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Simpson-like type inequalities for relative semi-(α, m)-logarithmically convex functions

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Abstract

In this paper, we derive a new integral identity concerning differentiable mappings defined on relative convex set. By using the obtained identity as an auxiliary result, we prove some new Simpson-like type inequalities for mappings whose absolute values of the first derivatives are relative semi-(α, m)-logarithmically convex. Several special cases are also discussed.

Keywords: Relative semi-(α, m)-logarithmically convex, Simpson's inequality, Hölder's inequality, Young inequality.
2010 MSC: 26A33, 26A51, 26D07, 26D20, 41A55.

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1. Introduction

Following inequalities are well-known in the literature termed as the Hermite-Hadamard's inequality and Simpson's inequality, respectively.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on $[u, v]$ with $u, v \in I$ and $u < v$. Then the following inequalities hold:*

$$f\left(\frac{u+v}{2}\right) \leqslant \frac{1}{v-u}\int_u^v f(x)dx \leqslant \frac{f(u)+f(v)}{2}. \tag{1.1}$$

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and*

$$\|f^{(4)}\|_\infty := \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty.$$

Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4. \tag{1.2}$$

Let us evoke some basic definitions as follows.

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Definition 1.3 ([2]). A function $f : [0, b] \longrightarrow (0, \infty)$ is said to be (α, m)-logarithmically convex if the inequality

$$f(tx + m(1-t)y) \leqslant [f(x)]^{t^{\alpha}}[f(y)]^{m(1-t^{\alpha})}$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$, and some fixed $(\alpha, m) \in (0, 1] \times (0, 1]$.

Note that for $(\alpha, m) \in \{(\alpha, 1), (1, m), (1, 1)\}$ one obtains the following classes of mappings: α-logarithmically convex, m-logarithmically convex and logarithmically convex, respectively.

Some interesting and important inequalities related to logarithmically convex functions can be found in [2, 14, 15, 31, 33].

Definition 1.4 ([35]). A set $M_{\varphi} \subseteq \mathbb{R}^n$ is said to be a relative convex (φ-convex) set, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$t\varphi(x) + (1-t)\varphi(y) \in M_{\varphi}, \forall\ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_{\varphi}, t \in [0, 1].$$

Definition 1.5 ([35]). A function f is said to be a relative convex (φ-convex) function on a relative convex (φ-convex) set M_{φ}, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leqslant tf(\varphi(x)) + (1-t)f(\varphi(y))$$

for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_{\varphi},\ t \in [0, 1]$.

Definition 1.6 ([3]). A function f is said to be a relative semi-convex on a relative convex set M_{φ}, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leqslant tf(x) + (1-t)f(y)$$

for all $x, y \in M_{\varphi},\ t \in [0, 1]$.

Definition 1.7 ([4]). A function $f : M_{\varphi} \to \mathbb{R}^+$ is said to be a relative semi-logarithmic convex on a relative convex set M_{φ}, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(t\varphi(x) + (1-t)\varphi(y)) \leqslant f(x)^t f(y)^{(1-t)}$$

for all $x, y \in M_{\varphi},\ t \in [0, 1]$.

In 2016, Shuang et al. [25] established the following lemma.

Lemma 1.8 ([25]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° with $a < b$. If $f' \in L_1([a, b])$, then*

$$\begin{aligned}\frac{1}{8}\Big[f(a) + 6f\Big(\frac{a+b}{2}\Big) + f(b)\Big] - \frac{1}{b-a}\int_a^b f(x)dx \\ = \frac{b-a}{4}\int_0^1 \Big[\Big(\frac{3}{4} - t\Big)f'\Big(ta + (1-t)\frac{a+b}{2}\Big) + \Big(\frac{1}{4} - t\Big)f'\Big(t\frac{a+b}{2} + (1-t)b\Big)\Big]dt.\end{aligned}$$

On the basis of the above obtained identity, they developed some Simpson type inequalities via (α, m)-convex.

In recent years, many researches generalized and extended the inequalities (1.1). For some recent related results, for example, see [1, 6, 10–13, 17–20, 23, 26, 29, 30, 32] and the references included there.

It is always been known that the Simpson type inequality plays a fundamental and important role in analysis. In particular, it is well applied in numerical integration. Due to this fact, the Simpson type integral inequality has attracted renewed attention and interest from researchers. A variety of refinements and generalizations with respect to the inequality (1.2) have been found (see for example, [5, 7–9, 16, 21, 22, 24, 27, 28, 34]).

Motivated by the inspiring idea in [12, 19] and by the ongoing research, in the present paper, the next section we are going to introduce a new concept, to be referred as the relative semi-(α, m)-logarithmically convex functions, and then we derive an integral identity. By using the obtained identity, we explore new Simpson-like type inequalities for mappings whose absolute value of first derivative is relative semi-(α, m)-logarithmically convex. Some interesting special cases of our main result are also considered.

2. New definitions and a lemma

In this section, we introduce the concept of relative semi-(α, m)-logarithmically convex functions.

From now onward, we take the notation $M_\varphi = I = [\varphi(a), \varphi(b)]$ with $0 < \varphi(a) < \varphi(b) < \infty$ be a relative convex set.

