



Approximation properties of solutions of a mean value type functional inequalities

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Abstract

We will prove the generalized Hyers-Ulam stability theorems of a mean value type functional equation, namely

$$f(x) - g(y) = (x - y)h(sx + sy),$$

which arises from the mean value theorem. As an application of our results, we introduce a characterization of quadratic polynomials. ©2017 All rights reserved.

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1. Introduction

The stability problem of functional equations seems to be first formulated by Ulam in 1940. Indeed he discussed a number of important unsolved problems (see [17]):

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, Hyers [6] gave a partial solution to the Ulam's problem for the case of approximate additive functions under the assumption that G_1 and G_2 are Banach spaces.

Considering this historical backgrounds, the additive Cauchy equation $f(x + y) = f(x) + f(y)$ on (G_1, G_2) is said to be stable in the sense of Hyers and Ulam. This terminology is also applied to the

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case of other functional equations. Moreover, Rassias [13] and Găvruta [3] generalized the stability result of Hyers for the additive functional equation. The stability phenomenon proved by Rassias and Găvruta is called the generalized Hyers-Ulam stability. For more detailed definitions of such terminologies, we refer the reader to [2, 7, 8, 9, 10].

The functional equation

$$f(x) - g(y) = (x - y)h(x + y), \quad (1.1)$$

where $x, y \in \mathbb{R}$ (the set of reals), arises from the mean value theorem and characterizes polynomials of degree one or two. This functional equation was originally treated by Aczél in 1963 and also independently by Haruki (see [1, 5]). A generalization of the functional equation (1.2) was treated by Kannappan et al. [11] (see also [15, Theorem 2.5] or [1, 5]). We summarize the result of Aczél regarding the equation (1.2) in the following theorem.

Theorem 1.1. *Let X be a field of characteristic different from 2. The functions $f, g, h : X \rightarrow X$ satisfy the functional equation (1.1) for all $x, y \in X$ if and only if there exist $a, b, c \in X$ such that*

$$f(x) = g(x) = ax^2 + bx + c \quad \text{and} \quad h(x) = ax + b$$

for all $x \in X$.

In Section 2 of this paper, we will prove the generalized Hyers-Ulam stability of the functional equation

$$f(x) - g(y) = (x - y)h(sx + sy), \quad (1.2)$$

where s is a nonzero parameter. Even though the functional equation (1.2) seems to be the same one as (1.1), we investigate the equation (1.2) as a special form of the general equation

$$f(x) - g(y) = (x - y)h(sx + ty).$$

The main result of this paper is a significant generalization of [10] as we shall see in Section 2. Section 3 is devoted to a characterization of quadratic polynomials.

2. Hyers-Ulam stability of (1.2)

In this section, let X be a commutative normed algebra over K with a unit element e (or a normed field of characteristic different from 2), where K is either \mathbb{R} or \mathbb{C} (see [14, Chapter 10] for more detailed definition of normed algebra).

As usual, we assume that a commutative normed algebra X has a sub-multiplicative norm $\|\cdot\|$, i.e., it satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\|,$$

for all $x, y \in X$, which makes the multiplication a continuous operation in X .

For a given function $\varphi : X \times X \rightarrow [0, \infty)$, we will use the following notation

$$\begin{aligned} \Phi_s(x, y) = & \varphi\left(\frac{x+y}{2s}, \frac{x-y}{2s}\right) + 2\varphi\left(\frac{x+y}{2s}, 0\right) + \varphi\left(\frac{x+y}{2s}, \frac{y-x}{2s}\right) \\ & + \varphi\left(\frac{y-x}{2s}, \frac{x-y}{2s}\right) + \varphi\left(\frac{y-x}{2s}, 0\right) + 2\varphi\left(0, \frac{x-y}{2s}\right) + \varphi\left(0, \frac{y-x}{2s}\right) \end{aligned}$$

for all $x, y \in X$, where s is a nonzero scalar parameter. Moreover, we assume that

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi_s(x, 2^n e) = 0 \quad (2.1)$$

for all $x \in X$.

