-1, 1]$.

3. Constructive method of solutions

In this section, we will prove that the existence of unimodal-even continuous solution of Eq. (1.1) by the constructive method, where $f(x)$ is a strictly decreasing odd continuous function on $[-1, 1]$ and satisfies $f(0) = 0, |f(x)| < |x|, (x \in [-1, 1], x \neq 0)$.

Theorem 3.1. *For arbitrary fixed strictly decreasing odd continuous function $f(x)$ on $[-1, 1]$ with $f(0) = 0, |f(x)| < |x|, (x \in [-1, 1], x \neq 0)$, denote $f(-1) = -f(1) = \lambda$. Suppose that φ_0 is an even continuous function on $[-1, -\lambda] \cup [\lambda, 1]$ and satisfies the following conditions:*

(i) *there exists an* $\alpha \in (\lambda, 1)$ *such that* $\varphi_0(-\alpha) = \varphi_0(\alpha) = 0$ *and* φ_0 *is strictly increasing on* $[-1, -\lambda]$ *and strictly decreasing on* $[\lambda, 1]$*;*

(ii) $\varphi_0^{p-1}(1) = f(1) = -\lambda, \quad \varphi_0^p(\lambda) = f(\varphi_0(1))$*;*

(iii) *denote* $J_0 = [\varphi_0(\lambda), 1], \quad J_i = \varphi_0^i(J_0)$*, then*

 (a) *for all* $i = 0, 1, \cdots, p-2$*, then* $J_i \subset (\lambda, 1]$ *or* $J_i \subset [-1, -\lambda)$*, and* $J_0, J_1, \cdots, J_{p-2}$ *are pairwise disjoint,*

 (b) *for all* $i = 0, 1, \cdots, p-2$*, then* $\varphi_0^i : J_0 \longmapsto J_i$ *is a homeomorphism,*

 (c) α *or* $-\alpha$ *is in the interior of* J_{p-2}*;*

(iv) *the equation* $\varphi_0^{p-1}(x) = f(x)$ *has only one solution* $x = 1$ *in* $(\alpha_0, 1]$*, where* $\alpha_0 \in J_0$ *with* $\varphi_0^{p-1}(\alpha_0) = 0$*,*

then there exists a unique unimodal-even continuous function φ *satisfying the equation*

$$\begin{cases} f(\varphi(x)) = \varphi^p(f(x)), & x \in [-1, 1], \\ \varphi(x) = \varphi_0(x), & x \in [-1, -\lambda] \cup [\lambda, 1]. \end{cases} \tag{3.1}$$

Conversely, if φ_0 *is the restriction on* $[-1, -\lambda] \cup [\lambda, 1]$ *of a unimodal-even continuous solution to Eq.* (1.1)*, then above conditions* (i)-(iv) *must hold.*

Proof. First of all, we can prove that if φ_0 satisfies the conditions (i)-(iv), then there exists a unique unimodal-even continuous function φ satisfying (3.1). Define

$$\psi = \varphi_0^{p-1}|_{[\varphi_0(\lambda), 1]}.$$

By conditions (iii) and (iv), ψ is a homeomorphism. And since

$$\psi(\varphi_0(\lambda)) = \varphi_0^{p-1}(\varphi_0(\lambda)) = f(\varphi_0(1)) > 0 > -\lambda = \varphi_0^{p-1}(1) = \psi(1),$$

thus ψ is strictly decreasing.

Let $h(x) = -f(x)$. Trivially, $\{h^k(1)\}$ is decreasing and $\lim_{k\to\infty} h^k(1) = 0$. Let

$$\Delta_k^+ = [h^{k+1}(1), h^k(1)], \quad \Delta_k^- = [-h^k(1), -h^{k+1}(1)], \quad \Delta_k = \Delta_k^+ \cup \Delta_k^- \quad (k = 0, 1, 2, \cdots),$$

then

$$[-1, 1] = \bigcup_{k=0}^{+\infty} \Delta_k.$$

We define φ on Δ_k by induction as follows.

Obviously, $\varphi = \varphi_0$ is well-defined on Δ_0. For $x \in \Delta_1 = [-h(1), -h^2(1)] \cup [h^2(1), h(1)]$, let

$$\varphi_1(x) = \psi^{-1}(f(\varphi_0(f^{-1}(x)))), \quad x \in \Delta_1. \tag{3.2}$$

Trivially, φ_1 is even continuous on Δ_1 and respectively strictly increasing and decreasing on Δ_1^- and Δ_1^+. For $x \in \Delta_0$ we have

$$\varphi_0^{p-1}(\varphi_1(f(x))) = f(\varphi_0(x)). \tag{3.3}$$

By condition (ii), we have

$$\psi(\varphi_0(\lambda)) = \varphi_0^{p-1}(\varphi_0(\lambda)) = \varphi_0^p(\lambda) = f(\varphi_0(1)).$$