Definition 2.1. A function $f : M_\varphi \to \mathbb{R}^+$ is said to be a relative semi-(α, m)-logarithmic convex on a relative convex set M_φ, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \leqslant [f(x)]^{t^\alpha}[f(y)]^{m(1-t^\alpha)}$$

for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$, $t \in [0, 1]$, and some fixed $(\alpha, m) \in (0, 1] \times (0, 1]$.

Definition 2.2. A function $f : M_\varphi \to \mathbb{R}^+$ is said to be a relative semi-α-logarithmic convex on a relative convex set M_φ, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leqslant f(y)\left[\frac{f(x)}{f(y)}\right]^{t^\alpha}$$

for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$, $t \in [0, 1]$, and some fixed $\alpha \in (0, 1]$.

Definition 2.3. A function $f : M_\varphi \to \mathbb{R}^+$ is said to be a relative semi-m-logarithmic convex on a relative convex set M_φ, if and only if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \leqslant [f(x)]^{t}[f(y)]^{m(1-t)}$$

for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$, $t \in [0, 1]$, and some fixed $m \in (0, 1]$.

Remark 2.4. In Definitions 2.1, 2.2 and 2.3, if we take φ is the identity function, then we have the definition of (α, m)-, α- and m-logarithmic convex, respectively.

To establish some new Simpson-like type inequalities for relative semi-(α, m)-logarithmically convex functions, we need the following lemma.

Lemma 2.5. *Let $f : I \subseteq \mathbb{R} \to (0, \infty)$ be a differentiable function on I° (interior of I) and let $\varphi : I \subseteq \mathbb{R} \to (0, \infty)$ be a continuous function with $\varphi(a) < \varphi(b)$. If $f' \in L([\varphi(a), \varphi(b)])$, then we have*

$$\begin{aligned}
&\frac{1}{8}\left[f(\varphi(a)) + 6f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + f(\varphi(b))\right] - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f(\varphi(x))}{\varphi(x)}d\varphi(x) \\
&= \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\left\{\sqrt{\varphi(a)}\int_0^1\left(\frac{3}{4} - t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\Big)dt\right. \qquad (2.1)\\
&\quad \left. + \sqrt{\varphi(b)}\int_0^1\left(\frac{1}{4} - t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\Big)dt\right\}.
\end{aligned}$$

Proof. Integrating by parts for $0 \leqslant t \leqslant 1$ leads to

$$\begin{aligned}
&\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(a)\varphi(b)}\int_0^1\left(\frac{3}{4}-t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}f'\left([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\right)dt\\
&=\int_0^1\left(\frac{t}{2}-\frac{3}{8}\right)d\left[f\left([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\right)\right]\\
&=\left(\frac{t}{2}-\frac{3}{8}\right)f\left([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\right)\Big|_0^1-\frac{1}{2}\int_0^1 f\left([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\right)dt\\
&=\frac{1}{8}f(\varphi(a))+\frac{3}{8}f\left(\sqrt{\varphi(a)\varphi(b)}\right)-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\sqrt{\varphi(a)\varphi(b)}}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\varphi(b)\int_0^1\left(\frac{1}{4}-t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}f'\left([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\right)dt\\
&=\int_0^1\left(\frac{t}{2}-\frac{1}{8}\right)d\left[f\left([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{\frac{2-t}{2}}\right)\right]\\
&=\left(\frac{t}{2}-\frac{1}{8}\right)f\left([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{\frac{2-t}{2}}\right)\Big|_0^1-\frac{1}{2}\int_0^1 f\left([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{\frac{2-t}{2}}\right)dt\\
&=\frac{3}{8}f\left(\sqrt{\varphi(a)\varphi(b)}\right)+\frac{1}{8}f(\varphi(b))-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\sqrt{\varphi(a)\varphi(b)}}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x).
\end{aligned}$$

Lemma 2.5 is thus proved. □

Remark 2.6. In Lemma 2.5, if we take φ is the identity function, then (2.1) becomes to the following identity

$$\begin{aligned}
&\frac{1}{8}\left[f(a)+6f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\\
&=\frac{\ln b-\ln a}{4}\sqrt{b}\left\{\sqrt{a}\int_0^1\left(\frac{3}{4}-t\right)\left(\frac{a}{b}\right)^{\frac{t}{2}}f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)dt+\sqrt{b}\int_0^1\left(\frac{1}{4}-t\right)\left(\frac{a}{b}\right)^{\frac{t}{2}}f'\left(a^{\frac{t}{2}}b^{1-\frac{t}{2}}\right)dt\right\}.
\end{aligned}$$

3. Main results

Using Lemma 2.5, in this section, integral inequalities of the Simpson-like type related to relative semi-(α, m)-logarithmically convex functions are presented.