For example, let p and θ be positive real numbers with $0 \leq p < 1$. If we temporarily define a function $\varphi : X \times X \rightarrow [0, \infty)$ by

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p),$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi_s(x, 2^n e) \leq \lim_{n \rightarrow \infty} \frac{3\theta}{2^{n+p-2}|s|^p} (\|x\|^p + 2^{pn}\|e\|^p) = 0.$$

The proof of the following theorem is strongly based on the proof of [10, Theorem 2] or [15, Theorem 2.3].

Theorem 2.1. *Let X be a commutative normed algebra over K with a unit element e (or a normed field of characteristic different from 2), where K is either \mathbb{R} or \mathbb{C} . Let s be a nonzero scalar parameter. Assume that a function $\varphi : X \times X \rightarrow [0, \infty)$ satisfies the condition (2.1). If functions $f, g, h : X \rightarrow X$ satisfy the inequality*

$$\|f(x) - g(y) - (x - y)h(sx + sy)\| \leq \varphi(x, y) \quad (2.2)$$

for all $x, y \in X$, then there exist uniquely determined $a, b \in X$ such that

$$\begin{aligned} \|f(x) - sax^2 - bx - g(0)\| &\leq \varphi(x, 0), \\ \|g(x) - sax^2 - bx - f(0)\| &\leq \varphi(0, x), \\ h(x) &= ax + b \end{aligned} \quad (2.3)$$

for all $x \in X$.

Proof. If we put $y = 0$ in (2.2), then we have

$$\|f(x) - xh(sx) - g(0)\| \leq \varphi(x, 0) \quad (2.4)$$

for each $x \in X$. Similarly, if we put $x = 0$ in (2.2), then we get

$$\|g(y) - yh(sy) - f(0)\| \leq \varphi(0, y) \quad (2.5)$$

for all $y \in X$. Further, by (2.2), (2.4) and (2.5), we obtain

$$\begin{aligned} &\|xh(sx) - yh(sy) - (x - y)h(sx + sy) - f(0) + g(0)\| \\ &\leq \|f(x) - g(y) - (x - y)h(sx + sy)\| + \|xh(sx) - f(x) + g(0)\| \\ &\quad + \|g(y) - yh(sy) - f(0)\| \\ &\leq \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) \end{aligned}$$

for all $x, y \in X$. If we replace x and y in the last inequality with $\frac{1}{s}x$ and $\frac{1}{s}y$, respectively, then we have

$$\begin{aligned} &\|xh(x) - yh(y) - (x - y)h(x + y) - sf(0) + sg(0)\| \\ &\leq |s|\varphi\left(\frac{x}{s}, \frac{y}{s}\right) + |s|\varphi\left(\frac{x}{s}, 0\right) + |s|\varphi\left(0, \frac{y}{s}\right) \end{aligned} \quad (2.6)$$

for all $x, y \in X$.

If h satisfies inequality (2.6), so does $h + c$, where c is an arbitrary element of X . Thus, we can assume that $h(0) = 0$. If we replace x by $-y$ in (2.6), then we get

$$\begin{aligned} &\|-yh(-y) - yh(y) - sf(0) + sg(0)\| \\ &\leq |s|\varphi\left(\frac{-y}{s}, \frac{y}{s}\right) + |s|\varphi\left(\frac{-y}{s}, 0\right) + |s|\varphi\left(0, \frac{y}{s}\right) \end{aligned} \quad (2.7)$$

for every $y \in X$. Moreover, if we replace y by $-y$ in (2.6), then

$$\begin{aligned} & \|xh(x) + yh(-y) - (x+y)h(x-y) - sf(0) + sg(0)\| \\ & \leq |s|\varphi\left(\frac{x}{s}, \frac{-y}{s}\right) + |s|\varphi\left(\frac{x}{s}, 0\right) + |s|\varphi\left(0, \frac{-y}{s}\right) \end{aligned} \quad (2.8)$$

for all $x, y \in X$. By (2.7) and (2.8), we further have

$$\begin{aligned} & \|xh(x) - yh(y) - (x+y)h(x-y) - 2sf(0) + 2sg(0)\| \\ & \leq \|xh(x) + yh(-y) - (x+y)h(x-y) - sf(0) + sg(0)\| \\ & \quad + \|-yh(-y) - yh(y) - sf(0) + sg(0)\| \\ & \leq |s|\varphi\left(\frac{x}{s}, \frac{-y}{s}\right) + |s|\varphi\left(\frac{-y}{s}, \frac{y}{s}\right) + |s|\varphi\left(\frac{x}{s}, 0\right) \\ & \quad + |s|\varphi\left(\frac{-y}{s}, 0\right) + |s|\varphi\left(0, \frac{y}{s}\right) + |s|\varphi\left(0, \frac{-y}{s}\right) \end{aligned} \quad (2.9)$$

for any $x, y \in X$.