By (3.2) and φ_0, φ_1 being even, noting that $x = h(1)$ we have

$$\varphi_1(-h(1)) = \varphi_1(h(1)) = \psi^{-1}(f(\varphi_0(1))) = \varphi_0(\lambda) = \varphi_0(h(1)) = \varphi_0(-h(1)), \tag{3.4}$$

i.e., φ_0 and φ_1 have the same value on the common endpoints of Δ_0 and Δ_1. Suppose that $\varphi(x)$ is well-defined as an even continuous function $\varphi_k(x)$ on Δ_k and respectively strictly increasing and decreasing function on Δ_k^- and Δ_k^+ for all $k \leqslant m$, where $m \geqslant 1$ is a certain integer. Let

$$\varphi_{m+1}(x) = \psi^{-1}(f(\varphi_m(f^{-1}(x)))), \quad (x \in \Delta_{m+1}), \tag{3.5}$$

then $\varphi(x)$ is well-defined as an even continuous function $\varphi_k(x)$ on Δ_k and respectively strictly increasing and decreasing function on Δ_k^- and Δ_k^+ for all $k \geqslant 1$, and for $x \in \Delta_k$ we have

$$\psi(\varphi_{k+1}(f(x))) = f(\varphi_k(x)). \tag{3.6}$$

Next, we prove φ_k and φ_{k+1} have the same value on the common endpoints of Δ_k and $\Delta_{k+1}(k = 1, 2, \cdots)$. For $k = 1, \cdots, m$, where $m \geqslant 1$ is a certain integer, we suppose that

$$\varphi_k(h^k(1)) = \varphi_{k-1}(h^k(1)). \tag{3.7}$$

By (3.5), noting that $x = h^{m+1}(1)$ we have

$$\begin{aligned} \varphi_{m+1}(-h^{m+1}(1)) &= \varphi_{m+1}(h^{m+1}(1)) \\ &= \psi^{-1}(f(\varphi_m(h^m(1)))) \\ &= \psi^{-1}(f(\varphi_{m-1}(h^m(1)))) = \varphi_m(h^{m+1}(1)) = \varphi_m(-h^{m+1}(1)), \end{aligned} \tag{3.8}$$

by $\varphi_{m-1}, \varphi_m, \varphi_{m+1}$ are even, i.e., φ_k and φ_{k+1} have the same value on the common endpoints of Δ_k and Δ_{k+1} $(k = 1, 2, \cdots)$. Let

$$\varphi(x) = \begin{cases} 1, & (x = 0), \\ \varphi_k(x), & (x \in \Delta_k). \end{cases}$$

By φ_k being even continuous on Δ_k and respectively strictly increasing and decreasing on Δ_k^- and Δ_k^+ and (3.4), (3.7), and (3.8), we have that φ is a unimodal-even continuous function on $[-1, 0) \cup (0, 1]$.

We now prove that φ is continuous at $x = 0$. Trivially, $\{h^k(\alpha)\}$ is strictly decreasing and $\lim_{k\to+\infty} h^k(\alpha) = 0$. And since φ is strictly decreasing in $(0, 1]$ and $\varphi_1(h(\alpha)) = \psi^-(f(\varphi(\alpha))) = \psi^-(0) = \alpha_0$, we have $\{\varphi_k(h^k(\alpha))\}|_{k=2}^{+\infty}$ is strictly increasing in $(\alpha_0, 1]$. Let

$$\lim_{k\to+\infty} \varphi_k(h^k(\alpha)) = \gamma,$$

then $\gamma \in (\alpha_0, 1]$. From (3.6), we have

$$\psi(\varphi_{k+1}(h^{k+1}(\alpha))) = f(\varphi_k(h^k(\alpha))),$$

i.e., $\varphi_0^{p-1}(\varphi_{k+1}(h^{k+1}(\alpha))) = f(\varphi_k(h^k(\alpha)))$. Letting $k \to \infty$, we get $\varphi_0^{p-1}(\gamma) = f(\gamma)$. By condition (iv), we know $\gamma = 1 = \varphi(0)$. This proves that φ is continuous at $x = 0$. Thus, φ is a unimodal-even continuous function on $[-1, 1]$. We have that φ satisfies (3.1) by (3.3) and (3.6), and $\varphi(x)$ is unique by Lemma 2.5.

Secondly, if φ_0 is the restriction on $[-1, -\lambda] \cup [\lambda, 1]$ of a unimodal-even continuous solution to Eq. (1.1), then conditions (i)-(iv) must hold by the lemmas in Section 2. This completes the proof. □

Example 3.2. Let $\varphi_0(x) : [-1, \ -1/5] \cup [1/5, \ 1] \longmapsto [-1, \ 1]$ is defined as

$$\varphi_0(x) = \begin{cases} x + \frac{4}{5}, & (-1 \leqslant x < -\frac{4}{5}), \\ \frac{10+\sqrt{85}}{15}x + \frac{40+4\sqrt{85}}{75}, & (-\frac{4}{5} \leqslant x \leqslant -\frac{1}{5}), \\ -\frac{10+\sqrt{85}}{15}x + \frac{40+4\sqrt{85}}{75}, & (\frac{1}{5} \leqslant x \leqslant \frac{4}{5}), \\ -x + \frac{4}{5}, & (\frac{4}{5} < x \leqslant 1). \end{cases}$$

It is trivial that φ_0 satisfies the conditions of Theorem 3.1 where $p = 2$, $f(x) = -x/5$ and $\lambda = -f(1) = 1/5, \alpha = 4/5$. Thereby we know it is the restriction on $[-1, \ -1/5] \cup [1/5, \ 1]$ of a unimodal-even continuous solution φ to Eq. (1.1).

4. Unimodal-even-smooth solutions

In this section, we will prove that the existence of unimodal-even-smooth solution of Eq. (1.1) by the constructive method. Where $f(x)$ is odd smooth function on $[-1,1]$ and satisfies $f(0)=0, -1<f'(x)<0, (x\in[-1,1])$.