Theorem 3.1. *Let $\varphi : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ be a continuous positive homogeneous function with $\varphi(a) < \varphi(b)$. Assume that $f : I \subseteq \mathbb{R} \to (0, \infty)$ is a differentiable mapping on I° (interior of I) such that $f' \in L([\varphi(a), \varphi(b)])$. If $|f'(\varphi(x))|^q$ for $q \geqslant 1$ is increasing and relative semi-(α, m)-logarithmically convex function on $\left[0, \varphi\left(\frac{b}{m}\right)\right]$ with $(\alpha, m) \in (0, 1] \times (0, 1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[(f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\
&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\left(\frac{b}{m}\right)\right|^m\\
&\times\left\{\sqrt{\varphi(a)}\vartheta_1^{1-\frac{1}{q}}\left(\varphi(a),\varphi(b)\right)\eta_0^{\frac{\alpha q}{2}}\left[\frac{-3+\eta_1^{\frac{\alpha q}{2}}}{2\alpha q\ln\eta_1}+\frac{8\eta_1^{\frac{3\alpha q}{8}}-4\eta_1^{\frac{\alpha q}{2}}-4}{(\alpha q\ln\eta_1)^2}\right]^{\frac{1}{q}}\right.\\
&\left.+\sqrt{\varphi(b)}\vartheta_2^{1-\frac{1}{q}}\left(\varphi(a),\varphi(b)\right)\left[\frac{-1+3\eta_1^{\frac{\alpha q}{2}}}{2\alpha q\ln\eta_1}+\frac{8\eta_1^{\frac{\alpha q}{8}}-4\eta_1^{\frac{\alpha q}{2}}-4}{(\alpha q\ln\eta_1)^2}\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.1}$$

where

$$\eta_0 = \frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}, \tag{3.2}$$

$$\eta_1 = \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{\alpha q}} \frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m} = \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{\alpha q}} \eta_0,$$

$$\vartheta_1\Big(\varphi(a), \varphi(b)\Big) = \frac{\big(\ln\varphi(b) - \ln\varphi(a)\big)\big(3 - \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}\big) + 16\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{3}{8}} - 8\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}} - 8}{2\big(\ln\varphi(b) - \ln\varphi(a)\big)^2},$$

and

$$\vartheta_2\Big(\varphi(a), \varphi(b)\Big) = \frac{\big(\ln\varphi(b) - \ln\varphi(a)\big)\big(1 - 3\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}\big) + 16\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{8}} - 8\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}} - 8}{2\big(\ln\varphi(b) - \ln\varphi(a)\big)^2}.$$

Proof. By Lemma 2.5, Young inequality, monotonically increasing and relative semi-(α, m)-logarithmically convexity of $|f'(\varphi(x))|$, we get

$$\begin{aligned}
&\left|\frac{1}{8}\Big[f(\varphi(a)) + 6f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + f(\varphi(b))\Big] - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f(\varphi(x))}{\varphi(x)} d\varphi(x)\right| \\
&\leqslant \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\Bigg\{\sqrt{\varphi(a)}\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'\Big([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\Big)\right| dt \\
&\quad + \sqrt{\varphi(b)}\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'\Big([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\Big)\right| dt\Bigg\} \\
&\leqslant \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\Bigg\{\sqrt{\varphi(a)}\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'\Big(\frac{1+t}{2}\varphi(a) + \frac{1-t}{2}\varphi(b)\Big)\right| dt \\
&\quad + \sqrt{\varphi(b)}\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'\Big(\frac{t}{2}\varphi(a) + \frac{2-t}{2}\varphi(b)\Big)\right| dt\Bigg\} \qquad (3.3) \\
&\leqslant \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\Bigg\{\sqrt{\varphi(a)}\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^\alpha}\left|f'\Big(\frac{b}{m}\Big)\right|^{m\left(1-\left(\frac{1+t}{2}\right)^\alpha\right)} dt \\
&\quad + \sqrt{\varphi(b)}\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left|f'(a)\right|^{(\frac{t}{2})^\alpha}\left|f'\Big(\frac{b}{m}\Big)\right|^{m(1-(\frac{t}{2})^\alpha)} dt\Bigg\} \\
&= \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\Big(\frac{b}{m}\Big)\right|^m\Bigg\{\sqrt{\varphi(a)}\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\right]^{\left(\frac{1+t}{2}\right)^\alpha} dt \\
&\quad + \sqrt{\varphi(b)}\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\right]^{\left(\frac{t}{2}\right)^\alpha} dt\Bigg\}.
\end{aligned}$$

Using power mean inequality, we get

$$\begin{aligned}
&\left|\frac{1}{8}\Big[f(\varphi(a)) + 6f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + f(\varphi(b))\Big] - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f(\varphi(x))}{\varphi(x)} d\varphi(x)\right| \\
&\leqslant \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\Big(\frac{b}{m}\Big)\right|^m \\
&\times\Bigg\{\sqrt{\varphi(a)}\left(\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{3}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\right]^{q\left(\frac{1+t}{2}\right)^\alpha} dt\right)^{\frac{1}{q}} \qquad (3.4) \\
&+ \sqrt{\varphi(b)}\left(\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{1}{4} - t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\right]^{q(\frac{t}{2})^\alpha} dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