On account of (2.6) and (2.9), we get

$$\begin{aligned} & \|(x+y)h(x-y) - (x-y)h(x+y) + sf(0) - sg(0)\| \\ & \leq \|-xh(x) + yh(y) + (x+y)h(x-y) + 2sf(0) - 2sg(0)\| \\ & \quad + \|xh(x) - yh(y) - (x-y)h(x+y) - sf(0) + sg(0)\| \\ & \leq |s|\varphi\left(\frac{x}{s}, \frac{y}{s}\right) + 2|s|\varphi\left(\frac{x}{s}, 0\right) + |s|\varphi\left(\frac{x}{s}, \frac{-y}{s}\right) + |s|\varphi\left(\frac{-y}{s}, \frac{y}{s}\right) \\ & \quad + |s|\varphi\left(\frac{-y}{s}, 0\right) + 2|s|\varphi\left(0, \frac{y}{s}\right) + |s|\varphi\left(0, \frac{-y}{s}\right) \end{aligned} \quad (2.10)$$

for every $x, y \in X$.

Substituting $u = x + y$ and $v = x - y$ in (2.10), we have

$$\|uh(v) - vh(u) + sf(0) - sg(0)\| \leq |s|\Phi_s(u, v)$$

for all $u, v \in X$. Furthermore, we obtain

$$\|uh(v) - vh(u)\| \leq |s|\Phi_s(u, v) + |s|\|f(0)\| + |s|\|g(0)\| \quad (2.11)$$

for any $u, v \in X$. Substituting $v = 2^n e$ in (2.11), we have

$$\|uh(2^n e) - 2^n eh(u)\| \leq |s|\Phi_s(u, 2^n e) + |s|\|f(0)\| + |s|\|g(0)\|$$

and hence

$$\left\| \frac{1}{2^n} uh(2^n e) - h(u) \right\| \leq \frac{|s|}{2^n} \left[\Phi_s(u, 2^n e) + \|f(0)\| + \|g(0)\| \right] \quad (2.12)$$

for all $u \in X$ and $n \in \mathbb{N}$.

In view of (2.1), if we let $n \rightarrow \infty$ in (2.12), then we get

$$h(u) = \lim_{n \rightarrow \infty} \frac{1}{2^n} uh(2^n e)$$

for each $u \in X$. If we put $u = e$ in the last equality, then we obtain

$$h(e) = \lim_{n \rightarrow \infty} \frac{1}{2^n} h(2^n e).$$

Since the multiplication is a continuous operation in X , we have

$$h(x) = ax$$

for all $x \in X$, where we set $a = h(e)$. If we do not assume $h(0) = 0$, then we have

$$h(x) = ax + b. \quad (2.13)$$

In view of (2.4) and (2.13), we obtain

$$\|f(x) - sax^2 - bx - g(0)\| \leq \varphi(x, 0)$$

for any $x \in X$. Similarly, by (2.5) and (2.13), we have

$$\|g(x) - sax^2 - bx - f(0)\| \leq \varphi(0, x)$$

for all $x \in X$.

Finally, we will prove the uniqueness of a and b . Assume that $a', b' \in X$ satisfy the inequalities and equality in (2.3). By the second inequality of (2.3), we have

$$\|s(a' - a)x^2 + (b' - b)x\| \leq 2\varphi(0, x)$$

for all $x \in X$. If we set $x = \frac{2^{n-1}}{s}e$, then we obtain

$$\|2^{n-1}(a' - a)e^2 + (b' - b)e\| \leq \frac{4|s|}{2^n} \varphi\left(0, \frac{2^{n-1}}{s}e\right) \leq \frac{4|s|}{2^n} \Phi_s(0, 2^n e)$$

for all $n \in \mathbb{N}$. In view of (2.1), if we let $n \rightarrow \infty$ in the last inequality, then we have

$$\lim_{n \rightarrow \infty} 2^{n-1}(a' - a) = b - b'.$$

Hence, we conclude that $a' - a = 0$ and $b - b' = 0$, which completes our proof. \square