Theorem 4.1. *For arbitrary fixed odd smooth function $f(x)$ on $[-1,1]$ with $f(0)=0, -1<f'(x)<0, (x\in[-1,1])$, denote $f(-1)=-f(1)=\lambda$ and $\alpha\in(\lambda,1)$. Suppose that φ_0 is an even smooth function on $[-1,f(\alpha)]\cup[-f(\alpha),1]$ and satisfies the following conditions:*

(i) $\varphi_0(-\alpha)=\varphi_0(\alpha)=0$ *and φ_0 is strictly increasing on $[-1,f(\alpha)]$ and strictly decreasing on $[-f(\alpha),1]$;*
(ii) $\varphi_0^{p-1}(1)=f(1)=-\lambda, \varphi_0^p(\lambda)=f(\varphi_0(1)), \varphi_0^p(-f(\alpha))=0$;
(iii) *denote $J_0=[\varphi_0(\lambda),1], J_i=\varphi_0^i(J_0)$, then*
  (a) *for all $i=0,1,\cdots,p-2$, then $J_i\subset(\lambda,1]$ or $J_i\subset[-1,-\lambda)$, and $J_0,J_1,\cdots,J_{p-2}$ are pairwise disjoint,*
  (b) *for all $i=0,1,\cdots,p-2$, then $\varphi_0^i : J_0\longmapsto J_i$ is a homeomorphism,*
  (c) *α or $-\alpha$ is in the interior of J_{p-2};*
(iv) *the equation $\varphi_0^{p-1}(x)=f(x)$ has only one solution $x=1$ on $(\varphi_0(-f(\alpha)),1]$;*
(v) $\varphi_0^p(x)=f(\varphi_0(f^{-1}(x))), \quad \forall x\in[-\lambda,f(\alpha)]\cup[-f(\alpha),\lambda]$;
(vi) $f'(1)+(\varphi_0^{p-1})'(1)\cdot f'(0)<0$.

Then there exists a unique unimodal-even-smooth function φ satisfying the equation

$$\begin{cases} f(\varphi(x))=\varphi^p(f(x)), & x\in[-1,1],\\ \varphi(x)=\varphi_0(x), & x\in[-1,f(\alpha)]\cup[-f(\alpha),1]. \end{cases} \tag{4.1}$$

Conversely, if φ_0 is the restriction on $[-1,f(\alpha)]\cup[-f(\alpha),1]$ of a unimodal-even-smooth solution to Eq. (1.1), then above conditions (i)-(v) *must hold.*

Proof. First of all, we can prove that if φ_0 satisfies the conditions (i)-(vi), then there exists a unique unimodal-even-smooth function φ satisfying (4.1). Define

$$\psi=\varphi_0^{p-1}|_{[\varphi_0(\lambda),1]}.$$

By conditions (iii) and (iv) , ψ is a homeomorphism and smooth. And since

$$\psi(\varphi_0(\lambda))=\varphi_0^{p-1}(\varphi_0(\lambda))=f(\varphi_0(1))>0>-\lambda=\varphi_0^{p-1}(1)=\psi(1),$$

thus ψ is strictly decreasing.

Let $h(x)=-f(x)$. Trivially, we have $\{h^k(1)\}$ and $\{h^k(\alpha)\}$ are decreasing, and $\lim_{k\to\infty}h^k(1)=0$, $\lim_{k\to\infty}h^k(\alpha)=0$. Let

$$\begin{aligned} \Delta_k^+&=[h^{k+1}(1),h^k(1)], & \Delta_k^-&=[-h^k(1),-h^{k+1}(1)],\\ \Delta_k^{\alpha,+}&=[h^{k+1}(\alpha),h^k(\alpha)], & \Delta_k^{\alpha,-}&=[-h^k(\alpha),-h^{k+1}(\alpha)],\\ \Delta_k&=\Delta_k^+\cup\Delta_k^-, & \Delta_k^{\alpha}&=\Delta_k^{\alpha,+}\cup\Delta_k^{\alpha,-}\quad(k=0,1,2,\cdots), \end{aligned}$$

then

$$[-1,1]=\bigcup_{k=0}^{+\infty}\Delta_k=\bigcup_{k=0}^{+\infty}\Delta_k^{\alpha}\bigcup[-1,-\alpha]\bigcup[\alpha,1].$$

For initial function $\varphi_0(x)$, we get the unimodal-even continuous solution $\tilde{\varphi}(x)$ of Eq. (1.1) by the constructive method from Theorem 3.1. Next we construct a solution of Eq. (1.1) by new constructive method.

We define $\bar{\varphi}$ on Δ_k^{α} by induction as follows:

$$\varphi_k(x)=\psi^{-1}(f(\varphi_{k-1}(f^{-1}(x)))),\quad (x\in\Delta_k^{\alpha},k\geqslant1),$$

then $\bar{\varphi}$ is well-defined as an even smooth function $\varphi_k(x)$ on Δ_k^α and respectively strictly increasing and decreasing function on $\Delta_k^{\alpha,-}$ and $\Delta_k^{\alpha,+}$ for all $k \geqslant 1$. And let

$$\bar{\varphi}(x) = \begin{cases} 1, & (x = 0), \\ \varphi_k(x), & (x \in \Delta_k^\alpha, k \geqslant 1), \\ \varphi_0(x), & (x \in [-1, f(\alpha)] \cup [-f(\alpha), 1]), \end{cases}$$

we have that $\bar{\varphi}$ is a unimodal-even continuous solution to Eq. (1.1) similar to Theorem 3.1 and we omit the rest of proof.