If $0<\mu\leqslant 1$ and $0<\alpha,s\leqslant 1$, then

$$\mu^{\alpha^s}\leqslant \mu^{\alpha s}. \tag{3.5}$$

When $\eta_0=\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\leqslant 1$, by (3.5), we get

$$\begin{aligned}\int_0^1\left|\frac{3}{4}-t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{q\left(\frac{1+t}{2}\right)^\alpha}dt&\leqslant\int_0^1\left|\frac{3}{4}-t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\eta_0^{\frac{\alpha q(1+t)}{2}}dt\\&=\eta_0^{\frac{\alpha q}{2}}\int_0^1\left|\frac{3}{4}-t\right|\eta_1^{\frac{\alpha q t}{2}}dt\\&=\eta_0^{\frac{\alpha q}{2}}\left[\frac{-3+\eta_1^{\frac{\alpha q}{2}}}{2\alpha q\ln\eta_1}+\frac{8\eta_1^{\frac{3\alpha q}{8}}-4\eta_1^{\frac{\alpha q}{2}}-4}{(\alpha q\ln\eta_1)^2}\right],\end{aligned}\tag{3.6}$$

similarly,

$$\int_0^1\left|\frac{1}{4}-t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{q\left(\frac{t}{2}\right)^\alpha}dt\leqslant\frac{-1+3\eta_1^{\frac{\alpha q}{2}}}{2\alpha q\ln\eta_1}+\frac{8\eta_1^{\frac{\alpha q}{8}}-4\eta_1^{\frac{\alpha q}{2}}-4}{(\alpha q\ln\eta_1)^2}. \tag{3.7}$$

Also

$$\int_0^1\left|\frac{3}{4}-t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}dt=\frac{\left(\ln\varphi(b)-\ln\varphi(a)\right)\left(3-\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}\right)+16\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{3}{8}}-8\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}-8}{2\left(\ln\varphi(b)-\ln\varphi(a)\right)^2}, \tag{3.8}$$

and

$$\int_0^1\left|\frac{1}{4}-t\right|\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}}dt=\frac{\left(\ln\varphi(b)-\ln\varphi(a)\right)\left(1-3\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}\right)+16\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{8}}-8\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{2}}-8}{2\left(\ln\varphi(b)-\ln\varphi(a)\right)^2}. \tag{3.9}$$

Using (3.6), (3.7), (3.8), (3.9) in (3.4), we obtain the desired inequality (3.1). This completes the proof. □

We now discuss some special cases of Theorem 3.1.

Corollary 3.2. *In Theorem 3.1, letting $q=1$, we have the following inequality for a relative semi-(α,m)-logarithmically convex*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\left(\frac{b}{m}\right)\right|^m\\&\quad\times\left\{\sqrt{\varphi(a)}\eta_0^{\frac{\alpha}{2}}\left[\frac{-3+\eta_2^{\frac{\alpha}{2}}}{2\alpha\ln\eta_2}+\frac{8\eta_2^{\frac{3\alpha}{8}}-4\eta_2^{\frac{\alpha}{2}}-4}{(\alpha\ln\eta_2)^2}\right]+\sqrt{\varphi(b)}\left[\frac{-1+3\eta_2^{\frac{\alpha}{2}}}{2\alpha\ln\eta_2}+\frac{8\eta_2^{\frac{\alpha}{8}}-4\eta_2^{\frac{\alpha}{2}}-4}{(\alpha\ln\eta_2)^2}\right]\right\},\end{aligned}$$

where

$$\eta_2=\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{\alpha}}\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}=\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{1}{\alpha}}\eta_0. \tag{3.10}$$

Corollary 3.3. *In Theorem 3.1, letting φ be the identity function and $\alpha=q=1$, we have the following inequality for an m-logarithmically convex*

$$\left|\frac{1}{8}\left[f(a)+6f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$
\begin{aligned}
&\leqslant \frac{\ln b-\ln a}{4}\sqrt{b}\left|f'\left(\frac{b}{m}\right)\right|^{m}\\
&\quad\times\left\{\sqrt{a\eta_0}\left[\frac{-3+\sqrt{\frac{a}{b}\eta_0}}{2(\ln a-\ln b+\ln\eta_0)}+\frac{8\left(\frac{a}{b}\eta_0\right)^{\frac{3}{8}}-4\sqrt{\frac{a}{b}\eta_0}-4}{(\ln a-\ln b+\ln\eta_0)^2}\right]\right.\\
&\quad\left.+\sqrt{b}\left[\frac{-1+3\sqrt{\frac{a}{b}\eta_0}}{2(\ln a-\ln b+\ln\eta_0)}+\frac{8\left(\frac{a}{b}\eta_0\right)^{\frac{1}{8}}-4\sqrt{\frac{a}{b}\eta_0}-4}{(\ln a-\ln b+\ln\eta_0)^2}\right]\right\},
\end{aligned}
$$

where η_0 *is defined by* (3.2).