In Theorem 2.1, the difference between f (or g) and a quadratic polynomial is bounded by $\varphi(x, 0)$ (or $\varphi(0, x)$), which is a significant improvement of [10, Theorem 2] even though the additional condition (2.1) is assumed in this paper. Moreover, h is explicitly determined in this paper, while it was given approximately by using an inequality in [10, Theorem 2].

If $\varphi : X \times X \rightarrow [0, \infty)$ satisfies the additional condition

$$\varphi(x, y) \geq \varphi(x, 0) + \varphi(0, y) + \varphi(0, 0), \quad (2.14)$$

as well as the condition (2.1), then the converse of Theorem 2.1 is also true. For example, if $X = \mathbb{R}$ and $\varphi(x, y) = \theta|x|^p|y|^q$ for some positive real numbers p, q and θ with $0 < p + q < 1$, then all the conditions in (2.1) and (2.14) are satisfied.

Theorem 2.2. *Let X be a commutative normed algebra over K with a unit element e (or a normed field of characteristic different from 2), where K is either \mathbb{R} or \mathbb{C} . Let s be a nonzero scalar parameter. Assume that a function $\varphi : X \times X \rightarrow [0, \infty)$ satisfies the conditions (2.1) and (2.14) for all $x, y \in X$. Then functions $f, g, h : X \rightarrow X$ satisfy the inequality (2.2) for all $x, y \in X$ if and only if there exist uniquely determined $a, b \in X$ such that the inequalities and equality in (2.3) hold for all $x \in X$.*

Proof. We only need to prove that if there exist uniquely determined $a, b \in X$ such that the inequalities and equality in (2.3) hold for all $x \in X$, then the functions $f, g, h : X \rightarrow X$ satisfy the inequality (2.2) for all $x, y \in X$.

Indeed, it follows from (2.3) and (2.14) that

$$\begin{aligned} \|f(x) - g(y) - (x - y)h(sx + sy)\| &\leq \|f(x) - sax^2 - bx - g(0)\| \\ &\quad + \|-g(y) + say^2 + by + f(0)\| + \|g(0) - f(0)\| \\ &\leq \varphi(x, 0) + \varphi(0, y) + \varphi(0, 0) \\ &\leq \varphi(x, y) \end{aligned}$$

for all $x, y \in X$, i.e., the functions f, g, h satisfy the inequality (2.2) for all $x, y \in X$. The proof is now complete. \square

If we set $\varphi(x, y) = \varepsilon$ in Theorem 2.1, then we obtain the Hyers-Ulam stability of the functional equation (1.2).

Corollary 2.3. *Let X be a commutative normed algebra over K with a unit element e (or a normed field of characteristic different from 2), where K is either \mathbb{R} or \mathbb{C} . For any previously given $\varepsilon > 0$, if functions $f, g, h : X \rightarrow X$ satisfy the inequality*

$$\|f(x) - g(y) - (x - y)h(sx + sy)\| \leq \varepsilon$$

for all $x, y \in X$ and for some nonzero scalar parameter s , then there exist uniquely determined $a, b \in X$ such that

$$\begin{aligned} \|f(x) - sax^2 - bx - g(0)\| &\leq \varepsilon, \\ \|g(x) - sax^2 - bx - f(0)\| &\leq \varepsilon, \\ h(x) &= ax + b \end{aligned}$$

for all $x \in X$.

In the following corollary, we prove the hyperstability of the functional equation

$$xh(x) - yh(y) = (x - y)h(x + y), \quad (2.15)$$

by using Theorem 2.1. That is, we will prove that every solution to a perturbed equation of (2.15) is just the solution to the exact equation (2.15). For more detailed definition of hyperstability, we refer the reader to [4, 12].