From $\tilde{\varphi}(x) = \bar{\varphi}(x) = \varphi_0(x)$ on $[\lambda, 1]$, we have $\tilde{\varphi}(x) = \bar{\varphi}(x)(x \in [-1, 1])$ by Lemma 2.5. We denote $\varphi(x) = \tilde{\varphi}(x) = \bar{\varphi}(x)$. Since $\varphi_0(x)$ and $f(x)$ are smooth, it follows that $\varphi(x)$ is respectively smooth on $\Delta_k^+, \Delta_k^-, \Delta_k^{\alpha,+}$, and $\Delta_k^{\alpha,-}$ by the constructive methods. And from

$$h^{k+1}(1) \in (h^{k+1}(\alpha), h^k(\alpha)), \qquad -h^{k+1}(1) \in (-h^k(\alpha), -h^{k+1}(\alpha)),$$
$$h^k(\alpha) \in (h^{k+1}(1), h^k(1)), \qquad -h^k(\alpha) \in (-h^k(1), -h^{k+1}(1)),$$

we have that $\varphi(x)$ is smooth respectively on $[-1, 0)$ and $(0, 1]$.

It is obvious that $\lim_{x\to 0^+} \varphi^{(n)}(x)$ and $\lim_{x\to 0^-} \varphi^{(n)}(x)$ exist. We prove

$$\varphi_+^{(n)}(0) = \lim_{x\to 0^+} \varphi^{(n)}(x) = 0, \quad \varphi_-^{(n)}(0) = \lim_{x\to 0^-} \varphi^{(n)}(x) = 0, \quad n \geqslant 1 \tag{4.2}$$

by induction as follows.

In view of

$$f(\varphi(x)) = \varphi^p(f(x)),$$

we have

$$f'(\varphi(x)) \cdot \varphi'(x) = (\varphi^{p-1})'\big(\varphi(f(x))\big) \cdot \varphi'(f(x)) \cdot f'(x).$$

Let $x \to 0^+$, we have

$$f'(1) \cdot \lim_{x\to 0^+} \varphi'(x) = (\varphi^{p-1})'(1) \cdot \lim_{x\to 0^-} \varphi'(x) \cdot f'(0), \tag{4.3}$$

and let $x \to 0^-$, we have

$$f'(1) \cdot \lim_{x\to 0^-} \varphi'(x) = (\varphi^{p-1})'(1) \cdot \lim_{x\to 0^+} \varphi'(x) \cdot f'(0). \tag{4.4}$$

By (4.3) and (4.4), we have

$$\left[|f'(1)|^2 - |(\varphi^{p-1})'(1) \cdot f'(0)|^2\right] \cdot \lim_{x\to 0^+} \varphi'(x) = 0.$$

Thus by $f'(1) + (\varphi^{p-1})'(1) \cdot f'(0) < 0$, we have

$$\lim_{x\to 0^+} \varphi'(x) = 0.$$

Similarly, we have

$$\lim_{x\to 0^-} \varphi'(x) = 0.$$

We assert that

$$f'(\varphi(x)) \cdot \varphi^{(n)}(x) = H_n(x) + (\varphi^{p-1})'\big(\varphi(f(x))\big) \cdot (f'(x))^n \cdot \varphi^{(n)}(f(x)), \tag{4.5}$$

and

$$\lim_{x\to 0^+} H_n(x) = 0, \tag{4.6}$$

where

$$\begin{cases} H_1(x) = 0, \\ H_n(x) = \sum_{j=1}^{n-1} G_j^{(n-1-j)}(x), \quad n \geqslant 2 \end{cases} \tag{4.7}$$

and

$$G_j(x) = \left((\varphi^{p-1})'(\varphi(f(x))) \cdot (f'(x))^j\right)' \cdot \varphi^{(j)}(f(x)) - f''(\varphi(x)) \cdot \varphi'(x) \cdot \varphi^{(j)}(x).$$

Next we prove the assertions (4.2), (4.5), and (4.6) by induction. It is trivial that (4.2), (4.5), and (4.6) are true for $n = 1$. Suppose that (4.2), (4.5), and (4.6) hold for all $n \leqslant s$, where $s \geqslant 1$ is a certain integer. Consider $n = s+1$. In view of (4.5) for $n = s$, we have

$$\begin{aligned} f'(\varphi(x)) \cdot \varphi^{(s+1)}(x) + f''(\varphi(x)) \cdot \varphi'(x) \cdot \varphi^{(s)}(x) = H_s'(x) + \left((\varphi^{p-1})'(\varphi(f(x))) \cdot (f'(x))^s\right)' \cdot \varphi^{(s)}(f(x)) \\ + (\varphi^{p-1})'(\varphi(f(x))) \cdot (f'(x))^{s+1} \cdot \varphi^{(s+1)}(f(x)). \end{aligned}$$

Let

$$G_s(x) = \left((\varphi^{p-1})'(\varphi(f(x))) \cdot (f'(x))^s\right)' \cdot \varphi^{(s)}(f(x)) - f''(\varphi(x)) \cdot \varphi'(x) \cdot \varphi^{(s)}(x)$$

and

$$H_{s+1}(x) = H_s'(x) + G_s(x).$$

By (4.7) for $n = s$, we get

$$H_{s+1}(x) = \sum_{j=1}^{s-1} G_j^{(s-j)}(x) + G_s(x) = \sum_{j=1}^{s} G_j^{(s-j)}(x),$$

i.e., (4.5) holds for $n = s+1$. In view of $(a \cdot b)^{(n)} = \sum_{i=0}^{n} C_n^i \cdot a^{(n-j)} \cdot b^{(j)}$, we have for $1 \leqslant j \leqslant s$,

$$\begin{aligned} G_j^{(s-j)}(x) = &\sum_{i=0}^{s-j} C_{s-j}^i \cdot \left[\left((\varphi^{p-1})'(\varphi(f(x))) \cdot (f'(x))^j\right)'\right]^{(s-j-i)} \cdot (\varphi^{(j)}(f(x)))^{(i)} \\ &- \sum_{i=0}^{s-j} C_{s-j}^i \cdot (f''(\varphi(x)) \cdot \varphi'(x))^{(s-j-i)} \cdot \varphi^{(i+j)}(x). \end{aligned}$$