Corollary 3.4. *In Theorem 3.1, letting* φ *be the identity function and* $m=q=1$*, we have the following inequality for an* α*-logarithmically convex*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+6f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\\
&\leqslant\frac{\ln b-\ln a}{4}\sqrt{b}|f'(b)|\\
&\quad\times\left\{\sqrt{a}\left[\frac{|f'(a)|}{|f'(b)|}\right]^{\frac{\alpha}{2}}\left[\frac{-3+\sqrt{\eta_3^{\alpha}}}{2\alpha\ln\eta_3}+\frac{8\left(\eta_3^{\alpha}\right)^{\frac{3}{8}}-4\sqrt{\eta_3^{\alpha}}-4}{(\alpha\ln\eta_3)^2}\right]\right.\\
&\quad\left.+\sqrt{b}\left[\frac{-1+3\sqrt{\eta_3^{\alpha}}}{2\alpha\ln\eta_3}+\frac{8\left(\eta_3^{\alpha}\right)^{\frac{1}{8}}-4\sqrt{\eta_3^{\alpha}}-4}{(\alpha\ln\eta_3)^2}\right]\right\},
\end{aligned}
$$

where

$$
\eta_3=\left(\frac{a}{b}\right)^{\frac{1}{\alpha}}\frac{|f'(a)|}{|f'(b)|}. \tag{3.11}
$$

Theorem 3.5. *Let* $\varphi:[a,b]\subseteq\mathbb{R}\to(0,\infty)$ *be a continuous positive homogeneous function with* $\varphi(a)<\varphi(b)$*. Assume that* $f:I\subseteq\mathbb{R}\to(0,\infty)$ *is a differentiable mapping on* I° *(interior of* I*) such that* $f'\in L([\varphi(a),\varphi(b)])$*. If* $|f'(\varphi(x))|^q$ *is increasing and relative semi-*(α,m)*-logarithmically convex function on* $\left[0,\varphi\left(\frac{b}{m}\right)\right]$ *for* $(\alpha,m)\in(0,1]\times(0,1]$ *,* $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$*, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\
&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{16}\left(\frac{1+3^{p+1}}{4(p+1)}\right)^{\frac{1}{p}}\left[\frac{2\eta_2^{\frac{\alpha q}{2}}-2}{\alpha q\ln\eta_2}\right]^{\frac{1}{q}}\sqrt{\varphi(b)}\left|f'\left(\frac{b}{m}\right)\right|^{m}\left[\eta_0^{\frac{\alpha}{2}}\sqrt{\varphi(a)}+\sqrt{\varphi(b)}\right],
\end{aligned}
\tag{3.12}
$$

where η_0 *and* η_2 *are defined by* (3.2) *and* (3.10)*, respectively.*

Proof. By Lemma 2.5, Hölder's inequality, Young inequality, monotonically increasing and relative semi-(α,m)-logarithmically convexity of $|f'(\varphi(x))|^q$, we get

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\
&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\\
&\quad\times\left\{\sqrt{\varphi(a)}\left(\int_0^1\left|\frac{3}{4}-t\right|^p dt\right)^{\frac{1}{p}}\left[\int_0^1\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{qt}{2}}\left|f'\left([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{\varphi(b)}\Big(\int_0^1\Big|\frac{1}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big|f'\Big([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\Big)\Big|^q dt\Big]^{\frac{1}{q}}\Big\}\\
&\leqslant \frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\Big\{\sqrt{\varphi(a)}\Big(\int_0^1\Big|\frac{3}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\\
&\times\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big|f'\Big(\frac{1+t}{2}\varphi(a)+\frac{1-t}{2}\varphi(b)\Big)\Big|^q dt\Big]^{\frac{1}{q}}\\
&+\sqrt{\varphi(b)}\Big(\int_0^1\Big|\frac{1}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big|f'\Big(\frac{t}{2}\varphi(a)+\frac{2-t}{2}\varphi(b)\Big)\Big|^q dt\Big]^{\frac{1}{q}}\Big\}\\
&\leqslant \frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\Big\{\sqrt{\varphi(a)}\Big(\int_0^1\Big|\frac{3}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\\
&\times\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}|f'(a)|^{q\left(\frac{1+t}{2}\right)^{\alpha}}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{mq\left(1-\left(\frac{1+t}{2}\right)^{\alpha}\right)}dt\Big]^{\frac{1}{q}}\\
&+\sqrt{\varphi(b)}\Big(\int_0^1\Big|\frac{1}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}|f'(a)|^{q(\frac{t}{2})^{\alpha}}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{mq(1-(\frac{t}{2})^{\alpha})}dt\Big]^{\frac{1}{q}}\Big\}\\
&=\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\
&\times\Big\{\sqrt{\varphi(a)}\Big(\int_0^1\Big|\frac{3}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\Big]^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\Big]^{\frac{1}{q}}\\
&+\sqrt{\varphi(b)}\Big(\int_0^1\Big|\frac{1}{4}-t\Big|^p dt\Big)^{\frac{1}{p}}\Big[\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\Big]^{q\left(\frac{t}{2}\right)^{\alpha}}dt\Big]^{\frac{1}{q}}\Big\}.
\end{aligned}\tag{3.13}
$$

If $0<\eta_0\leqslant 1$, by (3.5) we get

$$
\begin{aligned}
\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\Big]^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt &\leqslant \int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\eta_0^{\frac{\alpha q(1+t)}{2}}dt\\
&=\eta_0^{\frac{\alpha q}{2}}\int_0^1\eta_2^{\frac{\alpha qt}{2}}dt\\
&=\eta_0^{\frac{\alpha q}{2}}\frac{2\eta_2^{\frac{\alpha q}{2}}-2}{\alpha q\ln\eta_2},
\end{aligned}\tag{3.14}
$$

similarly,

$$
\int_0^1\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{qt}{2}}\Big[\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}\Big]^{q\left(\frac{t}{2}\right)^{\alpha}}dt\leqslant\frac{2\eta_2^{\frac{\alpha q}{2}}-2}{\alpha q\ln\eta_2}.\tag{3.15}
$$