Corollary 2.4. *Let X be a commutative normed algebra over K with a unit element e (or a normed field of characteristic different from 2), where K is either \mathbb{R} or \mathbb{C} . Assume that a function $\varphi : X \times X \rightarrow [0, \infty)$ satisfies the condition (2.1) for $s = 1$. If any function $h : X \rightarrow X$ satisfies the inequality*

$$\|xh(x) - yh(y) - (x - y)h(x + y)\| \leq \varphi(x, y)$$

for all $x, y \in X$, then

$$xh(x) - yh(y) = (x - y)h(x + y)$$

for all $x, y \in X$.

Proof. If we put $s = 1$, $f(x) = xh(x)$ and $g(y) = yh(y)$ for all $x, y \in X$, then the functions $f, g, h : X \rightarrow X$ satisfy inequality (2.2) for all $x, y \in X$. Therefore, the equality

$$h(x) = ax + b$$

holds for all $x \in X$. From the above equality, we get the desired equality

$$xh(x) - yh(y) - (x - y)h(x + y) = x(ax + b) - y(ay + b) - (x - y)(ax + ay + b) = 0$$

for all $x, y \in X$. \square

3. A characterization of quadratic polynomials

In this section, we introduce a characterization of polynomials of degree one or two as an application of Theorem 2.1.

Corollary 3.1. *Let $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ be a function satisfying the condition (2.1) for a previously given nonzero parameter s . A differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the inequality*

$$|f(x) - f(y) - (x - y)f'(sx + sy)| \leq \varphi(x, y) \quad (3.1)$$

for all $x, y \in \mathbb{R}$ if and only if there exist uniquely determined $a, b, c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} \frac{1}{2}ax^2 + bx + c & (\text{for } s = \frac{1}{2}), \\ bx + c & (\text{for } s \notin \{0, \frac{1}{2}\}), \end{cases}$$

for all $x \in \mathbb{R}$.

Proof. First, we assume that a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the inequality (3.1) for all $x, y \in \mathbb{R}$. We set $g = f$ and $h = f'$ in Theorem 2.1. Then there exist uniquely determined $a, b \in \mathbb{R}$ such that

$$\begin{cases} |f(x) - sax^2 - bx - f(0)| \leq \varphi(0, x), \\ f'(x) = ax + b \end{cases} \quad (3.2)$$

for all $x \in \mathbb{R}$. By using the relations of (3.2), we have

$$\left| \left(\frac{1}{2} - s \right) a \right| \leq \frac{1}{|x|^2} \varphi(0, x)$$

for any $x \in \mathbb{R}$. If we set $x = \frac{2^{n-1}}{s}$ for $n \in \mathbb{N}$, then we get

$$\left| \left(\frac{1}{2} - s \right) a \right| \leq \frac{s^2}{2^{n-2}} \frac{1}{2^n} \varphi \left(0, \frac{2^{n-1}}{s} \right) \leq \frac{s^2}{2^{n-2}} \frac{1}{2^n} \Phi_s(0, 2^n)$$

for each $n \in \mathbb{N}$. If we let $n \rightarrow \infty$ in the last inequality, then it follows from (2.1) that $\left| \left(\frac{1}{2} - s \right) a \right| = 0$, i.e., $s = \frac{1}{2}$ or $a = 0$. On account of (3.2), we see that

$$f(x) = \frac{1}{2}ax^2 + bx + c \quad (\text{for } s = \frac{1}{2}) \quad \text{or} \quad f(x) = bx + c \quad (\text{for } s \notin \{0, \frac{1}{2}\})$$

for all $x \in \mathbb{R}$.

Finally, we can easily verify the truth of the converse direction of our assertion. Hence, we omit the converse proof. \square

Szostok and Wąsowicz proved in their paper [16] that if functions $f, F : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the inequality

$$\left| F(x) - F(y) - (x - y)f\left(\frac{x + y}{2}\right) \right| \leq \varepsilon$$

for all $x, y \in \mathbb{R}$, then f satisfies the equation

$$G(x) - G(y) = (x - y)f\left(\frac{x + y}{2}\right)$$

for all $x, y \in \mathbb{R}$, where we set $G(x) = xf\left(\frac{x}{2}\right)$. This superstability result of Szostok and Wąsowicz is a special case of Corollary 3.1 with $s = \frac{1}{2}$.

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