And from (4.2) holding for $n \leqslant s$, we have

$$\lim_{x\to 0^+} G_j^{(s-j)}(x) = 0.$$

It follows that

$$\lim_{x\to 0^+} H_{s+1}(x) = 0,$$

i.e., (4.6) holds for $n = s+1$. In view of (4.5) for $n = s+1$ and letting $x \to 0^+$, we get

$$f'(1) \cdot \lim_{x\to 0^+} \varphi^{(s+1)}(x) = (\varphi^{p-1})'(1) \cdot (f'(0))^{s+1} \cdot \lim_{x\to 0^-} \varphi^{(s+1)}(x),$$

and letting $x \to 0^-$, we get

$$f'(1) \cdot \lim_{x\to 0^-} \varphi^{(s+1)}(x) = (\varphi^{p-1})'(1) \cdot (f'(0))^{s+1} \cdot \lim_{x\to 0^+} \varphi^{(s+1)}(x).$$

By the above two equations, we have

$$\left[|f'(1)|^2 - |(\varphi^{p-1})'(1) \cdot (f'(0))^{s+1}|^2\right] \cdot \lim_{x\to 0^+} \varphi^{(s+1)}(x) = 0.$$

Thus by $f'(1) + (\varphi^{p-1})'(1) \cdot f'(0) < 0$ and $0 < |f'(0)| < 1$, we have

$$\lim_{x\to 0^+} \varphi^{(s+1)}(x) = 0.$$

Similarly, we have

$$\lim_{x\to 0^-} \varphi^{(s+1)}(x) = 0,$$

i.e., (4.2) holds for $n = s + 1$. Thereby (4.2), (4.5), and (4.6) are proved by induction.

By the differential limiting theorem, we have

$$\varphi_+^{(n)}(0) = \lim_{x\to 0^+} \varphi^{(n)}(x) = 0, \quad \varphi_-^{(n)}(0) = \lim_{x\to 0^-} \varphi^{(n)}(x) = 0.$$

Thus

$$\varphi^{(n)}(0) = 0.$$

Thereby, $\varphi(x)$ is a unimodal-even-smooth solution of Eq. (1.1).

Secondly, if φ_0 is the restriction on $[-1, f(\alpha)] \cup [-f(\alpha), 1]$ of a unimodal-even-smooth solution to Eq. (1.1), then conditions (i)-(v) must hold by the lemmas in Section 2. This completes the proof. □

5. Appendix

Proof of Lemma 2.2.

(i) We now prove that for all $n \geqslant 0$ and each $x \in [-1, 1]$, we have

$$f^n(\varphi(x)) = \varphi^{p^n}(f^n(x)). \tag{5.1}$$

Obviously, (5.1) holds for $n = 1$ by Eq. (1.1). Suppose that (5.1) holds for $n \leqslant k$, where $k \geqslant 1$ is a certain integer. Therefore, by induction and (1.1), we have that

$$\begin{aligned}
\varphi^{p^{k+1}}(f^{k+1}(x)) &= (\varphi^{p^k})^p(f^{k+1}(x)) = (\varphi^{p^k})^{p-1} \circ \varphi^{p^k}(f^{k+1}(x)) \\
&= (\varphi^{p^k})^{p-1}(f^k(\varphi(f(x)))) = (\varphi^{p^k})^{p-2} \circ \varphi^{p^k}(f^k(\varphi(f(x)))) \\
&= (\varphi^{p^k})^{p-2}(f^k(\varphi^2(f(x)))) = \cdots = (\varphi^{p^k})^{p-i}(f^k(\varphi^i(f(x)))) \\
&= \cdots = f^k(\varphi^p(f(x))) = f^k(f(\varphi(x))) = f^{k+1}(\varphi(x)),
\end{aligned}$$

i.e., (5.1) holds for $n = k + 1$. Thereby, (5.1) is proved by induction. Let $x = 0$ in (5.1), we have that

$$f^n(1) = f^n(\varphi(0)) = \varphi^{p^n}(f^n(0)) = \varphi^{p^n}(0).$$

And trivially $\{|f^n(1)|\}$ is strictly decreasing and $\lim_{n\to\infty} |f^n(1)| = 0$. Thereby, we have that

$$\lim_{n\to\infty} \varphi^{p^n}(0) = \lim_{n\to\infty} f^n(1) = 0, \tag{5.2}$$

i.e., we proved that 0 is a recurrent but not periodic point of φ.