Also,

$$
\int_0^1\Big|\frac{3}{4}-t\Big|^p dt=\int_0^1\Big|\frac{1}{4}-t\Big|^p dt=\frac{1+3^{p+1}}{4^{p+1}(p+1)}.\tag{3.16}
$$

Using (3.14), (3.15) and (3.16) in (3.13), we obtain the desired inequality (3.12). This completes the proof. □

As special cases, we provide the following results for Theorem 3.5.

Corollary 3.6. *In Theorem 3.5, letting φ be the identity function, $\alpha = 1$ and $p = q = 2$, we have the following inequality for an m-logarithmically convex*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+6f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\\&\leqslant\frac{\sqrt{21b}(\ln b-\ln a)}{48}\left(\frac{\frac{a}{b}\eta_0-1}{\ln a-\ln b+\ln\eta_0}\right)^{\frac{1}{2}}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\eta_0^{\frac{1}{2}}\sqrt{a}+\sqrt{b}\right),\end{aligned}$$

where η_0 is defined by (3.2).

Corollary 3.7. *In Theorem 3.5, letting φ be the identity function, $m = 1$ and $p = q = 2$, we have the following inequality for an α-logarithmically convex*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+6f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\\&\leqslant\frac{\sqrt{21b}(\ln b-\ln a)}{48}\left(\frac{\eta_3^{\alpha}-1}{\alpha\ln\eta_3}\right)^{\frac{1}{2}}|f'(b)|\left(\left[\frac{|f'(a)|}{|f'(b)|}\right]^{\frac{\alpha}{2}}\sqrt{a}+\sqrt{b}\right),\end{aligned}$$

where η_3 is defined by (3.11).

Theorem 3.8. *Let $\varphi : [a, b] \subseteq \mathbb{R} \to (0,\infty)$ be a continuous positive homogeneous function with $\varphi(a) < \varphi(b)$. Assume that $f : I \subseteq \mathbb{R} \to (0,\infty)$ is a differentiable mapping on I° (interior of I) such that $f' \in L\big([\varphi(a), \varphi(b)]\big)$. If $|f'(\varphi(x))|$ is increasing and relative semi-(α, m)-logarithmically convex function on $\left[0, \varphi\left(\frac{b}{m}\right)\right]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$, and $\mu, \tau > 0$ with $\mu + \tau = 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\left(\frac{b}{m}\right)\right|^m\\&\times\left\{\sqrt{\varphi(a)}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)}+\eta_0^{\frac{\alpha}{2\tau}}\frac{2\tau^2(\eta_2^{\frac{\alpha}{2\tau}}-1)}{\alpha\ln\eta_2}\right]+\sqrt{\varphi(b)}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)}+\frac{2\tau^2(\eta_2^{\frac{\alpha}{2\tau}}-1)}{\alpha\ln\eta_2}\right]\right\},\end{aligned}\tag{3.17}$$

where η_0 and η_2 are defined by (3.2) and (3.10), respectively.

Proof. Continuing from inequality (3.3) in the proof of Theorem 3.1, using the well-known inequality $rt \leqslant \mu r^{\frac{1}{\mu}} + \tau t^{\frac{1}{\tau}}$ to the right-hand side of inequality (3.3), we get

$$\begin{aligned}&\left|\frac{1}{8}\left[f(\varphi(a))+6f\left(\sqrt{\varphi(a)\varphi(b)}\right)+f(\varphi(b))\right]-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}d\varphi(x)\right|\\&\leqslant\frac{\ln\varphi(b)-\ln\varphi(a)}{4}\sqrt{\varphi(b)}\left|f'\left(\frac{b}{m}\right)\right|^m\\&\times\left\{\sqrt{\varphi(a)}\left(\mu\int_0^1\left|\frac{3}{4}-t\right|^{\frac{1}{\mu}}dt+\tau\int_0^1\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2\tau}}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{\frac{1}{\tau}\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)\right.\\&\left.+\sqrt{\varphi(b)}\left(\mu\int_0^1\left|\frac{1}{4}-t\right|^{\frac{1}{\mu}}dt+\tau\int_0^1\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2\tau}}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{\frac{1}{\tau}\left(\frac{t}{2}\right)^{\alpha}}dt\right)\right\}.\end{aligned}\tag{3.18}$$

If $0 < \eta_0 \leqslant 1$, by (3.5), we get

$$\begin{aligned}\int_0^1 \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2\tau}} \left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{\frac{1}{\tau}\left(\frac{1+t}{2}\right)^\alpha} dt &\leqslant \int_0^1 \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2\tau}} \eta_0^{\frac{\alpha(1+t)}{2\tau}} dt \\ &= \eta_0^{\frac{\alpha}{2\tau}} \int_0^1 \eta_2^{\frac{\alpha t}{2\tau}} dt \\ &= \eta_0^{\frac{\alpha}{2\tau}} \frac{2\tau(\eta_2^{\frac{\alpha}{2\tau}} - 1)}{\alpha \ln \eta_2},\end{aligned} \tag{3.19}$$

similarly,

$$\int_0^1 \left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2\tau}} \left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{\frac{1}{\tau}\left(\frac{t}{2}\right)^\alpha} dt \leqslant \frac{2\tau(\eta_2^{\frac{\alpha}{2\tau}} - 1)}{\alpha \ln \eta_2}. \tag{3.20}$$

Also,

$$\int_0^1 \left|\frac{3}{4} - t\right|^{\frac{1}{\mu}} dt = \int_0^1 \left|\frac{1}{4} - t\right|^{\frac{1}{\mu}} dt = \frac{\mu(1 + 3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)}. \tag{3.21}$$

Using (3.19), (3.20) and (3.21) in (3.18), we obtain the desired inequality (3.17). This completes the proof. □

Let us discuss some special cases of Theorem 3.8.

Corollary 3.9. *In Theorem 3.8, letting φ be the identity function and $\alpha = 1$, we have the following inequality for an m-logarithmically convex*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a) + 6f\left(\sqrt{ab}\right) + f(b)\right] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} dx\right| \\ &\leqslant \frac{\ln b - \ln a}{4}\sqrt{b}\left|f'\left(\frac{b}{m}\right)\right|^m \\ &\times \left\{\sqrt{a}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)} + \eta_0^{\frac{1}{2\tau}} \frac{2\tau^2\left((\frac{a}{b}\eta_0)^{\frac{1}{2\tau}} - 1\right)}{\ln a - \ln b + \ln \eta_0}\right] + \sqrt{b}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)} + \frac{2\tau^2\left((\frac{a}{b}\eta_0)^{\frac{1}{2\tau}} - 1\right)}{\ln a - \ln b + \ln \eta_0}\right]\right\},\end{aligned}$$

where η_0 is defined by (3.2).

Corollary 3.10. *In Theorem 3.8, letting φ be the identity function and $m = 1$, we have the following inequality for an α-logarithmically convex*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a) + 6f\left(\sqrt{ab}\right) + f(b)\right] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} dx\right| \\ &\leqslant \frac{\ln b - \ln a}{4}\sqrt{b}|f'(b)| \\ &\times \left\{\sqrt{a}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)} + \left[\frac{|f'(a)|}{|f'(b)|}\right]^{\frac{\alpha}{2\tau}} \frac{2\tau^2(\eta_3^{\frac{\alpha}{2\tau}} - 1)}{\alpha \ln \eta_3}\right] + \sqrt{b}\left[\frac{\mu^2(1+3^{1+\frac{1}{\mu}})}{4^{1+\frac{1}{\mu}}(1+\mu)} + \frac{2\tau^2(\eta_3^{\frac{\alpha}{2\tau}} - 1)}{\alpha \ln \eta_3}\right]\right\},\end{aligned}$$

where η_3 is defined by (3.11).

Remark 3.11. In Corollary 3.9, if we take $\mu = \tau = \frac{1}{2}$, we have

$$\left|\frac{1}{8}\left[f(a) + 6f\left(\sqrt{ab}\right) + f(b)\right] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leqslant \frac{\ln b - \ln a}{4}\sqrt{b}\left|f'\left(\frac{b}{m}\right)\right|^m \left\{\sqrt{a}\left[\frac{7}{96} + \frac{\eta_0(\frac{a}{b}\eta_0 - 1)}{2(\ln a - \ln b + \ln \eta_0)}\right] + \sqrt{b}\left[\frac{7}{96} + \frac{\frac{a}{b}\eta_0 - 1}{2(\ln a - \ln b + \ln \eta_0)}\right]\right\},$$

where η_0 is defined by (3.2).

Remark 3.12. In Corollary (3.10), if we take $\mu = \tau = \frac{1}{2}$, we have

$$\left|\frac{1}{8}\left[f(a) + 6f\left(\sqrt{ab}\right) + f(b)\right] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leqslant \frac{\ln b - \ln a}{4}\sqrt{b}|f'(b)|\left\{\sqrt{a}\left[\frac{7}{96} + \left(\frac{|f'(a)|}{|f'(b)|}\right)^\alpha \frac{\eta_3^\alpha - 1}{2\alpha \ln \eta_3}\right] + \sqrt{b}\left[\frac{7}{96} + \frac{\eta_3^\alpha - 1}{2\alpha \ln \eta_3}\right]\right\},$$

where η_3 is defined by (3.11).

Finally we shall prove the following result.