(ii) Let $x = 0$ in (1.1), then

$$f(1) = f(\varphi(0)) = \varphi^p(f(0)) = \varphi^p(0) = \varphi^{p-1}(1).$$

Let $-\varphi^{p-1}(1) = -f(1) = \lambda > 0$. Since $\varphi(0) = 1, \varphi(|\varphi^{p-2}(1)|) = \varphi^{p-1}(1) = -\lambda < 0$ and $\varphi(x)$ is strictly decreasing on $[0,1]$, we have that $\varphi(x)$ has a unique point α on $[0,1]$ such that $\varphi(\alpha) = 0$. Firstly, we prove that $0 < \beta < \alpha$. Since $\varphi(0) = 1$, $\varphi(\alpha) = 0$, and $\varphi(x)$ is strictly decreasing on $[0,1]$, we have that $\varphi(x)$ has a unique fixed point on $(0,\alpha)$ and has not fixed point on $[\alpha,1]$, respectively. Suppose that $\varphi(x)$ has another fixed point q, then $q \in [-1,0]$. And by $\varphi(0) = 1$ and (5.2), we have $q \neq 0$ and $q \neq -1$. Thus, we have $q \in (-1,0)$. Since $\varphi(x)$ is strictly increasing in $[-1,0]$, it follows that $0 > q = \varphi(q) > \varphi(-1) = \varphi(1) > -1$. And by induction, for all $m \geqslant 0$, we have $0 > q = \varphi^m(q) > \varphi^m(1)$. Specially,

$$0 > q = \varphi^{p^n-1}(q) > \varphi^{p^n-1}(1) = \varphi^{p^n-1}(\varphi(0)) = \varphi^{p^n}(0).$$

This contradicts (5.2). Thus, we proved that $\varphi(x)$ has a unique fixed point $\beta = \varphi(\beta)$ in $[-1,1]$ and $0 < \beta < \alpha < 1$.

Secondly, we prove that $\lambda < \beta$. Suppose that $\lambda \geqslant \beta$. By $0 \leqslant -f(x) \leqslant -f(1) = \lambda$, we have that there exists $\gamma \in [0,1]$, such that $-f(\gamma) = \beta$, i.e., $\gamma = f^{-1}(-\beta)$. And by Eq. (1.1), we have that

$$\beta = \varphi^p(\beta) = \varphi^p(-f(\gamma)) = \varphi^p(f(\gamma)) = f(\varphi(\gamma)) = f(\varphi(f^{-1}(-\beta))).$$

Thus, $f^{-1}(\beta) = \varphi(f^{-1}(-\beta)) = \varphi(-f^{-1}(\beta)) = \varphi(f^{-1}(\beta))$. And since $\varphi(x)$ has a unique fixed point β in $[-1,1]$, we have $f^{-1}(\beta) = \beta$. Thus, $\beta = f(\beta)$. And by $f(0) = 0, -x < f(x) < 0, (x \in (0,1]), f(-x) = -f(x)$, we know $\beta = 0$. This contradicts $\varphi(0) = 1$. Thus, we proved that $\lambda < \beta$.

(iii) Firstly, we prove the sufficiency. By Eq. (1.1), we have $0 = f(\varphi(\alpha)) = \varphi^p(f(\alpha)) = \varphi^p(-f(\alpha))$. And since α and $-\alpha$ are the zero points of φ respectively on $[0,1]$ and $[-1,0]$, it follows that $|\varphi^{p-1}(-f(\alpha))| = |\varphi^{p-1}(f(\alpha))| = \alpha$. Thus the sufficiency is proved.

Secondly, we prove the necessity. Suppose that $|\varphi^i(x)| = \alpha$ for some $x \in [-\lambda,\lambda]$ and $0 \leqslant i \leqslant p-1$. Then $x \neq 0$. α and $-\alpha$ are not periodic points of $\varphi(x)$ by conclusion (i). We claim that $|x| \neq \lambda$. Suppose that $|x| = \lambda$. And since

$$\varphi^{p+1}(\alpha) = \varphi^p(\varphi(\alpha)) = \varphi^p(0) = \varphi^p(f(0)) = f(\varphi(0)) = f(1) = -\lambda,$$

then

$$\alpha = |\varphi^i(x)| = |\varphi^i(|x|)| = |\varphi^i(\lambda)| = |\varphi^i(-\lambda)| = |\varphi^{i+p+1}(\alpha)|.$$

This contradicts that α and $-\alpha$ are not periodic points. Thus $|x| \neq \lambda$. We prove that $|x| \notin (-f(\alpha),\lambda)$ in the following. Suppose that $|x| \in (-f(\alpha),\lambda)$, then φ^{i+1} is negative on $(-f(\alpha),|x|)$ or $(|x|,\lambda)$. This contradicts φ^p is not negative in $(-f(\alpha),\lambda)$ since Eq. (1.1) and $f \circ \varphi$ is not negative in $(-1,-\alpha)$. Thus $|x| \notin (-f(\alpha),\lambda)$. By the similar argument we can claim that $|x| \notin (0,-f(\alpha))$. So we have $x = f(\alpha)$ or $x = -f(\alpha)$. Since α and $-\alpha$ are not periodic points, then for all $j \neq i$, we know that $|\varphi^j(x)| \neq \alpha$. Thus, we have $|x| = -f(\alpha)$ and $i = p-1$. Thus the necessity is proved.

(iv) Firstly, we claim that

$$|\varphi^i(-f(\alpha))| = |\varphi^i(f(\alpha))| > \lambda, \quad \forall 1 \leqslant i \leqslant p-1. \tag{5.3}$$

Suppose that there exists some $1 \leqslant j \leqslant p-1$, such that $0 \leqslant |\varphi^j(f(\alpha))| = x \leqslant \lambda$. Then

$$|\varphi^{p-1-j}(x)| = |\varphi^{p-1-j}(\varphi^j(f(\alpha)))| = |\varphi^{p-1}(f(\alpha))| = \alpha$$

by conclusion (iii). This contradicts conclusion (iii). Thus (5.3) is proved.