Theorem 3.13. *Let $\varphi : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ be a continuous positive homogeneous function with $\varphi(a) < \varphi(b)$. Assume that $f, g : I \subseteq \mathbb{R} \to (0, \infty)$ is a differentiable mapping on I° (interior of I) such that $f, g \in L\big([\varphi(a), \varphi(b)]\big)$. If $f(\varphi(x))$ is increasing and relative semi-(α, m_1)-logarithmically convex and $g(\varphi(x))$ is increasing and relative semi-(α, m_2)-logarithmically convex on $\left[0, \varphi\left(\frac{b}{m_i}\right)\right]$ for $(\alpha, m_i) \in (0, 1] \times (0, 1]$ and $i = 1, 2$, then the following inequality holds:*

$$\frac{1}{\ln \varphi(b) - \ln \varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f(\varphi(x))g(\varphi(x))}{\varphi(x)}d\varphi(x) \leqslant \left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\Upsilon(\alpha),$$

where

$$\lambda = f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{-m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{-m_2},$$

and

$$\Upsilon(\alpha) = \begin{cases} 1, & \lambda = 1, \\ \frac{\lambda^\alpha - 1}{\alpha \ln \lambda}, & 0 < \lambda < 1, \\ \frac{\lambda - \lambda^{1-\alpha}}{\alpha \ln \lambda}, & \lambda > 1. \end{cases}$$

Proof. Let $\varphi(x) = [\varphi(a)]^t[\varphi(b)]^{1-t}$ and using Young inequality, we have

$$\begin{aligned} &\frac{1}{\ln \varphi(b) - \ln \varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f(\varphi(x))g(\varphi(x))}{\varphi(x)}d\varphi(x) \\ &= \int_0^1 f\left([\varphi(a)]^t[\varphi(b)]^{1-t}\right)g\left([\varphi(a)]^t[\varphi(b)]^{1-t}\right)dt \\ &\leqslant \int_0^1 f\Big(t\varphi(a) + (1-t)\varphi(b)\Big)g\Big(t\varphi(a) + (1-t)\varphi(b)\Big)dt. \end{aligned} \tag{3.22}$$

The relative semi-(α, m)-logarithmically convexity of $f(\varphi(x))$ and $g(\varphi(x))$ yields

$$f\left(t\varphi(a) + m_1(1-t)\varphi\left(\frac{b}{m_1}\right)\right) \leqslant [f(a)]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)}, \tag{3.23}$$

and

$$g\left(t\varphi(a)+m_2(1-t)\varphi\left(\frac{b}{m_2}\right)\right) \leqslant [g(a)]^{t^\alpha}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)}. \tag{3.24}$$

Using (3.23) and (3.24) in (3.22), we have

$$\begin{aligned}
&\int_0^1 f\Big(t\varphi(a)+(1-t)\varphi(b)\Big)g\Big(t\varphi(a)+(1-t)\varphi(b)\Big)dt \\
&\quad\leqslant \int_0^1 \big[f(a)\big]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)}\big[g(a)\big]^{t^\alpha}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)}dt \\
&\quad= \left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\int_0^1\left\{f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{-m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{-m_2}\right\}^{t^\alpha}dt.
\end{aligned}$$

When $\lambda = 1$, we have

$$\int_0^1 \lambda^{t^\alpha}dt = 1,$$

when $\lambda < 1$, we have

$$\int_0^1 \lambda^{t^\alpha}dt \leqslant \int_0^1 \lambda^{\alpha t}dt = \frac{\lambda^\alpha - 1}{\alpha\ln\lambda},$$

when $\lambda > 1$, we have

$$\int_0^1 \lambda^{t^\alpha}dt \leqslant \int_0^1 \lambda^{\alpha t+1-\alpha}dt = \frac{\lambda-\lambda^{1-\alpha}}{\alpha\ln\lambda}.$$

The proof of Theorem 3.13 is completed. □

It is easy to deduce the following corollaries.

Corollary 3.14. *In Theorem 3.13, if $f(\varphi(x))$ is increasing and relative semi-m_1-logarithmically convex and $g(\varphi(x))$ is increasing and relative semi-m_2-logarithmically convex on $\left[0, \varphi\left(\frac{b}{m_i}\right)\right]$ for $i = 1, 2$ and some fixed $m_1, m_2 \in (0, 1]$, then*

$$\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))g(\varphi(x))}{\varphi(x)}d\varphi(x) \leqslant \left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\Upsilon(1),$$

where Υ is defined as in Theorem 3.13.

Corollary 3.15. *In Theorem 3.13, if $f(\varphi(x))$ and $g(\varphi(x))$ are increasing and relative semi-(α, m)-logarithmically convex on $\left[0, \varphi\left(\frac{b}{m}\right)\right]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$, then*

$$\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))g(\varphi(x))}{\varphi(x)}d\varphi(x) \leqslant \left[f\left(\frac{b}{m}\right)\right]^{m}\left[g\left(\frac{b}{m}\right)\right]^{m}\Upsilon(\alpha),$$

where Υ is defined as in Theorem 3.13.

Corollary 3.16. *In Theorem 3.13, if $f(\varphi(x))$ and $g(\varphi(x))$ are increasing and relative semi-m-logarithmically convex on $\left[0, \varphi\left(\frac{b}{m}\right)\right]$ for some fixed $m \in (0, 1]$, then*

$$\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f(\varphi(x))g(\varphi(x))}{\varphi(x)}d\varphi(x) \leqslant \left[f\left(\frac{b}{m}\right)\right]^{m}\left[g\left(\frac{b}{m}\right)\right]^{m}\Upsilon(1),$$

where Υ is defined as in Theorem 3.13.

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