Secondly, we prove that

$$|\varphi^i(x)| > \lambda, \quad \forall x \in [f(\alpha),-f(\alpha)], \quad 1 \leqslant i \leqslant p-1. \tag{5.4}$$

Trivially, φ^i are homeomorphisms respectively on $[0, -f(\alpha)]$ and $[f(\alpha), 0]$ by conclusion (iii). It suffices from (5.3) to show that $|\varphi^i(0)| > \lambda$. Suppose that there exists some $1 \leqslant j \leqslant p-1$, such that $0 \leqslant |\varphi^j(0)| = x \leqslant \lambda$. Then

$$|\varphi^{p-j}(x)| = |\varphi^{p-j}(\varphi^j(0))| = |\varphi^p(0)| = |\varphi^p(f(0))| = |f(\varphi(0))| = \lambda.$$

Thus

$$|\varphi^j(\lambda)| = |\varphi^j(\varphi^{p-j}(x))| = |\varphi^p(x)| = |f(\varphi(f^{-1}(x)))| \leqslant |f(1)| = \lambda.$$

Since $\varphi^j(x)$ is also a homeomorphism in $[0, \lambda]$ by conclusion (iii), we have $|\varphi^j(-f(\alpha))| \leqslant \lambda$. This contradicts (5.3). Thus (5.4) is proved.

Thirdly, we prove that

$$|\varphi^i(x)| > -f(\alpha), \quad \forall x \in (-\lambda, f(\alpha)] \cup (-f(\alpha), \lambda], \quad 1 \leqslant i \leqslant p-1. \tag{5.5}$$

Suppose that there exists some $1 \leqslant j \leqslant p-1$ and $|x| \in (-f(\alpha), \lambda]$, such that $0 \leqslant |\varphi^j(x)| = y \leqslant -f(\alpha)$. Then

$$|\varphi^{p-j}(y)| = |\varphi^{p-j}(\varphi^j(x))| = |\varphi^p(x)| = |f(\varphi((f^{-1}(x))))| \leqslant |f(1)| = \lambda.$$

This contradicts (5.4). Thus (5.5) is proved.

(v) Suppose that there exist some $1 \leqslant j \leqslant p-1$ and $x \in [-\lambda, \lambda]$, such that x is a periodic point of φ with period j, i.e., $\varphi^j(x) = x$. Then we have $|x| \in (-f(\alpha), \lambda]$ by (5.4). Let

$$y = \min\{|x|, |\varphi(x)|, \cdots, |\varphi^{j-1}(x)|\},$$

then $y \in (-f(\alpha), \lambda]$ by (5.5). Since $f^{-1}(y) \in [-1, -\alpha)$, $\varphi(-\alpha) = 0 > -\alpha$, and by conclusion (ii), we have $0 > \varphi(f^{-1}(y)) > f^{-1}(y)$. Thus, by Eq. (1.1) and $f(x)$ being strictly decreasing odd function and $f(0) = 0$, we know that

$$|\varphi^p(y)| = |f(\varphi(f^{-1}(y)))| = f(\varphi(f^{-1}(y))) < f(f^{-1}(y)) = y.$$

This contradicts the definition of y. Thus we proved that φ has no periodic point of period i on $[-\lambda, \lambda]$. This completes the proof. □

Proof of Lemma 2.3.

(i) For all $i = 0, 1, \cdots, p-2$, then $\varphi^{i+1} : J \longmapsto J_i$ is a homeomorphism by Lemma 2.2 (iii). Thus $\varphi^i : J_0 \longmapsto J_i$ is also a homeomorphism.

(ii) Firstly, we prove that for all $i = 0, 1, \cdots, p-2$, we have $J_i \subset (\lambda, 1]$ or $J_i \subset [-1, -\lambda)$. We claim that $|\varphi^{i+1}(\lambda)| > \lambda$. Suppose that there exists some $1 \leqslant j \leqslant p-1$, such that $|\varphi^j(\lambda)| \leqslant \lambda$. Then by (5.3), we have that

$$|\varphi^j(-f(\alpha))| > \lambda > -f(\alpha).$$

Thus we know that $|\varphi^j|$ has a fixed point in $[-f(\alpha), \lambda]$. This contradicts Lemma 2.2 (v). Thus $|\varphi^{i+1}(\lambda)| > \lambda$. And since $\varphi^{i+1} : J \longmapsto J_i$ is a homeomorphism and by (5.4), we have that $J_i \subset (\lambda, 1]$ or $J_i \subset [-1, -\lambda)$.

Secondly, we prove that for all $0 \leqslant i \leqslant p-2$, J_i are pairwise disjoint. Suppose that there exists $0 \leqslant i < j \leqslant p-2$, such that $J_i \cap J_j = J_{ij} \neq \emptyset$. Let $y \in J_{ij}$, then there exists $x_i \in [0, \lambda], x_j \in [0, \lambda]$, such that $\varphi^{i+1}(x_i) = y = \varphi^{j+1}(x_j)$. Thus we have

$$|\varphi^{p-j+i}(x_i)| = |\varphi^{p-1-j}(\varphi^{i+1}(x_i))| = |\varphi^{p-1-j}(\varphi^{j+1}(x_j))| = |\varphi^p(x_j)| = |f(\varphi(f^{-1}(x_j)))| \leqslant |f(1)| = \lambda.$$

This contradicts $J_i \subset (\lambda, 1]$ or $J_i \subset [-1, -\lambda)$. Thus we proved that $J_0, \cdots, J_{p-2}$ are pairwise disjoint. This completes the proof. □

Proof of Lemma 2.4. Obviously, $x = 1$ is a solution of the equation $\varphi^{p-1}(x) = f(x)$ by Eq. (1.1). Suppose that $x = x_0$ is an arbitrary solution of this equation, i.e., $\varphi^{p-1}(x_0) = f(x_0)$. Since $(\varphi(-f(\alpha)), 1] = \varphi([0, -f(\alpha)])$, we have that there exists $y_0 \in [0, -f(\alpha)]$, such that $\varphi(y_0) = x_0$. Thus, $\varphi^{p-1}(\varphi(y_0)) = f(x_0)$. And by

Eq. (1.1), we get $f(\varphi(f^{-1}(y_0))) = \varphi^{p-1}(\varphi(y_0)) = f(x_0)$. It follows that $\varphi(f^{-1}(y_0)) = x_0$ since f is strictly monotone. Since $|f^{-1}(y_0)| \in [0, \alpha]$ and φ is strictly decreasing in $[0, \alpha]$, we have $y_0 = |f^{-1}(y_0)| = -f^{-1}(y_0)$, i.e., $f(y_0) = -y_0$. By $f(0) = 0, f(x) > -x(x \in (0, 1])$, we have $y_0 = 0$. It is proved that $x_0 = \varphi(y_0) = \varphi(0) = 1$. This completes the proof. □

Proof of Lemma 2.5. Since φ_1 and φ_2 are even, thus $\varphi_1(x) = \varphi_2(x)$ on $[-1, -\lambda]$. There exist $\alpha \in (\lambda, 1), \beta \in (\lambda, 1)$ such that $\varphi_i(\alpha) = 0, \varphi_i(\beta) = \beta$ $(i = 1, 2)$ by (2.1). Denote $\varphi_0(x) = \varphi_1(x) = \varphi_2(x)(x \in [-1, -\lambda] \cup [\lambda, 1])$, $\alpha_1 = \varphi_1(-f(\alpha)) = \varphi_1(f(\alpha)), \alpha_2 = \varphi_2(-f(\alpha)) = \varphi_2(f(\alpha))$. We now prove that $\alpha_1 = \alpha_2$. Trivially,

$$\alpha_1 > \varphi_1(-f(1)) = \varphi_0(\lambda) > \varphi_0(\beta) = \beta > \lambda,$$

by (2.1). Similarly, $\alpha_2 > \varphi_0(\lambda) > \lambda$. And by Lemma 2.3 and Lemma 2.2 (iii), we have that

$$\varphi_0^{p-1}(\alpha_1) = \varphi_1^{p-1}(\alpha_1) = \varphi_1^{p}(f(\alpha)) = 0.$$

Similarly, $\varphi_0^{p-1}(\alpha_2) = 0$. Since φ_0^{p-1} has a unique zero on $[\varphi_0(\lambda), 1]$, we conclude that $\alpha_1 = \alpha_2$.

Let $\alpha_0 = \alpha_1 = \alpha_2$, then $\alpha_0 \in [\varphi_0(\lambda), 1]$ and $\varphi_0^{p-1}(\alpha_0) = 0$. Define

$$\psi = \varphi_0^{p-1}|_{[\varphi_0(\lambda),1]}.$$

By Lemma 2.2 (iii) and Lemma 2.3, ψ is a homeomorphism. Since

$$\psi(\varphi_0(\lambda)) = \varphi_0^{p-1}(\varphi_0(\lambda)) = f(\varphi_0(1)) > 0 > -\lambda = \varphi_0^{p-1}(1) = \psi(1),$$

ψ is strictly decreasing. Let $h(x) = -f(x)$. Trivially, $\{h^k(1)\}$ is decreasing and

$$\lim_{k\to\infty} h^k(1) = 0.$$

Let

$$\Delta_k = [-h^k(1), -h^{k+1}(1)] \cup [h^{k+1}(1), h^k(1)], \quad (k = 0, 1, 2, \cdots).$$

Then $[-1, 1] = \bigcup_{k=0}^{+\infty} \Delta_k$.

We now prove $\varphi_1(x) = \varphi_2(x)$ on Δ_k by induction.

Obviously, $\varphi_1(x) = \varphi_2(x)$ on Δ_0. Suppose that $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \leqslant m$, where $m \geqslant 0$ is a certain integer. Let

$$\varphi(x) = \varphi_1(x) = \varphi_2(x), \quad x \in [-1, -h^{m+1}(1)] \cup [h^{m+1}(1), 1].$$

If $0 \leqslant x \leqslant h(1) = \lambda$, then $\varphi_i(-x) = \varphi_i(x) \geqslant \varphi_i(\lambda) = \varphi_0(\lambda) > \varphi_0(\beta) = \beta > \lambda$. Thus by Eq. (1.1) we have

$$\begin{aligned} f(\varphi(h^{-1}(x))) &= f(\varphi(-f^{-1}(x))) \\ &= f(\varphi_i(f^{-1}(x))) = \varphi_i^{p-1}(\varphi_i(x)) \\ &= \varphi_0^{p-1}(\varphi_i(x)) = \psi(\varphi_i(x)), \quad (i = 1, 2, x \in \Delta_{m+1}). \end{aligned} \tag{5.6}$$

Then equation (5.6) is equivalent to

$$\varphi_i(x) = \psi^{-1}(f(\varphi(h^{-1}(x)))), \quad (i = 1, 2, x \in \Delta_{m+1}).$$

Thus $\varphi_1(x) = \varphi_2(x)$ on Δ_{m+1}. By induction, $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \geqslant 0$. This completes the proof. □

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